

Goldbach conjecture : a typical statistical matter.

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Abstract By selecting different sets of numbers, we show to some extent that the Goldbach conjecture is mainly a statistical matter.

Conjecture de Goldbach : une affaire de statistiques.

Résumé En choisissant différents ensembles de nombres, nous montrons à quel point la conjecture de Goldbach est essentiellement une affaire de statistiques.

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SIGNS ET ABBREVIATIONS

N	Set of natural integers
p	Any prime number
P	Set of prime numbers
\wedge	Exponent sign ($x^\wedge n = x^n$)
\	Such as
\in	Sign of membership to a set ($n \in N, p \in P$)

REFERENCES

- [1] http://fr.wikipedia.org/wiki/Conjecture_de_Goldbach
- [2] http://fr.wikipedia.org/wiki/Th%C3%A9orie_des_nombres_premiers

1 Formulation of the conjecture

We focus here on Goldbach's conjecture [1] in its so-called 'strong' version, namely :

Every even integer greater than 3 can be written as the sum of two prime numbers

This problem has been checked for even integers up to $4 \cdot 10^{18}$.

These checks may be accompanied by a precise count of the number of solutions.

What will occupy us here is this quantitative aspect, rather than the qualitative one, which is to prove that at least a solution exists.

2 Formula of Hardy-Littlewood

This formula stems from a criterion of rarefaction of prime numbers and a test of divisibility of the current even number.
Let us write

$$2n = p+q \quad (1)$$

for "Goldbach equation" and let us have $\#(2n)$ the number of solutions of this equation.

Asymptotically, the quantity of prime numbers is given by the formula proved independently by Hadamard and De La Vallée Poussin in 1896 [2] :

$$\pi_1(m) \approx m/\ln(m) \quad (2)$$

and thus, asymptotically, the gap between prime numbers is about $\ln(m)$ at abscissa m .

One can also recursively approximate the position of a prime number p_i by the formulas :

$$q_i = q_{i-1} + \ln(q_{i-1}), q_1 = 2, p_i \approx q_i \quad (3)$$

For small prime numbers, the relative deviation is rather large, reducing slowly (namely at a logarithmic pace) :

p	q	Error	Relative error
2	2,0	0,0	0,00%
11	6,6	4,4	40,05%
101	77,6	23,4	23,20%
1009	933,2	75,8	7,51%
10007	9840,0	167,0	1,67%
100003	99560,6	442,4	0,44%
1000003	998204,0	1799,0	0,18%

Then crossing the prime numbers, the average amount of couples of primes (effect due to the depletion of primes) corresponding to the equation $2n = p+q$ is given by :

$$\pi_2(2n) \approx 2n/\ln^2(2n) \quad (4)$$

For the divisibility criterion, we use the concepts of localization (modulo p).

Let us have q the odd prime divisors of $2n$. Let p be a prime number. We project the set P of all the prime numbers onto the congruencies modulo p classes :

$$\begin{array}{ccc} \text{modulo} & & \\ P & \rightarrow & \{0, 1, 2, \dots, p-1\} \\ p_i & & p_i \bmod p \end{array} \quad (5)$$

This application projects a unique number to 0. It is p . The other classes are images in same density of all the other primes. By assigning a probability density to the quantities of numbers projected on each of the congruencies 0, 1, 2,..., $p-1$ and arbitrarily summing all of the densities up to p (i.e. to get an average density of 1 for each class of congruency), we get correspondence :

Congruencies	0	1	2	...	$p-1$
Normalized densities of probability	$\rightarrow 0$	$\rightarrow p/(p-1)$	$\rightarrow p/(p-1)$		$\rightarrow p/(p-1)$

For each variable, the representative classes are locally :

		$q_1 \bmod p$			
		1	2	...	$p-1$
$q_2 \bmod p$	1	2	3	...	0
	2	3	4	...	1

	$p-2$	$p-1$	0	...	$p-1$
	$p-1$	0	$p+1$...	$p-2$

We get then for the classes collected within the table :

$$\begin{aligned}\#\{n = 0 \bmod p\} &= p-1 \quad (\text{main secondary diagonal}) \\ \#\{n \neq 0 \bmod p\} &= p-2 \quad (\text{other secondary diagonals})\end{aligned}$$

This gives the density, versus some factor, of the numbers n at sequence p (including for $p = 2$). The overall proportion is then rendered by the product of these values for $p = 2$ to ∞ .

To get the singular series (the enshrined mathematical term), it is sufficient to adjust the mean frequency to 1. Among the classes $[0, 1, 2, p-1]$, there are the target 0 with cardinal $\#(0)$ and some $p-2$ other targets with cardinal $\#\{c \neq 0\}$. The adjustment factor f is given thus by $f(1.\#(0)+(p-1).\#\{c \neq 0\}) = p$ number of elements, that is $f((p-1)+(p-1).(p-2)) = p$, and finally $f = p/(p-1)^2$.

Hence :

$$\begin{aligned}\#_{\text{adjusted}}(n = 0 \bmod p) &= f.(p-1) = p/(p-1) = 1+1/(p-1) \\ \#_{\text{adjusted}}(n \neq 0 \bmod p) &= f.(p-2) = p.(p-2)/(p-1)^2 = 1-1/(p-1)^2\end{aligned}$$

Using these factors, the Goldbach enumeration is then given, by :

$$\pi(p-q=2n) \approx \prod_{\substack{p \nmid n \\ p \geq 3}} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p \nmid n \\ p \geq 3}} \left(1 + \frac{1}{(p-1)}\right) \frac{2n}{\ln^2(n)} \quad (6)$$

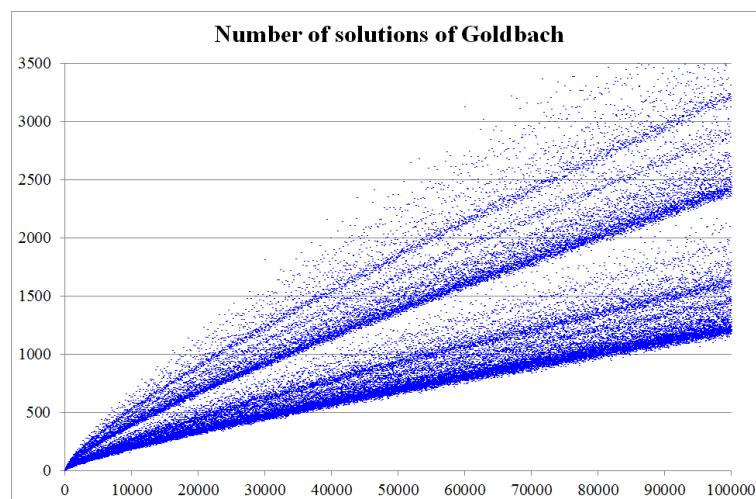
That is also

$$\pi(p+q=2n) \approx \prod_{\substack{p \geq 3 \\ p \nmid n \\ p \geq 3}} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p \geq 3 \\ p \nmid n \\ p \geq 3}} \left(\frac{p-1}{p-2}\right) \frac{2n}{\ln^2(n)} \quad (7)$$

This local method can be generalized to many other Diophantine equations. Besides, Hardy and Littlewood's formula was established in 1923 in a more general framework (k prime numbers decomposition) on the basis of the method of the circle.

3 Graphical matters

The number of solutions for $2n < 100\,000$ is represented below in the form of the 'Goldbach Comet'. One can see concentrations of cardinals by characteristic striations. We will see later the (presumed) origin of these concentrations.



4 Simulation methods

We now perform some manipulation on the basic sets to locate the Hardy Littlewood enumeration in more or less wider

environment (remaining with two variables but not necessarily of prime numbers).

4.1 Depletion method

Method of depletion

We start with a set including all odd numbers initially and observe how the number of solutions is evolving. Here, we could imagine doing this with the Eratosthenes sieve but removing first the highest multiples of primes. We consider an alternative, quite different, method here.

Let us have the set R of numbers r

$$R = \{r \mid r = a.(n_1+n_2)+b.n_1.n_2\}$$

where (a,b) is a couple of natural numbers given in advance, and n_1 and n_2 run over the natural numbers (including 0). We consider then S, the complementary set of natural numbers of R in N :

$$S = N - R = \{s \in N \setminus s \neq a.(n_1+n_2)+b.n_1.n_2\}$$

Finally, one collects the list of integers T

$$T = \{t \mid t = 2s+9\} \quad (8)$$

We give below the table in the beginning of this list of numbers t for different pairs (a,b). One chooses (a,b) such that a and b are coprimes (and here $a = b+1$) :

