Théorie des nombres / *Number Theory*

Locus of the Riemann Zeta functions zeros

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Abstract Using a study of truncated Zeta and likewise functions, we add arguments towards Riemann hypothesis. Thus, the real values of s in ζ(s) and likewise functions are correlated to a concept of density.

Lieu des zéros des fonctions Zêta de Riemann

Résumé Par une étude des fonctions Zêta tronquées et de fonctions analogues, nous apportons des arguments à l'hypothèse de Riemann. Ainsi, les valeurs réelles de s dans ζ(s) et des fonctions analogues sont corrélées à une notion de densité.

Statut Preprint.
Date 02/06/2010

Background

Let us have s = a+i.b a complex number.
The Zeta function is defined for Re(s) > 0 by the extension of the Dirichlet series :

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^s} \qquad (1)$$

Let us write

$$\zeta_n(s) = \frac{1}{1-2^{1-s}} \sum_{m=1}^{n} \frac{(-1)^{m-1}}{m^s} \qquad (2)$$

We get the ζ(s) function when n tends towards ∞ in $\zeta_n(s)$.
We are interested by the $\zeta_n(s)$ zeroes. The ratio $1/(1-2^{1-s})$ equals 0 only if $s \to \pm\infty$ (with adequate conditions on parameter b). This limit case is of no peculiar interest Thus, we consider later on, no more $\zeta_n(s)$, but $\zeta^*_n(s) = \zeta_n(s).(1-2^{1-s})$ investigating the zeros of $\zeta_n(s)$.
We have then :

$$\zeta^*_n(s) = \sum_{m=1}^{n} m^{-a}.(-1)^{m-1}.\cos(b.\ln(m)) + i. \sum_{m=1}^{n} m^{-a}.(-1)^{m-1}.\sin(b.\ln(m)) \qquad (3)$$

To get zeros of $\zeta_n(s)$ is then equivalent to solve the two equations :

$$1 + \sum_{m=2}^{n} m^{-a}.(-1)^{m-1}.\cos(b.\ln(m)) = 0 \qquad (4)$$

$$\sum_{m=2}^{n} m^{-a}.(-1)^{m-1}.\sin(b.\ln(m)) = 0 \qquad (5)$$

We will use a physics analogy by speaking, for a given n, of an "n-bodies problem" while we try to solve simultaneously (4) and (5). We will call m the mass of such body, a the energy, b the phase, $(a^2+b^2)^{1/2}$ the module of the associated wave, f and g the force fields, independent here of a, with $f(b,m) = (-1)^{m-1}.\cos(b.\ln(m))$ and $g(b,m) = (-1)^{m-1}.\sin(b.\ln(m))$. The bodies are plunged in an empty space (hence the equalities to 0) and we observe the behaviour of the energy levels and phases. The energy levels (or the system) are called degenerated when the parameter a is not constant. Otherwise, the energy levels (or the system) is said to be non-degenerated.

Note : Subsequently, we will talk about n-bodies instead of n+1-bodies problems, as the body of mass m = 1 implied by the constant 1 of equation (4) is not included in this number n. Effectively, the body m = 1 does not introduce any indeterminate and it is better not to count it for the physics analogies that we do here.

One-body problem

Truncated function Zeta zeros meet the following equalities :

$$\begin{aligned} &1+m^{-a}.(-1)^{m-1}\cos(b.\ln(m)) = 0 \\ &m^{-a}.(-1)^{m-1}.\sin(b.\ln(m)) = 0 \end{aligned} \qquad (6)$$

It follows immediately (m > 1) :

$$\begin{aligned} &a = 0 \\ &b = i.\pi \ \frac{\text{or}(m = 0 \bmod 2,0,1)+2k}{\ln(m)} \end{aligned} \qquad (7)$$

The zeros forms an enumerable infinite set in an explicit way. The real part of the function zeros is a fixed point equal to 0. The imaginary parts of zeros are distributed uniformly with distance $2\pi/\ln(m)$ on the imaginary axis line. This system is of the non-degenerated type.

Two-bodies problem

Let us have two distinct bodies of mass r and mass s ($r \neq s$). The function zeros obey to the equalities :

$$1+r^{-a}.(-1)^{r-1}.\cos(b.\ln(r))+s^{-a}.(-1)^{s-1}.\cos(b.\ln(s)) = 0 \qquad (8)$$
$$r^{-a}.(-1)^{r-1}.\sin(b.\ln(r))+s^{-a}.(-1)^{s-1}.\sin(b.\ln(s)) = 0 \qquad (9)$$

Real part

Using (9), we get $s^{-a} = (-1)^{r-s+1}.r^{-a}.\sin(b.\ln(r))/\sin(b.\ln(s))$. Replacing that in (8), it follows $1+(-1)^{r-1}.r^{-a}.\cos(b.\ln(r))+(-1)^{r}.r^{-a}.\sin(b.\ln(r))/\sin(b.\ln(s)).\cos(b.\ln(s)) = 1+(-1)^{r}.r^{-a}.(-\sin(b.\ln(s)).\cos(b.\ln(r))+\sin(b.\ln(r)).\cos(b.\ln(s)))/\sin(b.\ln(s)) = 1+(-1)^{r}.r^{-a}.\sin(b.\ln(r)-b.\ln(s))/\sin(b.\ln(s)) = 0$, thus $r^{-a} = -(-1)^{r}.\sin(b.\ln(s))/\sin(b.\ln(r/s))$. This expression is correct only if :

$$(-1)^{r-1}.\sin(b.\ln(s))/\sin(b.\ln(r/s)) > 0 \qquad (10)$$

The masses r and s can be permuted in the initial equations. So we shall verify also :

$$(-1)^{s-1}.\sin(b.\ln(r))/\sin(b.\ln(s/r)) > 0 \qquad (11)$$

We get then (a,b) solutions that satisfy to :

$$a = \frac{\ln((-1)^{r}.\sin(b.\ln(s/r))/\sin(b.\ln(s)))}{\ln(r)} = \frac{\ln((-1)^{s}.\sin(b.\ln(r/s))/\sin(b.\ln(r)))}{\ln(s)} \qquad (12)$$

Alternatively, we have also directly from relation (9) :

$$a = \frac{\ln((-1)^{s-r+1}.\sin(b.\ln(r))/\sin(b.\ln(s)))}{\ln(r/s)} = \frac{\ln((-1)^{r-s+1}.\sin(b.\ln(s))/\sin(b.\ln(r)))}{\ln(s/r)} \qquad (13)$$

We produce the third terms of relations (12) and (13) immediately from the second members of these relations with the arguments of permutability of r and s masses. However, we note that the two last members of relation (13) are identical. So we get by trial and error the parameter b only from relations (12) :

$$\ln(s).\ln((-1)^{r}.\sin(b.\ln(s/r))/\sin(b.\ln(s))) = \ln(r).\ln((-1)^{s}.\sin(b.\ln(r/s))/\sin(b.\ln(r))) \qquad (14)$$

Then to a solution b corresponds a couple (a,b) given by the first equality.
Conversely, we can write b as a function of a by using $\cos^2(x)+\sin^2(x) = 1$ in equations (8). Thus, with k a relative integer :

$$(-1)^{s-1}.\cos(b.\ln(s)) = \frac{r^{-2a}-s^{-2a}-1}{2s^{-a}} \qquad \text{hence } b = \frac{\pm\text{acos}((-1)^{s-1}.(r^{-2a}-s^{-2a}-1)/(2s^{-a}))+2k.\pi}{\ln(s)} \qquad (15)$$

Since again r and s can be swapped in the initial equations, we get also (k' a relative integer) :

$$(-1)^{r-1}.\cos(b.\ln(r)) = \frac{s^{-2a}-r^{-2a}-1}{2r^{-a}} \qquad \text{hence } b = \frac{\pm \text{acos}((-1)^{r-1}.(s^{-2a}-r^{-2a}-1)/(2r^{-a}))+2k'.\pi}{\ln(r)} \qquad (16)$$

To find k and k' to infer b has nothing trivial here and does not matter much since the issue is already "resolved" above. However, following boundaries related to trigonometric functions are crucial :

$$-1 \leq \frac{r^{-2a}-s^{-2a}-1}{2s^{-a}} \leq 1 \qquad (17)$$

$$-1 \leq \frac{s^{-2a}-r^{-2a}-1}{2r^{-a}} \leq 1 \qquad (18)$$

This means that parameter a stays in finite range of values.
For example :

r	s	min(a)	max(a)
2	3	-1	≈ 0.788
2	4	≈ -0.694	≈ 0.694
2	5	≈ -0.564	≈ 0.639
2	10	≈ -0.358	≈ 0.519
2	100	≈ -0.163	≈ 0.339
2	1000	≈ -0.106	≈ 0.261
2	10000	≈ -0.078	≈ 0.215
2	100000	≈ -0.062	≈ 0.184
100000	10000000000	≈ -0.042	≈ 0.042

When s >> r (or r >> s), the admissible range of parameter a tends towards an accumulation point equal to 0.

Indeed, if a > 0 and s >> r > 1 then $(r^{-2a}-s^{-2a}-1)/(2s^{-a}) = ((s/r^2)^a-1/s^a-s^a)/2 = (-s^a.(1-1/r^{2a})-1/s^a)/2 \rightarrow -\infty-0 \rightarrow -\infty$ which is inconsistent with the interval [-1,1] to which expression $(r^{-2a}-s^{-2a}-1)/(2s^{-a})$ is constrained according to relation 17. If a < 0 and s >> r > 1 then $(r^{-2a}-s^{-2a}-1)/(2s^{-a}) = ((1/s^{-a}).(r^{-2a}-1)-s^{-a})/2 \rightarrow +0-\infty \rightarrow -\infty$ which is again inconsistent with the interval [-1,1]. Finally, if a = 0 then $(r^{-2a}-s^{-2a}-1)/(2s^{-a}) = -1/2$ which is effectively in the admissible interval. Thus, a = 0 is the only solution to our asymptotic problem.