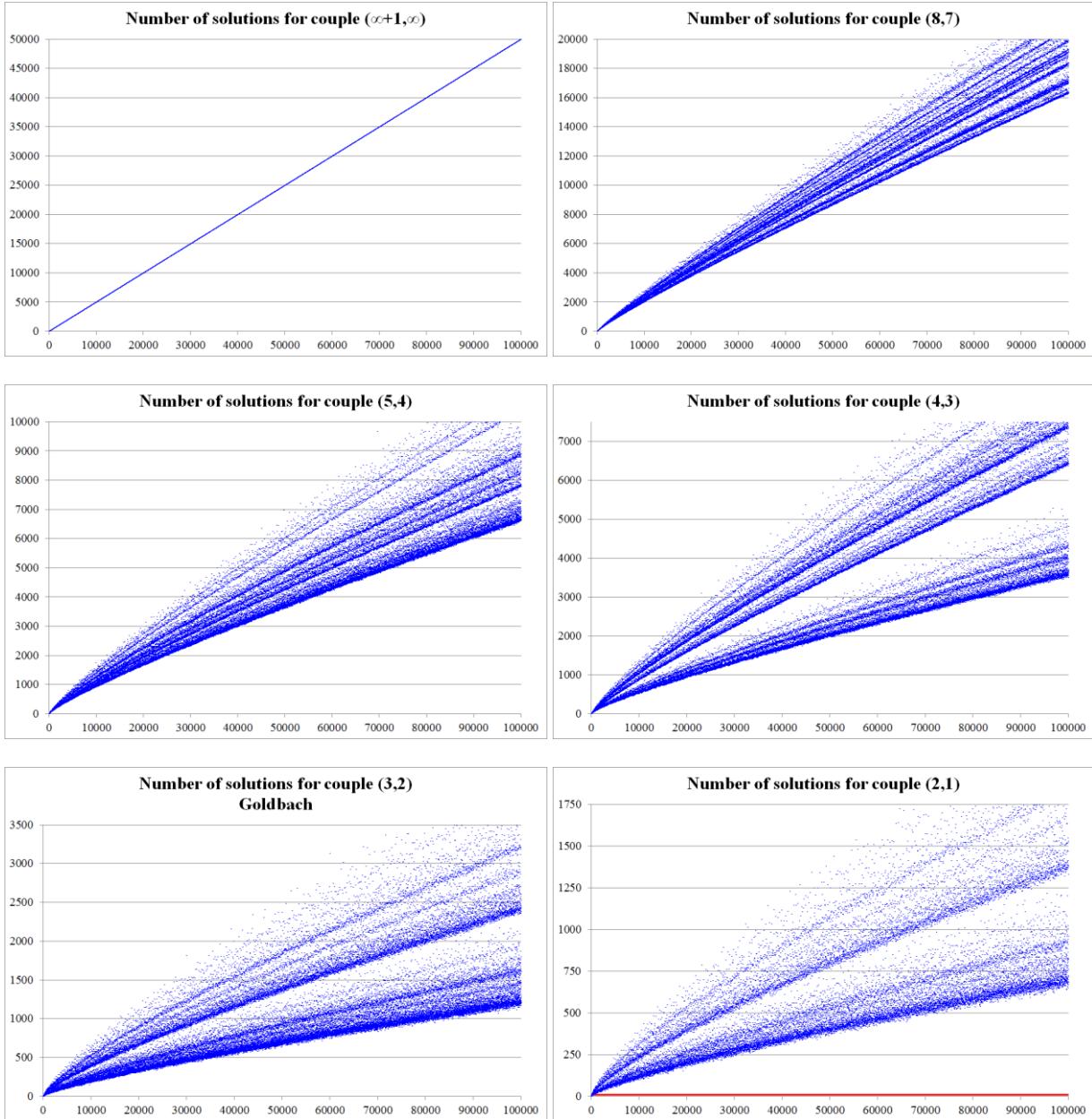
(2,1)	(3,2)	(4,3)	(5,4)	(8,7)	$(\infty+1, \infty)$
11	11	11	11	11	11
15	13	13	13	13	13
23	17	15	15	15	15
27	19	19	17	17	17
35	23	21	21	19	19
39	29	23	23	21	21
47	31	27	25	23	23
59	37	29	27	27	25
63	41	35	31	29	27
75	43	37	33	31	29
83	47	39	35	33	31
87	53	43	41	35	33
95	59	47	43	37	35
107	61	51	45	39	37
119	67	53	47	43	39
123	71	55	51	45	41
135	73	61	53	47	43
143	79	63	57	49	45
147	83	67	61	51	47
159	89	69	63	53	49
167	97	71	65	59	51
179	101	75	67	61	53
195	103	77	71	63	55
203	107	79	75	65	57
207	109	83	77	67	59
215	113	91	83	69	61
219	127	93	85	71	63
227	131	95	87	75	65
255	137	99	93	77	67
263	139	103	95	79	69
275	149	107	97	81	71
279	151	109	101	83	73
299	157	117	103	87	75

The attentive reader observes that the column (3.2) gives all the prime numbers starting at 11, because it is the complementary of the set of numbers of the type $(3+2n_1).(3+2n_2)$ but retaining only the odd numbers.

On the other hand, the columns on the left are less abundant in numbers than the columns on the right. Nevertheless, a column to the right of column (3.2) does not necessarily contain all prime numbers greater than 11.

We then cross (adding) these numbers (pp,qp) to get $2,np = pp+qp$ and count #(2,np) as in the case Goldbach enumeration.

The charts below give corresponding abundances starting to the right of the table.



We can observe the similar look of the different representations of these enumerations and the evolution from the straight line to the Goldbach cloud. It is important to note that the scatter plot evolves without excessive dispersion, and especially at lower values, throughout the process of depletion.

If it were possible to quantify precisely the evolution of these dispersions, it would be eventually possible to prove Goldbach's conjecture (and going much further).

The Goldbach cloud is not here the ultimate image. The last chart is special. Indeed, as all initial numbers p and q are equal to 3 modulo 4, there is no solution to $2np = p + q$ for $2np \equiv 0 \pmod{4}$. On the other hand, all numbers $2np \equiv 2 \pmod{4}$, starting at 22, gets at least one solution (up to infinity a priori).

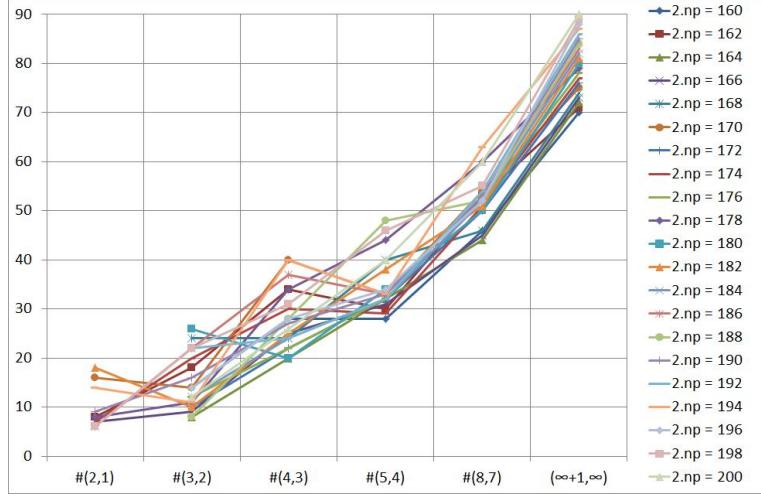
$2np$	$\#(2,1)$	$\#(3,2)$	$\#(4,3)$	$\#(5,4)$	$\#(8,7)$	$(\infty+1, \infty)$
22	1	1	1	1	1	1
24		2	2	2	2	2
26	2	1	3	3	3	3
28		2	2	4	4	4
30	1	4	3	3	5	5
32		2	4	4	6	6
34	2	3	6	5	7	7
36		4	4	6	6	8
38	4	1	5	8	7	9
40		4	6	6	8	10
42	2	6	7	7	9	11

2,np	#(2,1)	#(3,2)	#(4,3)	#(5,4)	#(8,7)	$(\infty+1,\infty)$
44		2	4	8	10	12
46	3	3	5	9	11	13
48		6	8	10	12	14
50	6	4	10	7	14	15
52		4	6	8	12	16
54	3	8	7	9	13	17
56		4	8	12	14	18
58	4	5	11	14	15	19
60		10	6	10	16	20
62	6	3	10	11	17	21
64		6	10	12	18	22
66	2	8	14	13	19	23
68		2	8	16	16	24
70	5	8	9	11	17	25
72		10	10	12	18	26
74	8	5	15	16	19	27
76		6	12	16	20	28
78	3	10	11	20	21	29
80		6	12	14	24	30
82	4	7	18	15	26	31
84		14	12	16	22	32
86	8	5	13	17	23	33
88		6	14	22	24	34
90	4	16	24	15	25	35
92		6	14	18	26	36
94	5	7	15	17	27	37
96		12	14	18	28	38
98	10	6	20	24	31	39
100		10	12	18	26	40
102	6	14	15	17	27	41
104		6	18	18	28	42
106	6	7	23	19	29	43
108		12	14	26	30	44
110	10	8	17	24	34	45
112		10	16	20	34	46
114	4	16	24	21	38	47
116		8	16	22	32	48
118	7	9	22	30	33	49
120		22	18	22	34	50
122	12	7	25	21	35	51
124		10	14	22	36	52
126	3	20	17	25	37	53
128		6	18	36	38	54
130	8	12	28	25	43	55
132		16	20	24	36	56
134	12	7	19	25	37	57
136		8	18	30	38	58
138	4	14	27	34	39	59
140		12	16	26	42	60
142	8	11	23	25	41	61
144		18	22	26	42	62
146	12	9	36	32	48	63
148		10	18	36	40	64
150	7	24	23	27	41	65
152		6	22	26	42	66
154	8	12	31	27	45	67
156		18	22	28	44	68
158	14	7	23	36	45	69
160		14	28	28	46	70
162	8	18	34	30	53	71
164		8	20	32	44	72
166	7	9	25	31	45	73
168		24	24	40	46	74
170	16	14	40	33	54	75
172		10	22	32	50	76
174	7	20	30	29	51	77
176		12	22	32	52	78
178	8	11	34	44	60	79
180		26	20	34	50	80
182	18	10	25	38	51	81
184		12	24	32	52	82
186	6	22	37	33	53	83

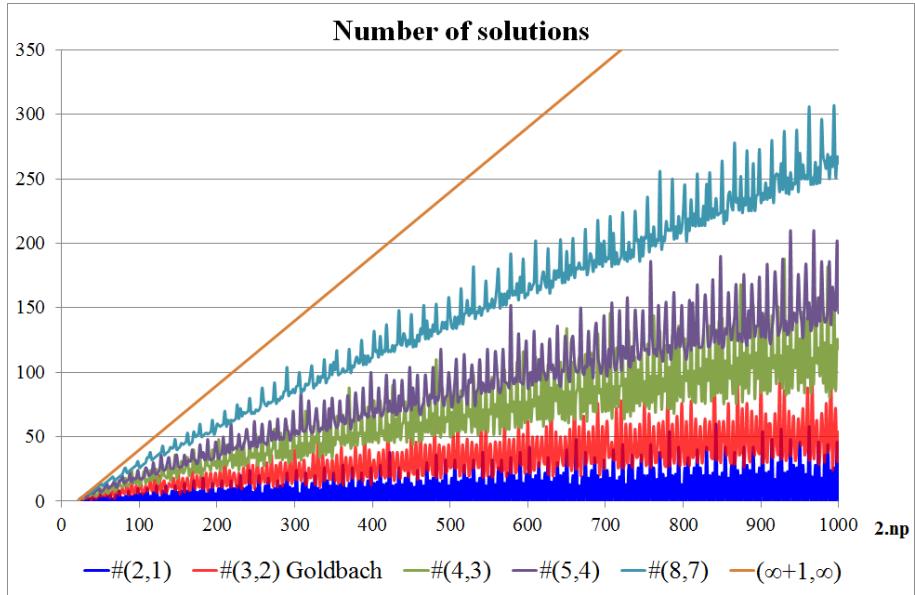
2.np	#(2,1)	#(3,2)	#(4,3)	#(5,4)	#(8,7)	$(\infty+1,\infty)$
188		8	28	48	52	84
190	9	16	27	33	53	85
192		22	24	32	54	86
194	14	11	40	33	63	87
196		14	28	34	52	88
198	6	22	31	46	55	89
200		12	26	40	60	90

Note: The column (3.2) shows less solutions than exact Goldbach enumeration as the list primes opens here by 11 (instead of 3).