When s and r are of the same order of magnitude and tend towards infinity, the admissible range of parameter a defined by (17) and (18) does not tend any more towards a fixed point, but varies in interval [-1,0[.

r	s	min(a)	max(a)
2	3	-1	≈ 0.788
10	11	-1	≈ 0.295
100	101	-1	≈ 0.150
1000	1001	-1	≈ 0.100
10000	10001	-1	≈ 0.075
100000	100001	-1	≈ 0.060
1000000	1000001	-1	≈ 0.050
$+\infty$	$+\infty+1$	-1	0^+
$+\infty$	$+\infty+k$	-1	0^+

The two limits -1 and 0 values are determined without difficulty from relation 17 (and correspond to the same boundary inside equation 17). However, if we get back to the initial equations $1+r^{-a}.(-1)^{r-1}.\cos(b.\ln(r))+s^{-a}.(-1)^{s-1}.\cos(b.\ln(s)) = 0$ and $r^{-a}.(-1)^{r-1}.\sin(b.\ln(r))+s^{-a}.(-1)^{s-1}.\sin(b.\ln(s)) = 0$ and do only make parameter s tend towards infinity (but not r), these equations cannot verify unless a = 0. This phenomenon is due to the presence of mass m = 1 which involves an implicit interval of type (17) or (18) with three parameters 1, r and s (instead of two parameters r and s of former explicit relations).

Thus, the admissible range of **a tends towards an accumulation point when the relative masses of bodies diverge**. In the presence of the 1-mass body, always present (by construction), the divergence of one or the other or of the two masses r and s involves the convergence of a towards a unique value. The constraint that acts on the interval of definition of the system formed by the equations 8 and 9 is simply the reduced [1,1] interval of the sin and cos functions, such constaint increasing as the relative difference between masses increases.

Imaginary part

Beyond the problem of the admissible range, let us interest to the solutions structure, that is to the values of b. Let us suppose first :

$$r \approx s$$
$$r \rightarrow \infty$$

Then $r/s \rightarrow 1$ and $\ln(r/s) \rightarrow 0$, so that $\sin(b.\ln(r/s)) \rightarrow b.\ln(r/s)$.

Then :

$$a \approx \frac{\ln((-1)^r.b.\ln(s/r)/\sin(b.\ln(s)))}{\ln(r)} \approx \frac{\ln((-1)^s.b.\ln(r/s)/\sin(b.\ln(r)))}{\ln(s)}$$

So that

$$a \approx \frac{\ln(b.\ln(s/r)/\sin((-1)^r.b.\ln(s)))}{\ln(r)} \approx \frac{\ln(b.\ln(s/r)/\sin((-1)^s.b.\ln(r)))}{\ln(s)}$$

It follows using $\ln(r) \approx \ln(s)$ and $\ln(s/r) = -\ln(r/s)$:

$$(-1)^r.b.\ln(s) \approx (-1)^s.b.\ln(r)+2k.\pi$$

This allows writing with k et k' relative integers :

$$(-1)^r.b.\ln(s) \approx -(-1)^s.b.\ln(r)+2k.\pi$$

or

$$(-1)^r.b.\ln(s) \approx (-1)^s.b.\ln(r)+(2k'+1).\pi$$

From where :

$$b \approx (-1)^s.2k.\pi/(\ln(r)+(-1)^{r-s}.\ln(s))$$

or

$$b \approx (-1)^s.(2k'+1).\pi/(-\ln(r)+(-1)^{r-s}.\ln(s))$$

It results two typical cases, depending on r-s parity, when we do represent a as a function of b using relations (12) thus leading to table 1 underneath :

r-s	b		Behaviour near origin	Behaviour at half-period
0 mod 2	$(-1)^s.(2k'+1).\pi/\ln(s/r)$	$(-1)^s.2k.\pi/\ln(r.s)$	Schifting (lag)	Stacking (in phase)
1 mod 2	$(-1)^s.2k.\pi/\ln(r/s)$	$(-1)^{s-1}.(2k'+1).\pi/\ln(r.s)$	Stacking (in phase)	Schifting (lag)

For r = 999, s = 1001 as an example, we represent the curves $a = \ln(((-1)^r.\sin(b.\ln(s/r))/\sin(b.\ln(s)))/\ln(r)$ in red and $a = \ln(((-1)^s.\sin(b.\ln(r/s))/\sin(b.\ln(r)))/\ln(s)$ in blue at different scales to reflect the behaviour at each scale.

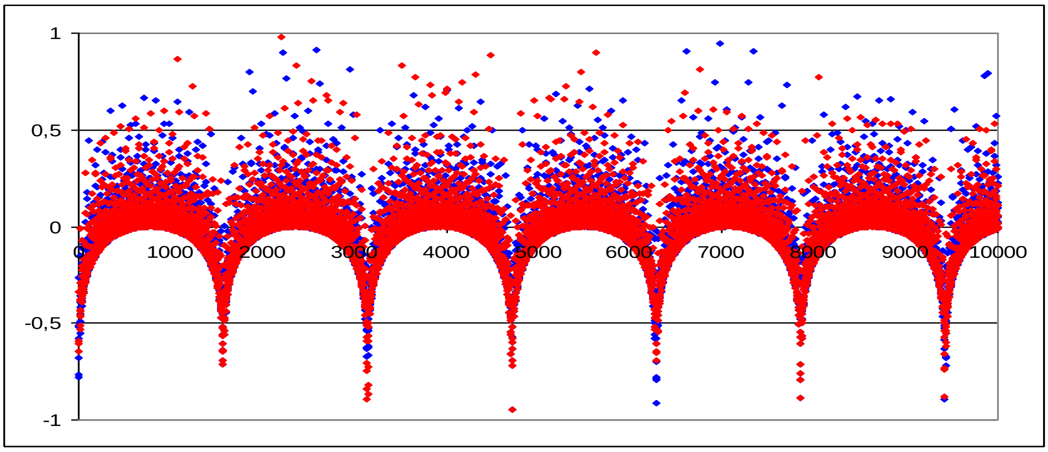


Chart 1

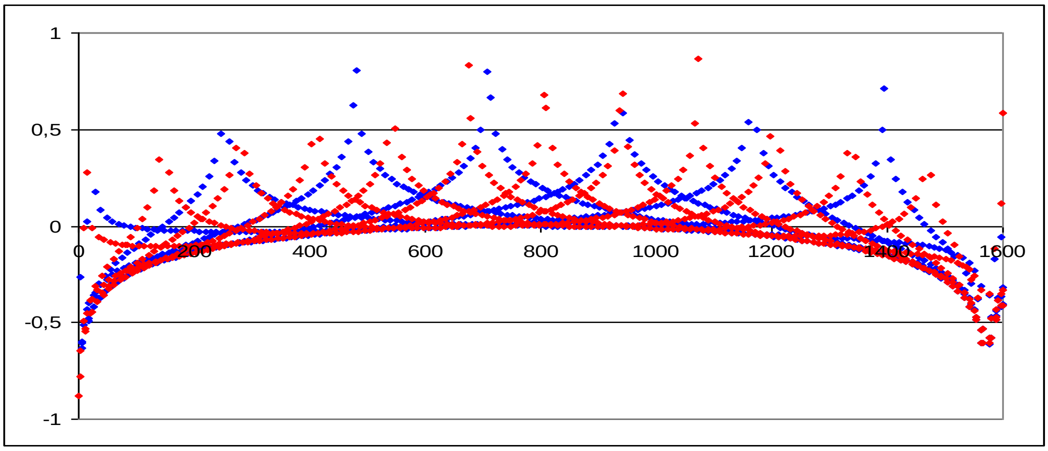


Chart 2

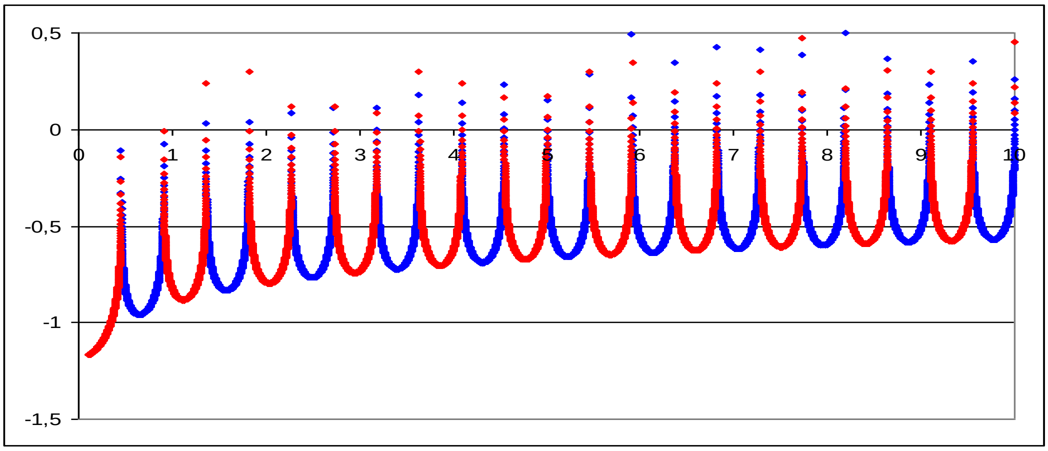


Chart 3

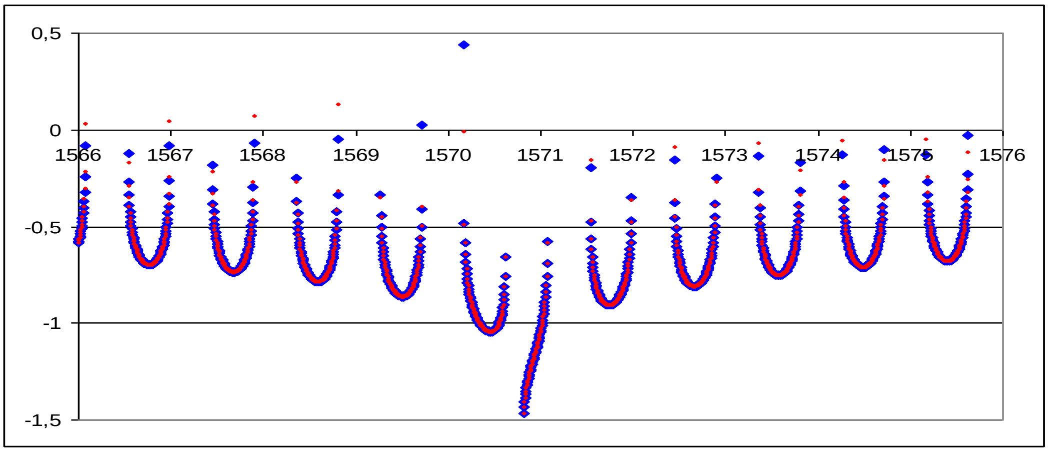
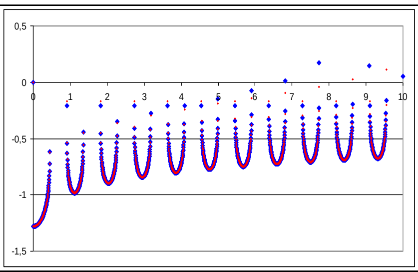
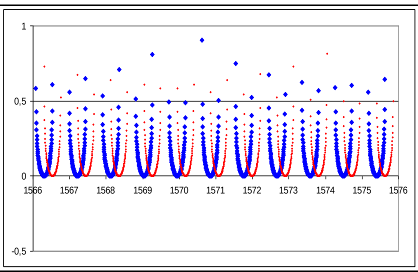
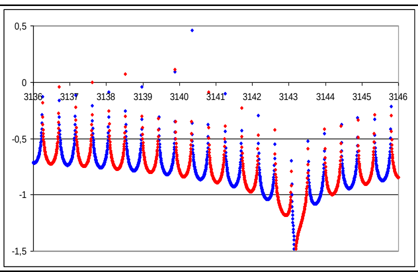
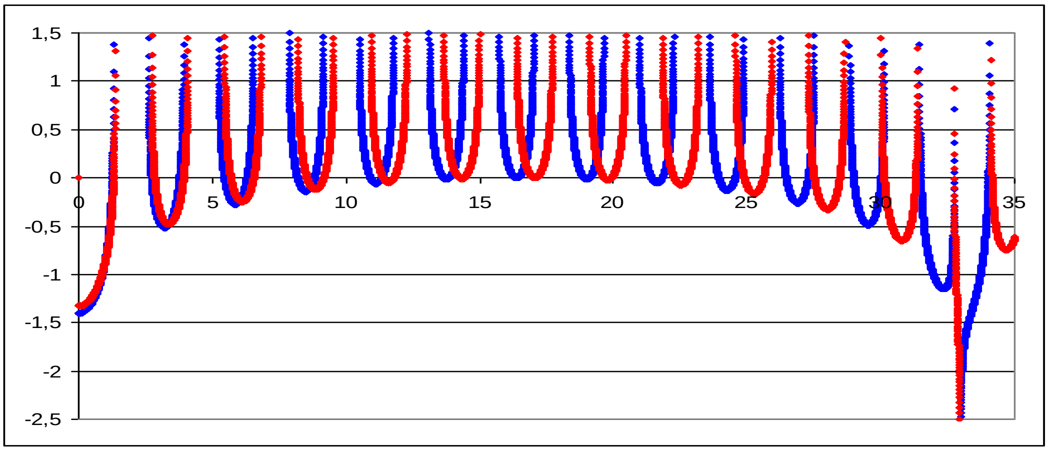


Chart 4

Chart 1 shows the solutions locus on a large scale but does not allow distinguishing them clearly. We however note a repeating schema (which is not necessarily identical) which period is (nearly) imposed by the first expression of b in table 1. In the hereby presented case, we have $2\pi/(\ln(s/r) = 2\pi/(\ln(1001/999) \approx 3142$ (half-period 1571) which is the approximate value that appears in chart 1.
When we use a smaller scale, so looking at chart 2, we start to discover a lower envelop and some friezes above. These drawings have no meaning because the points in question are not connected with each other (by the equations that define them). These drawings are incidental.
The intersections between the blue and red curves become readable thanks to chart 3 scale. The reader understands then that the lower envelop (in charts 1 and 2) gives the sought solutions. We observe also that $\Delta b \approx 2\pi/\ln(r.s)$ (≈ 0.45 here) is an appropriate interval. We have a good agreement with the expected behaviour when r and s are heavy masses, except near 0 where Δb range is doubled.
In the case of an even gap of masses, we see on chart 4 a phase difference of the two curves near the origin which is gradually erodes on the half-period way.
Conversely, if the mass difference is odd, we have a phasing at the origin and a phase difference at half-period and intermediary situations in between the two figures).