The charts below give a sample of changes in the quantities of solutions for some 2.np values given in advance.



The numbers of solutions, based on given pairs (a, b), are also represented below :



Note also here that peak values are mainly directed upside, what is particularly visible for the couples (5.4) and (8.7).

In the appendix 1, we present another set of results based on the choice of pairs (a,b) with $a > 1$, $b = 1$. Again, we have systematically solutions for the equation $2np = pp+qp$, except for $a = 2$ (case already seen above). This set however does not cross the Goldbach case.

These results do open a generalization of the Goldbach conjecture :

Conjecture

Any set formed by crossing (summing) of T generators based on the couple of integers (a,b) admits at least one solution for $2np = pq+qp$, when a and b are coprimes and $a \geq 3$, $b \geq 1$.

4.2 Method of regular gaps

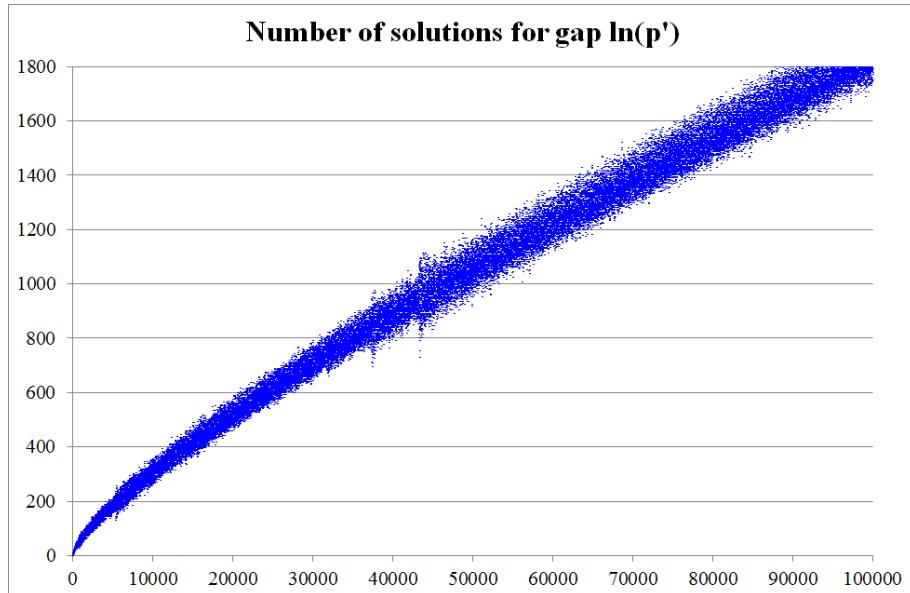
We recalled above is approximately the value of the prime numbers p_i by the recursive formula :

$$q_i = q_{i-1} + \ln(q_{i-1}), q_1 = 2, p_i \approx q_i \quad (9)$$

We use this type of recursive formula then selecting the nearest odd number to the found successive value. If the number of solutions is essentially a matter of statistics, we should find an analogy with distribution $\#(2n)$.

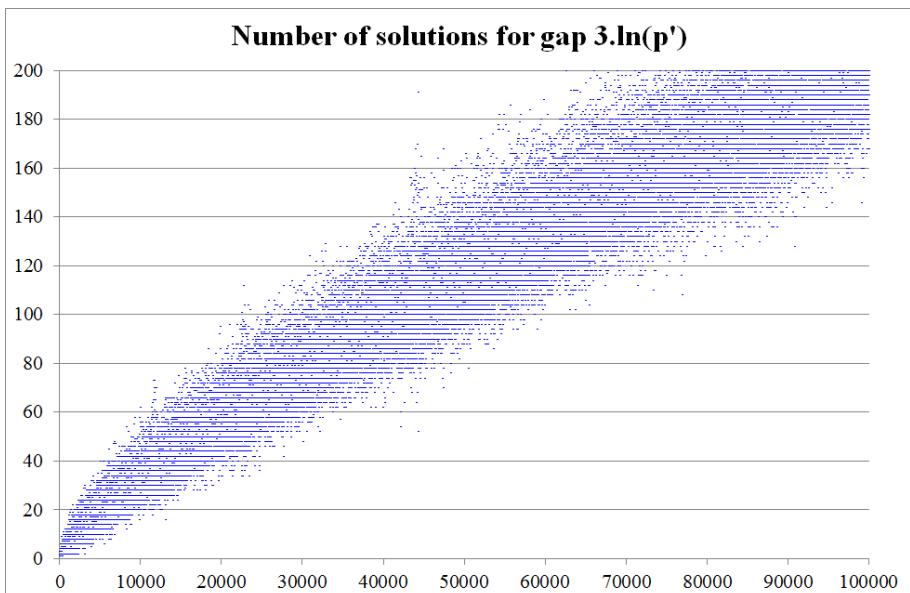
We have chosen three recursive models:

Model 1 : $q_i = q_{i-1} + \ln(q_{i-1})$, $q_1 = 2$, $p_i' = \text{rounded modulo 2 of } (q_i)$



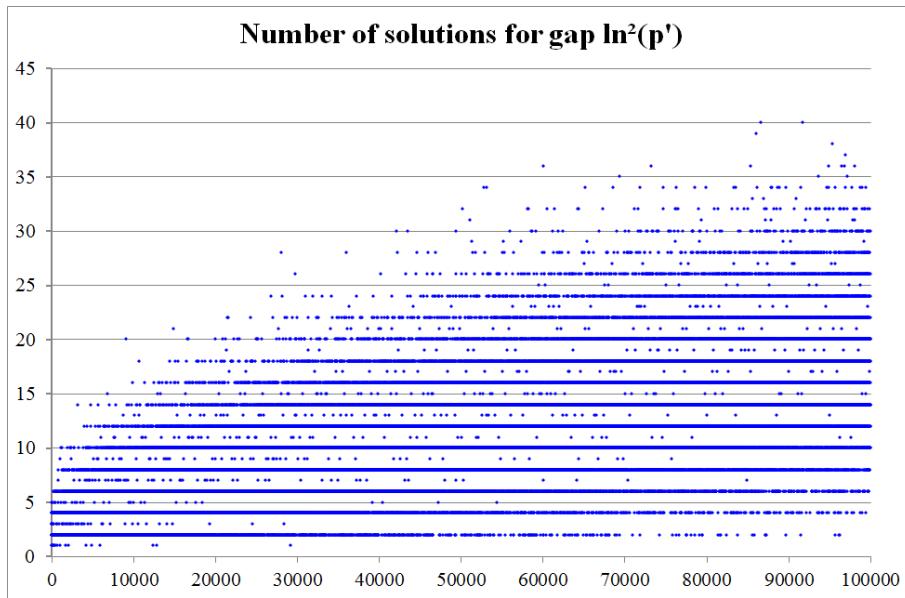
In this model, that is similar in numbers density, the concentrations by streaks disappear. The cloud focuses around a mean curve with a fairly steady progression.

Model 2 : $q_i = q_{i-1} + 3 \cdot \ln(q_{i-1})$, $q_1 = 2$, $p_i' = \text{rounded modulo 2 of } (q_i)$



We start here with a model with almost three times less generating numbers than in the case of Goldbach conjecture. We see that despite this wide depletion, the cloud of points fruits still generously.

Model 3 : $q_i = q_{i-1} + \ln^2(q_{i-1})$, $q_1 = 2$, $p_i' = \text{rounded modulo 2 of } (q_i)$



This model is inspired by Cramer's conjecture stating that the maximum gap between two primes is $O(\ln^2(p_i))$ at the p_i abscissa and is rarely higher than $\ln^2(p_i)$.

It is therefore more than an extreme case. One pictured here that the set of solutions is often empty and that the number of solutions of favourable cases ascend sparingly.

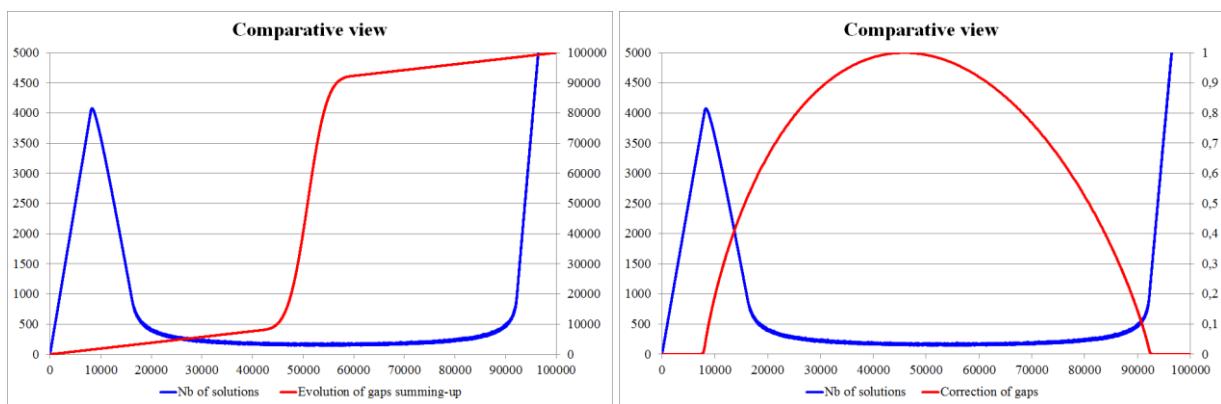
4.3 Method of the extreme gaps

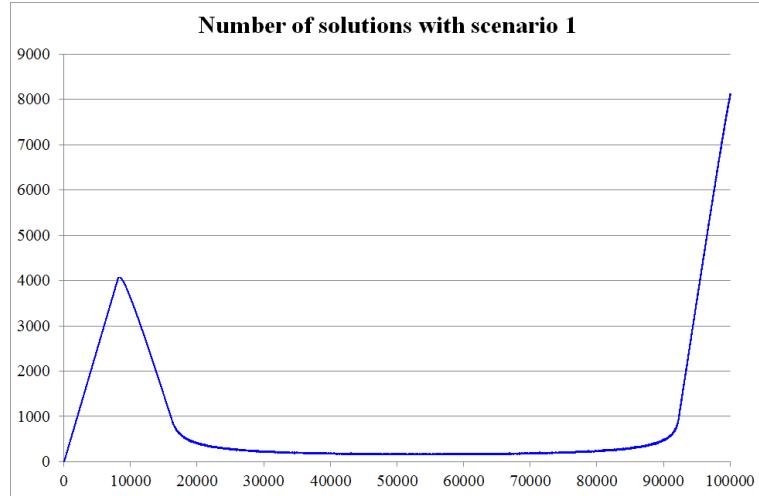
In this paragraph, we choose a generating set having, within a given interval (here the interval from 0 to 100000), the same number of odd items as in the initial set (the P set).