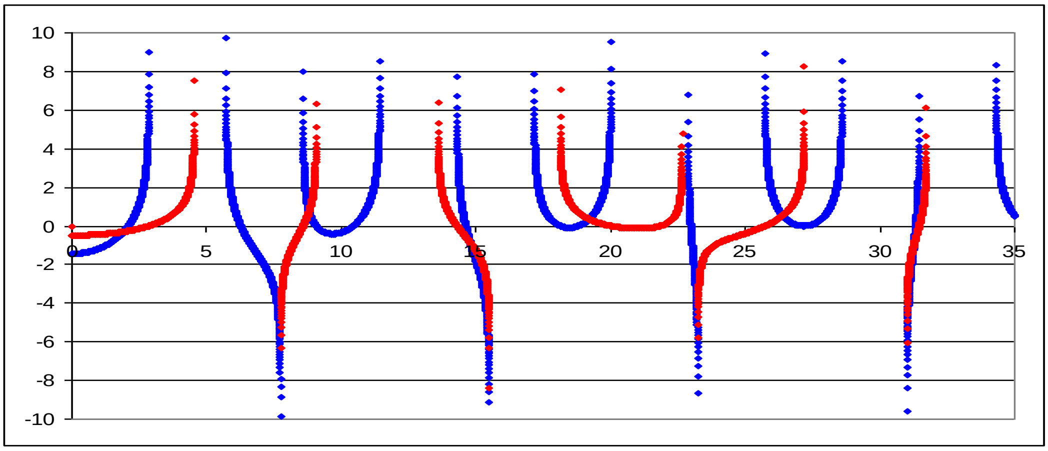
		
r = 1000, s = 1001, $b \approx 0$	r = 1000, s = 1001, $b \approx (\pi/2)/\ln(1001/1000) \approx 1571$	r = 1000, s = 1001, $b \approx \pi/\ln(1001/1000) \approx 3143$

We show underneath the two modes on the same chart using small and close r and s : r = 10, s = 11, $\pi/\ln(11/10) \approx 33$, $2\pi/\ln(11.10) \approx 1.34$.



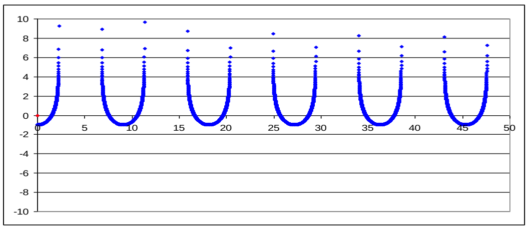
When the bodies' masses are relatively remote, the interferences shift more quickly (which is not the case for close masses). The range of the values of a varies accordingly : because of the quasi-coincidence of the two curves in the chart above, the value of parameter a goes down near to -1 regularly (in the s = r±1 cases). Because of the offset, the number of solutions is up to 2 times smaller than the number based on the $\Delta b \approx 2\pi/\ln(r.s)$ formula.

The limit case r = 2, s = 3 is finally the most anachronistic ($2\pi/\ln(3/2) \approx 15{,}49$, $2\pi/\ln(3.2) \approx 3{,}07$).

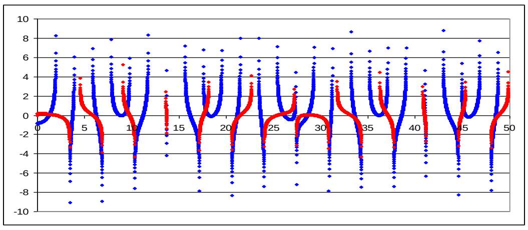


Here (between x-axis 14 and 16 for example), we see what looks like two intersections for the same curves. In fact, the curves do not intersect and the minimum value of a is effectively -1 as shown above.

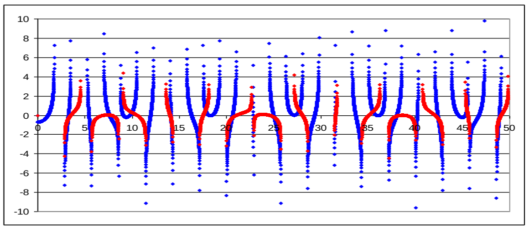
We give an additional sample of the two typical situations where the reader will notice an alternation of the coincidences and the schiftings near the origin.



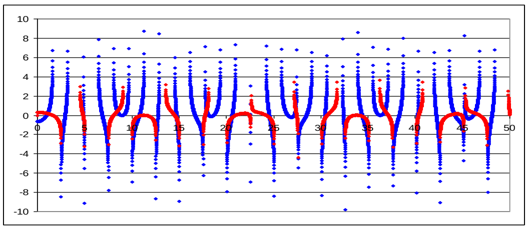
r = 2, s = 4



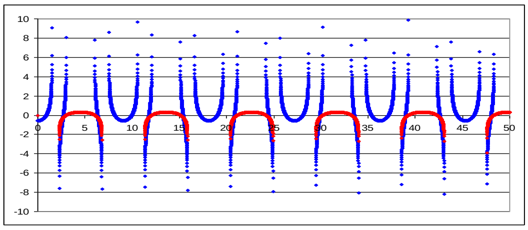
r = 2, s = 5



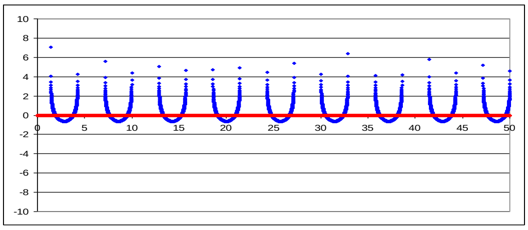
r = 2, s = 6



r = 2, s = 7

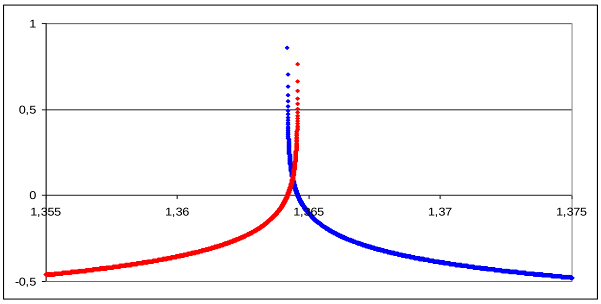


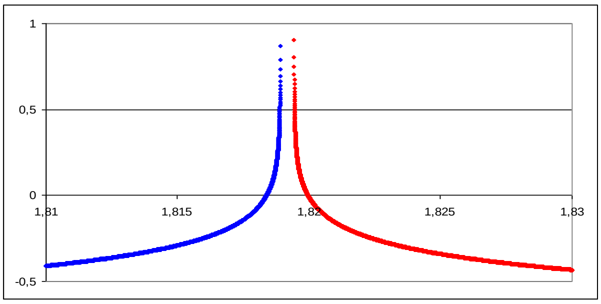
r = 2, s = 8



r = 3, s = 9

To finish with, the reader can note that $2\pi/\ln(r.s)$ is not the solutions period. Indeed, we gathered approximate equations to get to our conclusions. The locus of $a = \ln((-1)^r.\sin(b.\ln(s/r))/\sin(b.\ln(s)))/\ln(r)$ and of $a = \ln((-1)^s.\sin(b.\ln(r/s))/\sin(b.\ln(r)))/\ln(s)$ do get near at the indicated period, but nothing imposes an intersection. In fact, the approximate period (for large r) is $4\pi/\ln(r.s)$. Going back to example r = 999, s = 1001 with adequate scaling, we observe alternatively an intersection and a misfit of the two curves.





In peculiar cases, it may happen that we get no solution (a,b) at all. Indeed let us return to the condition defined by relation 10 :

$$\sin((-1)^{r-1}.b.\ln(s))/\sin(b.\ln(r/s)) > 0$$

If $(-1)^{r-1}.b.\ln(s) = -b.\ln(r/s)$, this condition cannot be met. In this case $(-1)^r.\ln(s)+\ln(s) = \ln(r)$, so that :

$$r = s^{1+si(r = 0 \bmod 2,1,-1)}$$

This equation will realise only if :

$$r = s^2 \text{ and } r = 0 \bmod 2 \qquad (19)$$

Starting from relation 11 (or permuting roles of r and s), it follows alternatively :

$$s = r^2 \text{ and } r = 0 \bmod 2 \qquad (20)$$

We do observe the void of solutions in the above example (r,s) = (2,4) with the absence of the red curve.

These remarks made, an important point is that we still get, as for the one-body problem, an infinite enumerable set of couples (a,b) resulting from the repetition of above presented interference patterns (the repetition is not identical but with a slight shift) when solutions exist.

Three-bodies problem

The zeros of the function obey to the equalities ($r \neq s$, $r \neq u$, $s \neq u$) :

$$\begin{aligned} &1+r^{-a}.(-1)^r.\cos(b.\ln(r))+s^{-a}.(-1)^s.\cos(b.\ln(s))+u^{-a}.(-1)^u.\cos(b.\ln(u)) = 0 \\ &r^{-a}.(-1)^r.\sin(b.\ln(r))+s^{-a}.(-1)^s.\sin(b.\ln(s))+u^{-a}.(-1)^u.\sin(b.\ln(u)) = 0 \end{aligned} \qquad (21)$$

As in the usual physical case (despite the contributions of Karl Sundman), the problem has (a priori) no simpler explicit formulations than those given by relations (21). Thus, we cannot explicitly produce an expression for the admissible range of parameter **a**, likewise relations (17) and (18), but this range exists for any given (r, s, u) triplet. Numerically, we see that this admissible range obeys the same criteria as in the two-bodies case, that means it tends towards an accumulation point when the respective masses (r, s, u) of the bodies diverge. The enumerable quantity may be presumed from cases of one or two bodies.

The lack of explicit expressions does not however prevent to get useful conclusions. Specifically, let us place in the case of finite bodies, except u :

We rewrite relation (21) as :

$$(1/u^{-a})+(-1)^r.(r/u)^{-a}.\cos(b.\ln(r))+(-1)^s.(s/u)^{-a}.\cos(b.\ln(s)) = -(-1)^u.\cos(b.\ln(u)) \quad (22)$$
$$(-1)^r.(r/u)^{-a}.\sin(b.\ln(r))+(-1)^s.(s/u)^{-a}.\sin(b.\ln(s)) = -(-1)^u.\sin(b.\ln(u)) \quad (23)$$

The right members of these relations are not simultaneously equal to 0 as $\cos^2(x)+\sin^2(x) =1$ and do not diverge, for the same reason. The left members, in the same time, tend towards 0 if $a < 0$, or towards $\pm\infty$ if $a > 0$. If $a = 0$, then we get

$$1+(-1)^r.\cos(b.\ln(r))+(-1)^s.\cos(b.\ln(s)) = -(-1)^u.\cos(b.\ln(u)) \quad (24)$$
$$(-1)^r.\sin(b.\ln(r))+(-1)^s.\sin(b.\ln(s)) = -(-1)^u.\sin(b.\ln(u)) \quad (25)$$

This system of equations may have solutions, the only solutions to the asymptotic problem.