Thus from 0 to 100000 we have 9595 odd primes. So let's choose a set with 9595 odd numbers between 0 and 100000 to in order to cross them and display the number of solutions $2.np$. This choice will not be done randomly, otherwise we would end up in a close $\ln(p)$ recursive case. We choose instead extreme situations, namely ranges of systematic gap 2 and ranges of $\ln^2(p)$ systematic gaps in order to check if this type of concentration is likely to reduce the number of solutions to 0 for sufficiently high $2n$.

We have chosen 4 scenarios :

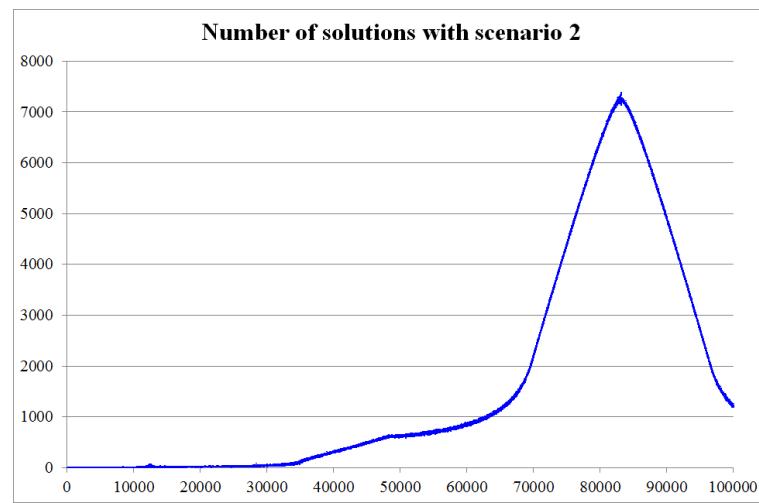
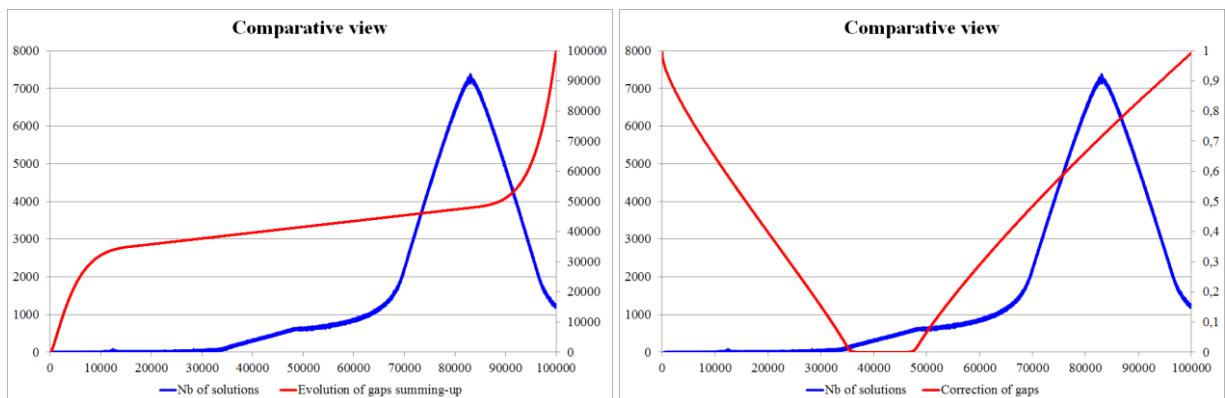
Model 1 : gaps of 2 near the origin and near maximum selected abscissa, intermediary gaps about $\ln^2(p')$





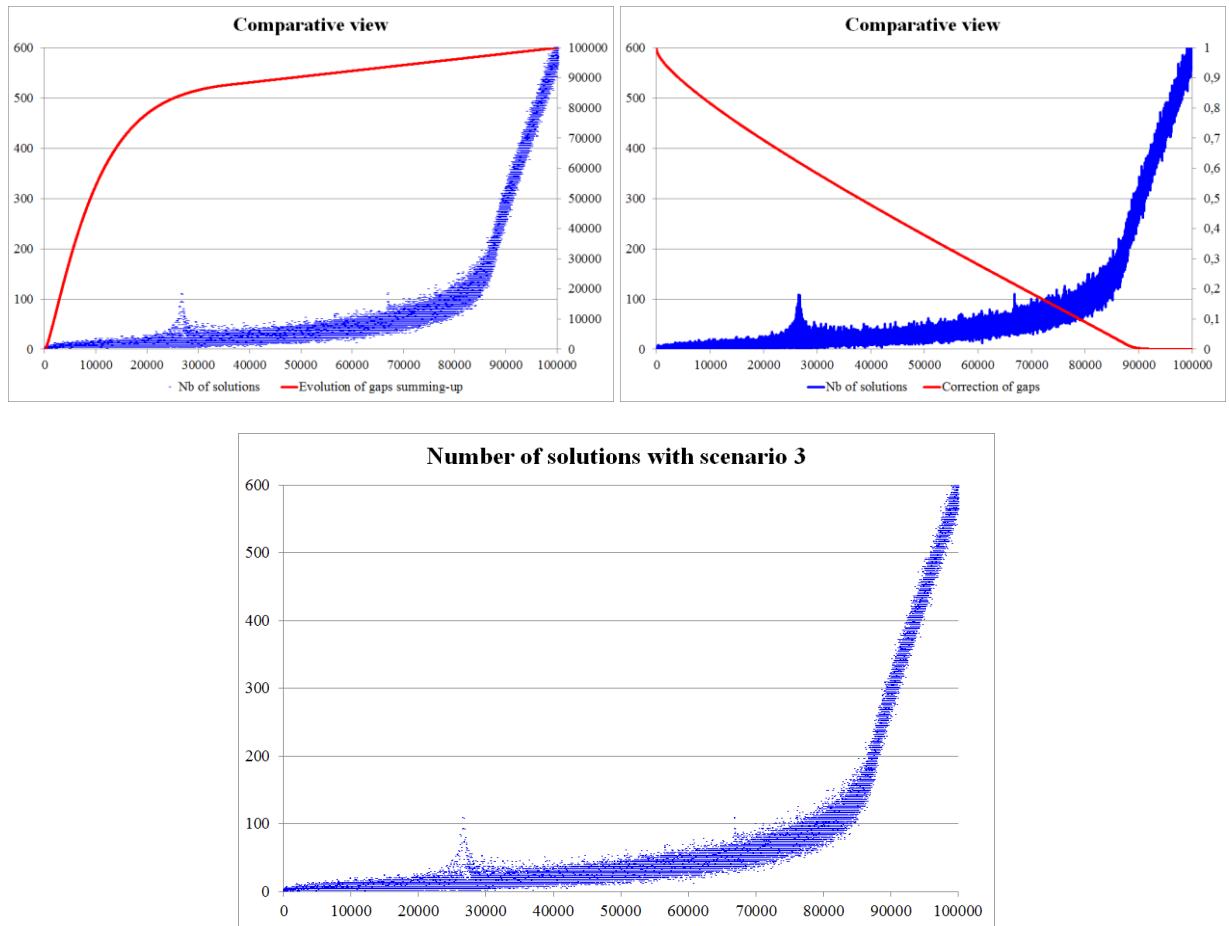
The number of solutions in the median strip is at least equal to 144 (for $2n = 100000$). When $2n$ increase (no chart presented), it is likely that this floor value would increase.

Model 2 : gaps about $\ln^2(p)$ near the origin and near maximum selected abscissa, intermediary gaps of 2



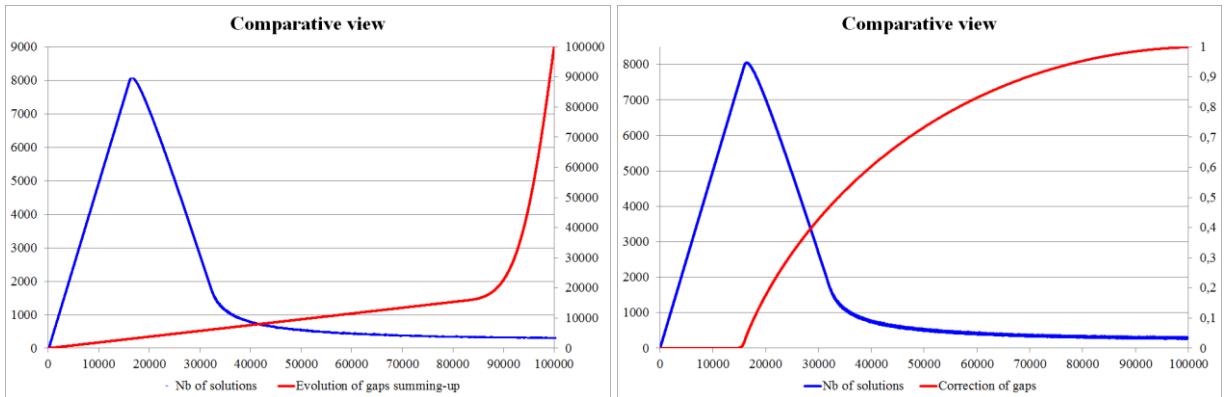
The number of solutions is growing very slowly near the origin, and then collapses after a peak. However, at the end of interval, the number of solutions remains above 1182, a quantity which is more important than the intermediate minima in the previous example.

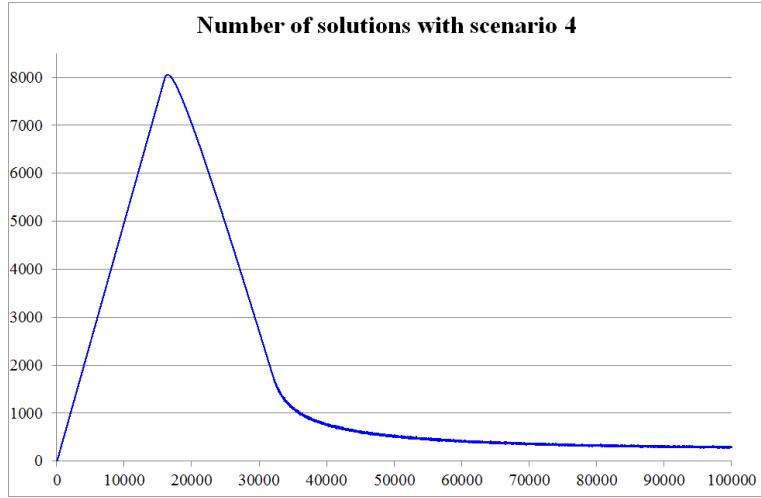
Model 3 : gaps about $\ln^2(p')$ near the origin, final gaps of 2



The increase in the number of solutions near the origin is feeble, then energizes at high values of $2 \cdot np$.

Model 4 : gaps of 2 near the origin, final gaps about $\ln^2(p')$





The quantitative result is analogous to model 1 except that the intermediate range with low number of solutions is now at the end of the abscissa range. The minimum number of solutions is there at 266 (greater than 144 observed for model 1).

Let us note finally that the number of solutions shows little dispersion, except for the case of model 3.

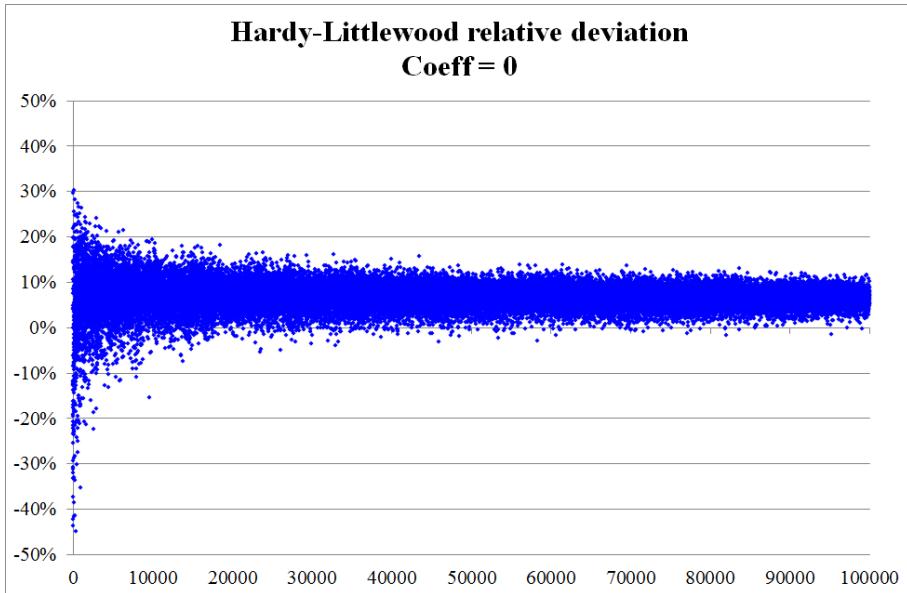
In all cases, the number of solutions is much greater than 1 for $2.np$ in the high bracket and militates in favour of Goldbach's conjecture.

5 The Goldbach comet

5.1 Relative error to the Hardy-Littlewood formula

Let us have $\#(2n)$ the exact number of solutions of Goldbach and let us have $\#(2np)$ the number of solutions given by the asymptotic formula of Hardy-Littlewood. We write down the percentage ratio

$$ER\% = (\#(2n) - \#(2np)) / \#(2n)$$



The relative deviation is important near the small values of $2n$ and then gradually decreases. The deviation shows then mainly an excess of exact solutions to the asymptotic Hardy-Littlewood evaluation.

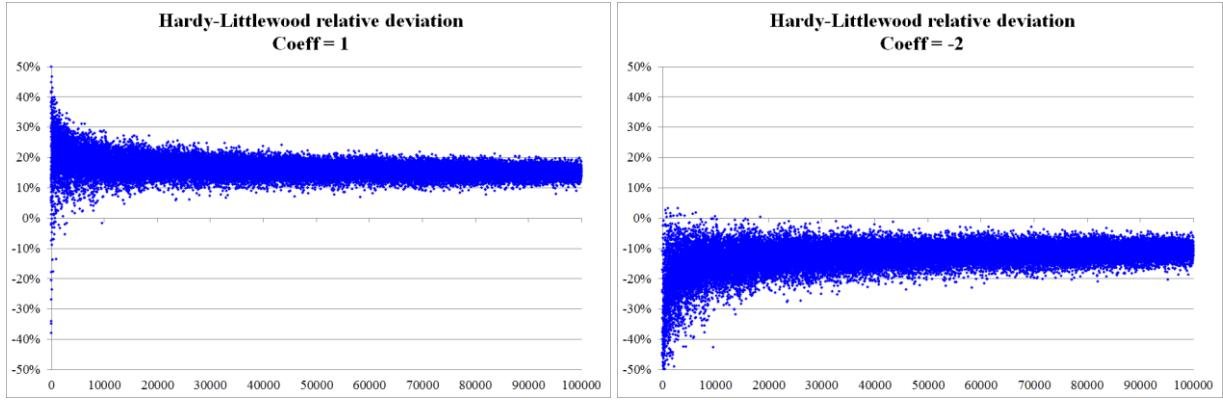
5.2 Correction of relative error

Let us have

$$\#(2npc) = \#(2np) \cdot (1 - coeff / \ln(n)) \quad (10)$$

where $coeff$ is a real number of adjustment. This amounts to a correction $coeff / \ln^3(n)$ in the Hardy-Littlewood formula.

The illustration then gives:



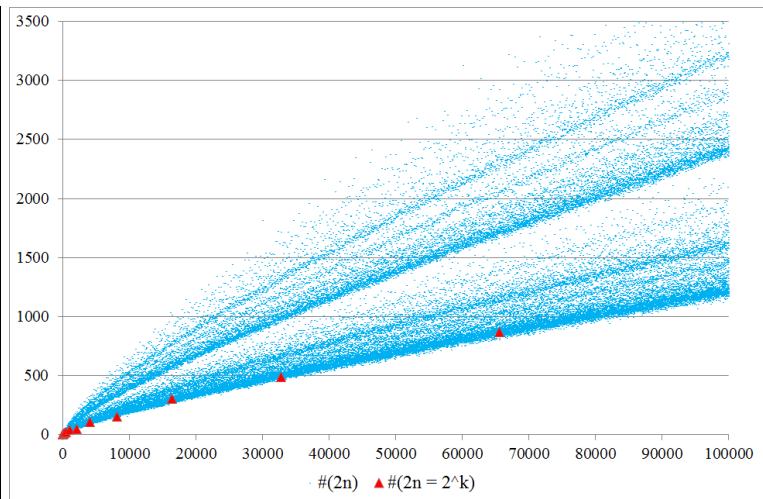
By varying coefficients from 1 to -2, the cloud of points is, for $2n$ large enough, up or down of the $y = 0$ axis, thus giving a lower and upper bounds for the number of solutions (at least in the observed range). A range of values of coefficients of this magnitude probably gives similar results asymptotically.

5.3 Minima et maxima de solutions

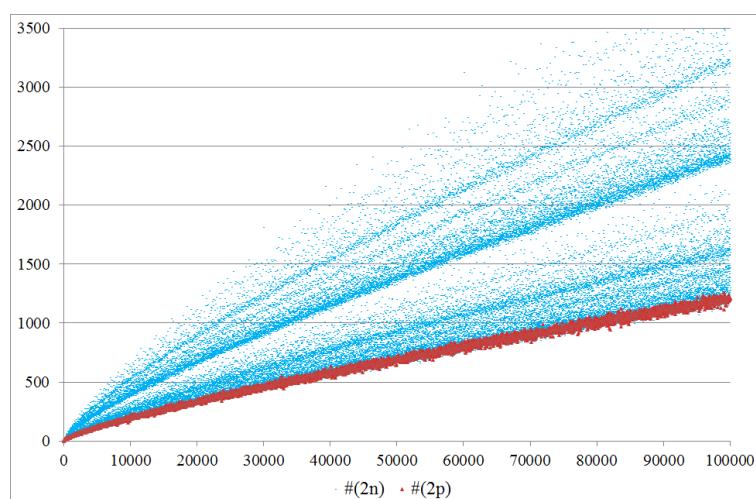
The Hardy-Littlewood enumeration relies, as we saw above, on ratios $\Pi(p-1)/(p-2)$ where p describes the prime odd divisors of n . When these ratios are low, asymptotically, representative points in the cloud are therefore, a priori, in the lower part. When the ratios are high on the contrary, representative points are in the upper section.

Note that multiple divisors have no influence on the ratios, but as $2n/\ln^2(n)$ evolves at the same time, representative points will a priori offset right, therefore towards the lower part. This is particularly the case for $2n = 2^k$.

$2n$	$\#(2n)$
8	2
16	4
32	4
64	10
128	6
256	16
512	22
1024	44
2048	50
4096	106
8192	152
16384	302
32768	488
65536	870

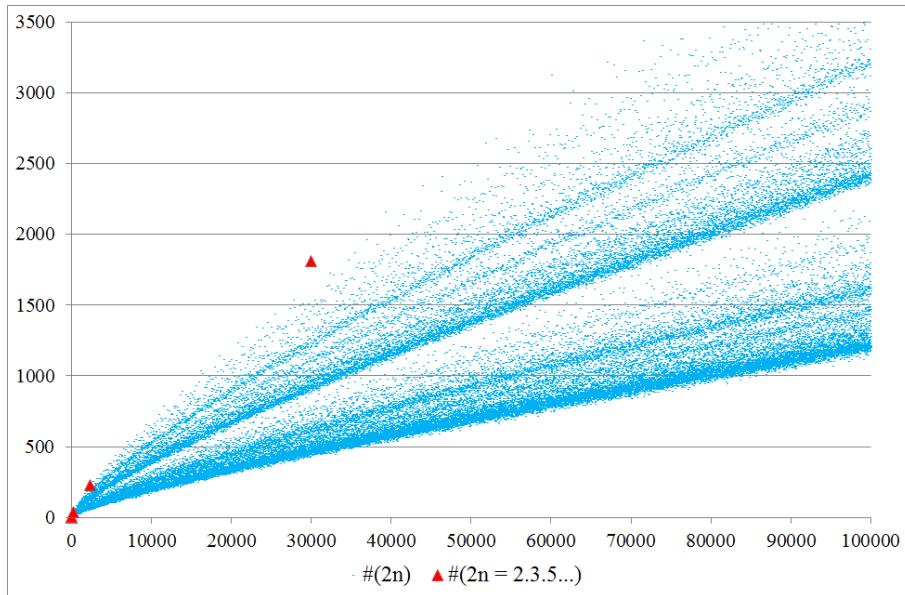


When number $2n$ is formed with large odd divisors (the ideal case being $2p$), the representative point is shifted to the lower positions also.



Conversely, with multiple factors, the representative points are shifted upward. However, if this list of divisors is small, the ratio decreased (what is antagonist). The first effect remains nevertheless predominant. In particular, the cardinals $\#(2n)$ with $2n = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \dots$ are in the upper section.

$2n$	$\#(2n)$
2	0
6	1
30	6
210	38
2310	228
30030	1810

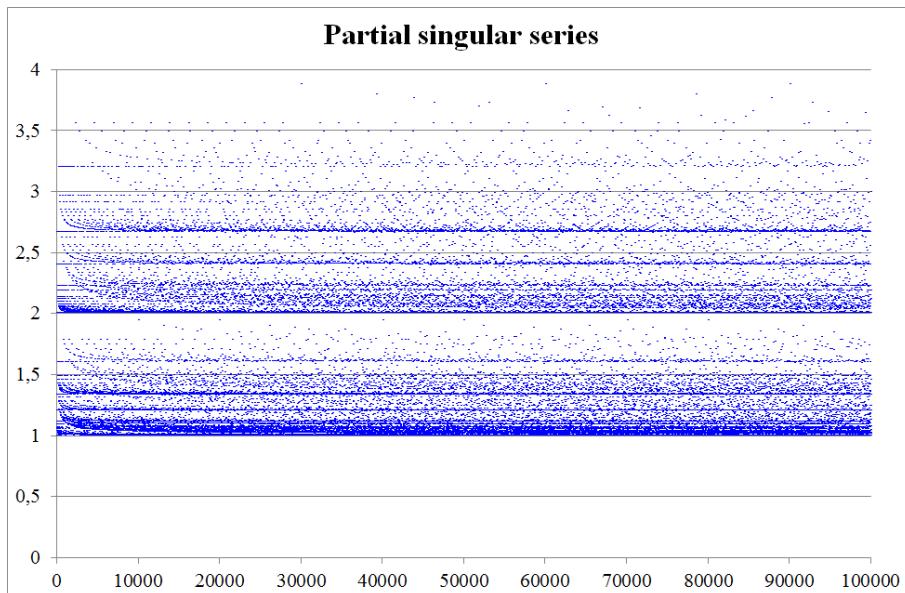


The above charts show that to seek approximatively the minimum envelope of the number of solutions of Goldbach is essentially equivalent to study either the set $2n = 2p$, or the set $2n = 2^k$. For the maximum envelope, one will study the progression to $2n = 2$, $2n = 2 \cdot 3$, $2n = 2 \cdot 3 \cdot 5$, $2n = 2 \cdot 3 \cdot 7$, $2n = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \dots$

5.4 Origin of striation in the cloud

To finish with, let us examine the origin of striations in the Goldbach cloud.

These accretion lines originate in the products $\Pi(p - 1)/(p-2)$ bearing on the prime divisors of n , products that we will call partial singular series.

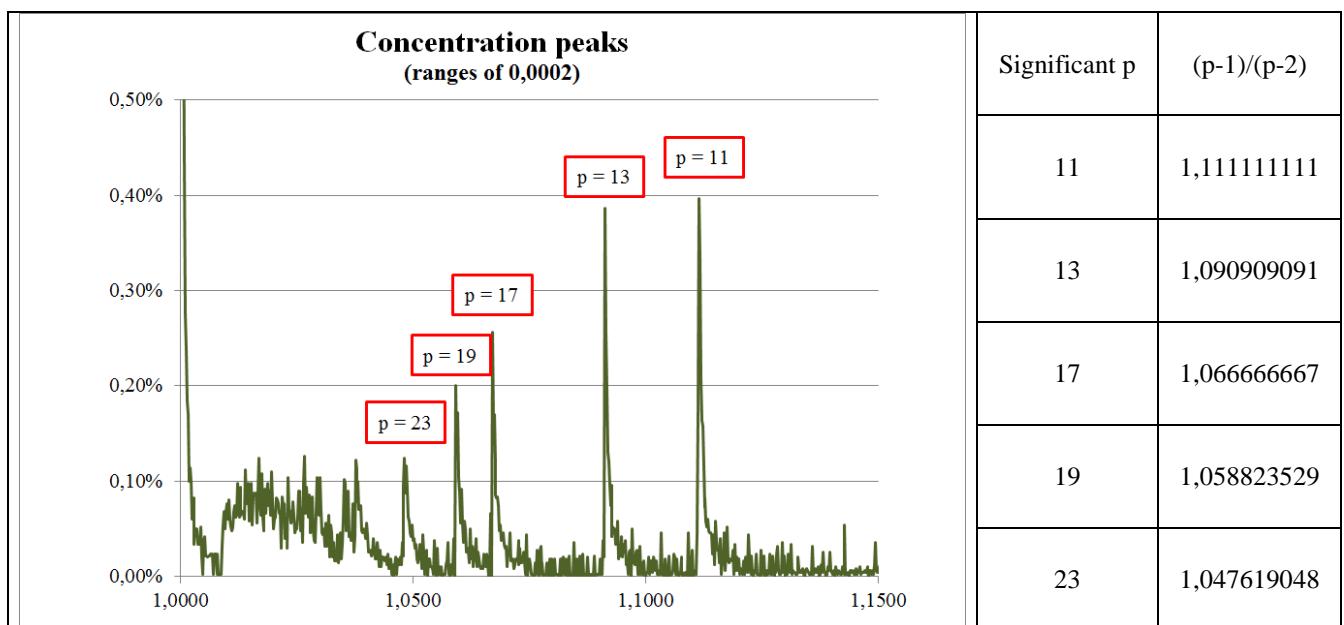
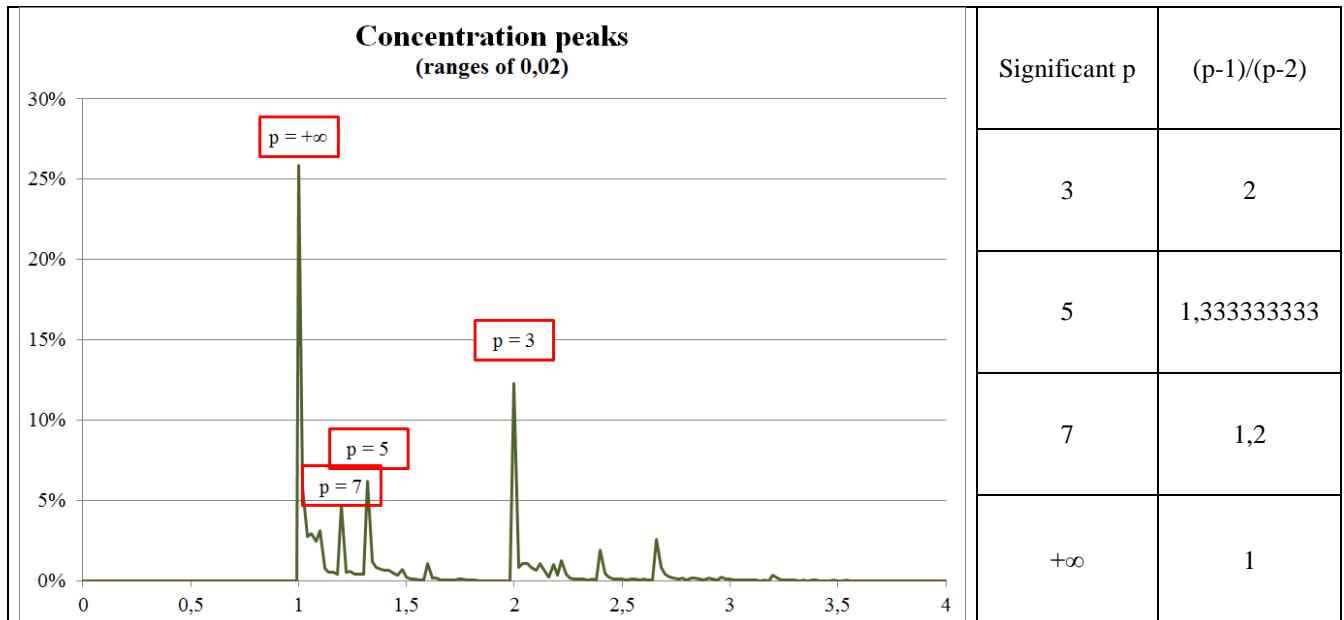


Taking the frequency of representative points in fixed intervals ranges at of that report, we get these concentrations in the form of peaks. The principal peaks (see below) correspond to ratios $(p-1)/(p-2)$ concerning only a unique p value with $p = 3$,

$p = 5$, $p = 7$, $p = 11$, etc., in that order. We call these numbers significant p (which correspond to all odd prime numbers). We end up, following increasing values of p , to the left-most peak with a sum-up of none-differentiated data at the scale of current numeric precision, that we note $p = +\infty$.

These peaks shift gradually to a value greater than $(p-1)/(p-2)$ by virtue of the contribution of additional ratios when more distinct dividers p exist in n (first 2 dividers, then 3, etc.). The detail of contribution becomes quickly difficult to grasp.

Appendix 2 includes detailed data for the two charts below.



6 Conclusion

The look of the comet of Goldbach is entirely coherent with a statistical anticipation. The comet is essentially minored by the enumerations corresponding to $2n = 2p$ and $2n = 2^k$ and majored by the enumerations corresponding to $2n = 2.3.5.7.11\dots$

It is possible to construct analogue clouds to the Goldbach cloud based on generators different of prime numbers, clouds with systematic solutions as in the initial case (the Goldbach case).

With equal number of generators in a given majored range, offsetting the generators positions in order to get extreme situations, the number of solutions remains superior to 1, except sometimes near the origin (question which is answered by direct calculation).

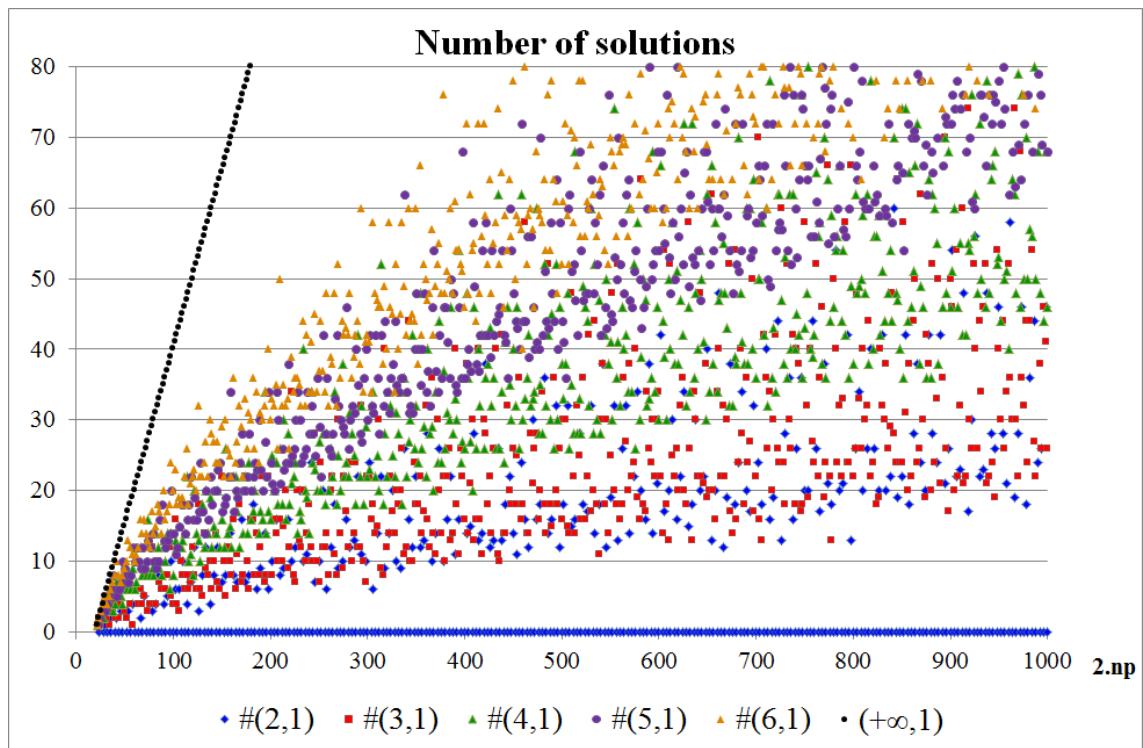
All of this remarks point to the direction of an inevitable theorem of Goldbach.

Appendix 1

Depletion method series (a,b = 1)

2,np	#(2,1)	#(3,1)	#(4,1)	#(5,1)	#(6,1)	(∞ ,1)
22	1	1	1	1	1	1
24		2	2	2	2	2
26	2	1	3	3	3	3
28		2	2	4	4	4
30	1	4	3	3	5	5
32		2	4	4	4	6
34	2	1	6	5	5	7
36		4	4	6	6	8
38	4	3	3	8	7	9
40		2	4	6	8	10
42	2	4	7	5	10	11
44		2	6	6	8	12
46	3	4	5	7	7	13
48		6	4	10	8	14
50	6	3	8	9	9	15
52		2	6	8	10	16
54	3	8	8	7	13	17
56		4	6	8	12	18
58	4	1	9	12	11	19
60		6	6	10	10	20
62	6	6	7	12	11	21
64		4	10	10	12	22
66	2	6	12	9	16	23
68		4	8	12	14	24
70	5	5	9	9	16	25
72		8	8	10	14	26
74	8	3	16	13	13	27
76		4	8	12	12	28
78	3	12	9	14	17	29
80		4	8	10	14	30
82	4	4	14	9	15	31
84		10	12	10	18	32
86	8	7	9	16	17	33
88		6	8	18	14	34
90	4	8	13	13	20	35
92		6	10	12	16	36
94	5	8	14	13	17	37
96		12	6	14	18	38
98	10	5	14	20	24	39
100		4	8	14	20	40
102	6	16	13	11	23	41
104		4	12	12	18	42
106	6	3	15	15	19	43
108		10	12	18	18	44
110	10	8	13	18	19	45
112		6	14	14	22	46
114	4	8	22	16	26	47
116		6	10	18	18	48
118	7	7	11	22	21	49
120		12	8	16	22	50
122	12	8	20	19	23	51
124		6	14	16	20	52
126	3	18	13	17	32	53
128		6	10	24	22	54
130	8	5	18	17	22	55
132		14	12	14	24	56
134	12	9	20	20	27	57
136		8	14	16	22	58
138	4	10	19	22	30	59
140		6	12	18	28	60
142	8	10	15	20	27	61
144		14	14	20	24	62
146	12	7	20	23	27	63
148		6	12	24	24	64
150	7	22	16	17	29	65
152		10	12	18	22	66
154	8	5	26	19	30	67

2.np	#(2,1)	#(3,1)	#(4,1)	#(5,1)	#(6,1)	($+\infty$,1)
156		12	14	22	26	68
158	14	10	15	34	25	69
160		8	12	22	22	70
162	8	16	22	17	36	71
164		6	22	20	24	72
166	7	11	17	19	27	73
168		18	12	24	32	74
170	16	7	24	28	29	75
172		8	14	20	22	76
174	7	22	22	20	33	77
176		10	14	22	26	78
178	8	6	24	28	31	79
180		16	16	20	30	80
182	18	16	17	22	36	81
184		8	20	24	26	82
186	6	14	23	23	34	83
188		8	14	30	24	84
190	9	12	17	23	31	85
192		22	16	20	30	86
194	14	7	34	27	32	87
196		8	14	22	34	88
198	6	22	17	34	40	89
200		8	16	24	30	90



Appendix 2

Number and frequency of points in a given range (of 0.02) for partial singular series

Interval	Amount of points	Frequency	Significant p	(p-1)/(p-2)
[1 ; 1,02[12915	25,83%	$+\infty$	1
[1,02 ; 1,04[2902	5,80%		
[1,04 ; 1,06[1395	2,79%		
[1,06 ; 1,08[1462	2,92%		
[1,08 ; 1,1[1240	2,48%		
[1,1 ; 1,12[1557	3,11%		
[1,12 ; 1,14[382	0,76%		
[1,14 ; 1,16[271	0,54%		
[1,16 ; 1,18[274	0,55%		
[1,18 ; 1,2[208	0,42%		
[1,2 ; 1,22[2349	4,70%	7	1.2
[1,22 ; 1,24[273	0,55%		
[1,24 ; 1,26[294	0,59%		
[1,26 ; 1,28[214	0,43%		
[1,28 ; 1,3[219	0,44%		
[1,3 ; 1,32[202	0,40%		
[1,32 ; 1,34[3090	6,18%	5	1.333333333
[1,34 ; 1,36[601	1,20%		
[1,36 ; 1,38[418	0,84%		
[1,38 ; 1,4[356	0,71%		
[1,4 ; 1,42[319	0,64%		
[1,42 ; 1,44[325	0,65%		
[1,44 ; 1,46[249	0,50%		
[1,46 ; 1,48[188	0,38%		
[1,48 ; 1,5[365	0,73%		
[1,5 ; 1,52[112	0,22%		
[1,52 ; 1,54[48	0,10%		
[1,54 ; 1,56[48	0,10%		
[1,56 ; 1,58[19	0,04%		
[1,58 ; 1,6[26	0,05%		
[1,6 ; 1,62[553	1,11%		
[1,62 ; 1,64[101	0,20%		
[1,64 ; 1,66[74	0,15%		
[1,66 ; 1,68[28	0,06%		
[1,68 ; 1,7[28	0,06%		
[1,7 ; 1,72[38	0,08%		
[1,72 ; 1,74[23	0,05%		
[1,74 ; 1,76[42	0,08%		
[1,76 ; 1,78[55	0,11%		
[1,78 ; 1,8[20	0,04%		
[1,8 ; 1,82[19	0,04%		
[1,82 ; 1,84[12	0,02%		
[1,84 ; 1,86[8	0,02%		
[1,86 ; 1,88[8	0,02%		
[1,88 ; 1,9[9	0,02%		
[1,9 ; 1,92[1	0,00%		
[1,92 ; 1,94[6	0,01%		
[1,94 ; 1,96[0	0,00%		
[1,96 ; 1,98[1	0,00%		
[1,98 ; 2[0	0,00%		
[2 ; 2,02[6148	12,30%	3	2
[2,02 ; 2,04[429	0,86%		
[2,04 ; 2,06[551	1,10%		
[2,06 ; 2,08[547	1,09%		
[2,08 ; 2,1[413	0,83%		
[2,1 ; 2,12[334	0,67%		
[2,12 ; 2,14[552	1,10%		
[2,14 ; 2,16[310	0,62%		
[2,16 ; 2,18[127	0,25%		

Interval	Amount of points	Frequency	Significant p	(p-1)/(p-2)
[2,18 ; 2,2[517	1,03%		
[2,2 ; 2,22[164	0,33%		
[2,22 ; 2,24[620	1,24%		
[2,24 ; 2,26[194	0,39%		
[2,26 ; 2,28[93	0,19%		
[2,28 ; 2,3[65	0,13%		
[2,3 ; 2,32[52	0,10%		
[2,32 ; 2,34[56	0,11%		
[2,34 ; 2,36[40	0,08%		
[2,36 ; 2,38[52	0,10%		
[2,38 ; 2,4[21	0,04%		
[2,4 ; 2,42[962	1,92%		
[2,42 ; 2,44[226	0,45%		
[2,44 ; 2,46[119	0,24%		
[2,46 ; 2,48[63	0,13%		
[2,48 ; 2,5[67	0,13%		
[2,5 ; 2,52[54	0,11%		
[2,52 ; 2,54[22	0,04%		
[2,54 ; 2,56[54	0,11%		
[2,56 ; 2,58[71	0,14%		
[2,58 ; 2,6[30	0,06%		
[2,6 ; 2,62[71	0,14%		
[2,62 ; 2,64[29	0,06%		
[2,64 ; 2,66[26	0,05%		
[2,66 ; 2,68[1298	2,60%		
[2,68 ; 2,7[415	0,83%		
[2,7 ; 2,72[200	0,40%		
[2,72 ; 2,74[102	0,20%		
[2,74 ; 2,76[86	0,17%		
[2,76 ; 2,78[70	0,14%		
[2,78 ; 2,8[91	0,18%		
[2,8 ; 2,82[27	0,05%		
[2,82 ; 2,84[84	0,17%		
[2,84 ; 2,86[97	0,19%		
[2,86 ; 2,88[50	0,10%		
[2,88 ; 2,9[33	0,07%		
[2,9 ; 2,92[98	0,20%		
[2,92 ; 2,94[53	0,11%		
[2,94 ; 2,96[39	0,08%		
[2,96 ; 2,98[116	0,23%		
[2,98 ; 3[53	0,11%		
[3 ; 3,02[47	0,09%		
[3,02 ; 3,04[23	0,05%		
[3,04 ; 3,06[17	0,03%		
[3,06 ; 3,08[17	0,03%		
[3,08 ; 3,1[11	0,02%		
[3,1 ; 3,12[23	0,05%		
[3,12 ; 3,14[12	0,02%		
[3,14 ; 3,16[0	0,00%		
[3,16 ; 3,18[14	0,03%		
[3,18 ; 3,2[0	0,00%		
[3,2 ; 3,22[167	0,33%		
[3,22 ; 3,24[72	0,14%		
[3,24 ; 3,26[40	0,08%		
[3,26 ; 3,28[28	0,06%		
[3,28 ; 3,3[21	0,04%		
[3,3 ; 3,32[27	0,05%		
[3,32 ; 3,34[0	0,00%		
[3,34 ; 3,36[16	0,03%		
[3,36 ; 3,38[0	0,00%		
[3,38 ; 3,4[21	0,04%		
[3,4 ; 3,42[22	0,04%		
[3,42 ; 3,44[1	0,00%		

Interval	Amount of points	Frequency	Significant p	(p-1)/(p-2)
[3,44 ; 3,46[1	0,00%		
[3,46 ; 3,48[0	0,00%		
[3,48 ; 3,5[27	0,05%		
[3,5 ; 3,52[0	0,00%		
[3,52 ; 3,54[0	0,00%		
[3,54 ; 3,56[31	0,06%		
[3,56 ; 3,58[1	0,00%		
[3,58 ; 3,6[0	0,00%		
[3,6 ; 3,62[2	0,00%		
[3,62 ; 3,64[1	0,00%		
[3,64 ; 3,66[4	0,01%		
[3,66 ; 3,68[1	0,00%		
[3,68 ; 3,7[2	0,00%		
[3,7 ; 3,72[0	0,00%		
[3,72 ; 3,74[3	0,01%		
[3,74 ; 3,76[0	0,00%		
[3,76 ; 3,78[2	0,00%		
[3,78 ; 3,8[2	0,00%		
[3,8 ; 3,82[0	0,00%		
[3,82 ; 3,84[0	0,00%		
[3,84 ; 3,86[0	0,00%		
[3,86 ; 3,88[3	0,01%		
[3,88 ; 3,9[0	0,00%		
[3,9 ; 3,92[0	0,00%		
[3,92 ; 3,94[0	0,00%		
[3,94 ; 3,96[0	0,00%		
[3,96 ; 3,98[0	0,00%		
[3,98 ; 4[0	0,00%		
[4 ; [0	0,00%		

Number and frequency of points in a given range (of 0.0002) for partial singular series

In this table, using a 100 times greater accuracy, we observe that strias are often offset (as previously announced) slightly down of the significant p's (data in columns of amounts and of frequencies increasing after these p's).

Note: The lines with reduced quantities have been removed for brevity.

Interval	Amount of points	Frequency	Significant p	(p-1)/(p-2)
...		
[1,0474 ; 1,0476[6	0,01%		
[1,0476 ; 1,0478[18	0,04%	23	1,047619048
[1,0478 ; 1,048[11	0,02%		
[1,048 ; 1,0482[48	0,10%		
[1,0482 ; 1,0484[62	0,12%		
[1,0484 ; 1,0486[44	0,09%		
[1,0486 ; 1,0488[58	0,12%		
[1,0488 ; 1,049[46	0,09%		
[1,049 ; 1,0492[31	0,06%		
[1,0492 ; 1,0494[29	0,06%		
[1,0494 ; 1,0496[23	0,05%		
[1,0496 ; 1,0498[21	0,04%		
[1,0498 ; 1,05[27	0,05%		
[1,05 ; 1,0502[19	0,04%		
[1,0502 ; 1,0504[20	0,04%		
[1,0504 ; 1,0506[15	0,03%		
[1,0506 ; 1,0508[17	0,03%		
[1,0508 ; 1,051[9	0,02%		
[1,051 ; 1,0512[7	0,01%		
...		
[1,0586 ; 1,0588[5	0,01%		
[1,0588 ; 1,059[20	0,04%	19	1,058823529
[1,059 ; 1,0592[0	0,00%		

Interval	Amount of points	Frequency	Significant p	(p-1)/(p-2)
[1,0592 ; 1,0594[100	0,20%		
[1,0594 ; 1,0596[65	0,13%		
[1,0596 ; 1,0598[86	0,17%		
[1,0598 ; 1,06[52	0,10%		
[1,06 ; 1,0602[44	0,09%		
[1,0602 ; 1,0604[28	0,06%		
[1,0604 ; 1,0606[46	0,09%		
[1,0606 ; 1,0608[28	0,06%		
[1,0608 ; 1,061[18	0,04%		
[1,061 ; 1,0612[29	0,06%		
[1,0612 ; 1,0614[25	0,05%		
[1,0614 ; 1,0616[13	0,03%		
[1,0616 ; 1,0618[15	0,03%		
[1,0618 ; 1,062[13	0,03%		
[1,062 ; 1,0622[7	0,01%		
[1,0622 ; 1,0624[25	0,05%		
[1,0624 ; 1,0626[9	0,02%		
[1,0626 ; 1,0628[16	0,03%		
[1,0628 ; 1,063[14	0,03%		
[1,063 ; 1,0632[5	0,01%		
[1,0632 ; 1,0634[9	0,02%		
[1,0634 ; 1,0636[13	0,03%		
[1,0636 ; 1,0638[20	0,04%		
[1,0638 ; 1,064[5	0,01%		
[1,064 ; 1,0642[13	0,03%		
[1,0642 ; 1,0644[9	0,02%		
[1,0644 ; 1,0646[5	0,01%		
[1,0646 ; 1,0648[4	0,01%		
[1,0648 ; 1,065[5	0,01%		
[1,065 ; 1,0652[4	0,01%		
[1,0652 ; 1,0654[12	0,02%		
[1,0654 ; 1,0656[6	0,01%		
[1,0656 ; 1,0658[5	0,01%		
[1,0658 ; 1,066[12	0,02%		
[1,066 ; 1,0662[6	0,01%		
[1,0662 ; 1,0664[0	0,00%		
[1,0664 ; 1,0666[6	0,01%		
[1,0666 ; 1,0668[33	0,07%	17	1,066666667
[1,0668 ; 1,067[0	0,00%		
[1,067 ; 1,0672[128	0,26%		
[1,0672 ; 1,0674[80	0,16%		
[1,0674 ; 1,0676[85	0,17%		
[1,0676 ; 1,0678[59	0,12%		
[1,0678 ; 1,068[43	0,09%		
[1,068 ; 1,0682[41	0,08%		
[1,0682 ; 1,0684[42	0,08%		
[1,0684 ; 1,0686[37	0,07%		
[1,0686 ; 1,0688[24	0,05%		
[1,0688 ; 1,069[27	0,05%		
[1,069 ; 1,0692[19	0,04%		
[1,0692 ; 1,0694[21	0,04%		
[1,0694 ; 1,0696[23	0,05%		
[1,0696 ; 1,0698[17	0,03%		
[1,0698 ; 1,07[15	0,03%		
[1,07 ; 1,0702[16	0,03%		
[1,0702 ; 1,0704[5	0,01%		
[1,0704 ; 1,0706[13	0,03%		
[1,0706 ; 1,0708[12	0,02%		
[1,0708 ; 1,071[23	0,05%		
[1,071 ; 1,0712[8	0,02%		
[1,0712 ; 1,0714[8	0,02%		
[1,0714 ; 1,0716[9	0,02%		
...		

Interval	Amount of points	Frequency	Significant p	(p-1)/(p-2)
[1,0904 ; 1,0906[1	0,00%		
[1,0906 ; 1,0908[1	0,00%		
[1,0908 ; 1,091[19	0,04%	13	1,090909091
[1,091 ; 1,0912[10	0,02%		
[1,0912 ; 1,0914[193	0,39%		
[1,0914 ; 1,0916[127	0,25%		
[1,0916 ; 1,0918[98	0,20%		
[1,0918 ; 1,092[65	0,13%		
[1,092 ; 1,0922[60	0,12%		
[1,0922 ; 1,0924[48	0,10%		
[1,0924 ; 1,0926[37	0,07%		
[1,0926 ; 1,0928[48	0,10%		
[1,0928 ; 1,093[21	0,04%		
[1,093 ; 1,0932[26	0,05%		
[1,0932 ; 1,0934[25	0,05%		
[1,0934 ; 1,0936[25	0,05%		
[1,0936 ; 1,0938[17	0,03%		
[1,0938 ; 1,094[17	0,03%		
[1,094 ; 1,0942[29	0,06%		
[1,0942 ; 1,0944[9	0,02%		
[1,0944 ; 1,0946[13	0,03%		
[1,0946 ; 1,0948[11	0,02%		
...		
[1,1096 ; 1,1098[0	0,00%		
[1,1098 ; 1,11[2	0,00%		
[1,11 ; 1,1102[14	0,03%		
[1,1102 ; 1,1104[0	0,00%		
[1,1104 ; 1,1106[3	0,01%		
[1,1106 ; 1,1108[2	0,00%		
[1,1108 ; 1,111[1	0,00%		
[1,111 ; 1,1112[22	0,04%	11	1,111111111
[1,1112 ; 1,1114[86	0,17%		
[1,1114 ; 1,1116[198	0,40%		
[1,1116 ; 1,1118[167	0,33%		
[1,1118 ; 1,112[103	0,21%		
[1,112 ; 1,1122[82	0,16%		
[1,1122 ; 1,1124[79	0,16%		
[1,1124 ; 1,1126[58	0,12%		
[1,1126 ; 1,1128[36	0,07%		
[1,1128 ; 1,113[41	0,08%		
[1,113 ; 1,1132[29	0,06%		
[1,1132 ; 1,1134[26	0,05%		
[1,1134 ; 1,1136[30	0,06%		
[1,1136 ; 1,1138[24	0,05%		
[1,1138 ; 1,114[23	0,05%		
[1,114 ; 1,1142[23	0,05%		
[1,1142 ; 1,1144[22	0,04%		
[1,1144 ; 1,1146[12	0,02%		
[1,1146 ; 1,1148[22	0,04%		
[1,1148 ; 1,115[6	0,01%		
[1,115 ; 1,1152[29	0,06%		
[1,1152 ; 1,1154[19	0,04%		
[1,1154 ; 1,1156[14	0,03%		
[1,1156 ; 1,1158[15	0,03%		
[1,1158 ; 1,116[7	0,01%		
[1,116 ; 1,1162[20	0,04%		
[1,1162 ; 1,1164[1	0,00%		
[1,1164 ; 1,1166[7	0,01%		
[1,1166 ; 1,1168[8	0,02%		