Thus, **a tends to accumulation point 0 when the <u>relative</u> masses of bodies diverge**.

Finite universe problem

In this universe, where the bodies have all different integer masses, we have :

$$1+\sum_m m^{-a}.(-1)^m.\cos(b.\ln(m)) = 0 \quad (26)$$

and

$$\sum_m m^{-a}.(-1)^m.\sin(b.\ln(m)) = 0 \quad (27)$$

We proceed like on the preceding paragraph rearranging these equations thus leaving the finite masses bodies in the first member of the equation and putting the bodies with masses tending towards infinity in the second member and making then the adequate division. The asymptotic problem has then solutions only if a tend towards 0.

Note :
The (a,b) solutions are enumerable (but this point is still to be proved).

<u>Lemma (of finite universe)</u>

The real part of truncated zeta function zeros tends towards an **accumulation point 0 if** some **masses** of the finite universe **tend towards infinity** (in presence of a finite mass).

Let us write $s = a+i.b$, $D(a)$ the domain of definition of a and $u \gg n > 1$, then

$$\sum_{m=1}^{n} \frac{(-1)^{m-1}}{m^s} + \frac{(-1)^{u-1}}{u^s} = 0 \quad \Rightarrow \quad D(a) \rightarrow \text{unique point} \qquad . (28)$$

<u>Numerical example with 24 (+1) bodies.</u>

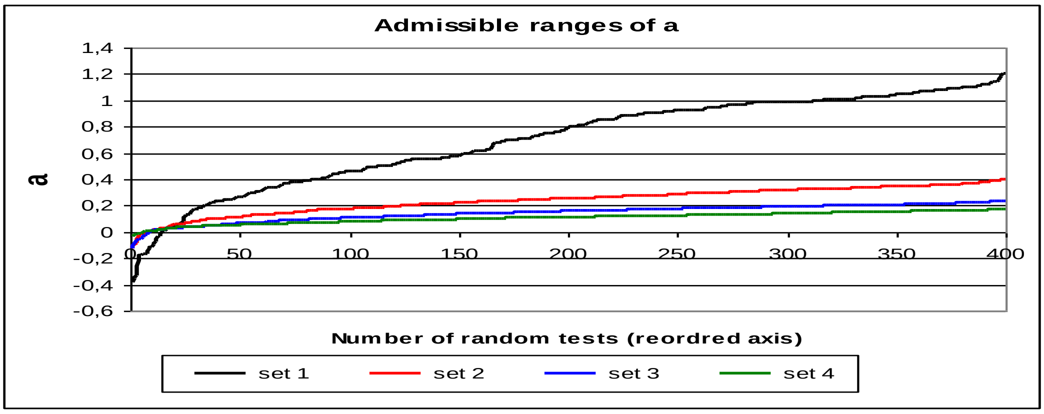
In the following example, we have sought randomly b values such as (abs is the absolute value of the expression in parentheses) :

$$\text{abs}(1+\sum m^{-a}.(-1)^m.\cos(b.\ln(m)))+\text{abs}(\sum m^{-a}.(-1)^m.\sin(b.\ln(m))) < 0.01$$

for the following bodies :

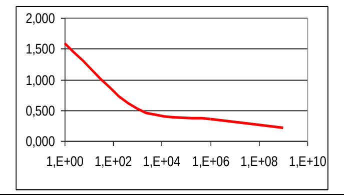
Set 1 : {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25}
Set 2 : {2, 1002, 2002, 3002, 4002, 5002, 6002, 7002, 8002, 9002, 10002, 11002, 12002,13002, 14002, 15002, 16002, 17002, 18002, 19002, 20002, 21002, 22002, 23002}
Set 3 : {2, 1000002, 2000002, 3000002, 4000002, 5000002, 6000002, 7000002, 8000002, 9000002, 100000002, 11000002, 12000002, 13000002, 14000002, 15000002, 16000002, 17000002, 18000002, 19000002, 200000002, 21000002, 22000002, 23000002}
Set 4 : {2, 1000000002, 2000000002, 3000000002, 4000000002, 5000000002, 6000000002, 7000000002, 8000000002, 9000000002, 100000000002, 11000000002, 12000000002, 13000000002, 14000000002, 15000000002, 16000000002, 17000000002, 18000000002, 19000000002, 200000000002, 21000000002, 22000000002, 23000000002}

The admissible ranges of parameter a, for 400 successful tests, are given by graphics :



The admissible range of parameter a reduces, in a logarithmic pace thus very slowly, as the intervals between the bodies masses increase :

Sets	Masses distances	Range a (approximate)	Range size (approximate)
1	1	-0,381 à 1,201	1,582
2	10^3	-0,124 à 0,396	0,520
3	10^6	-0,124 à 0,234	0,358
4	10^9	-0,029 à 0,178	0,207



The trend shows a convergence of parameter a admissible range to an accumulation point (equal to 0) in agreement with the previous lemma.

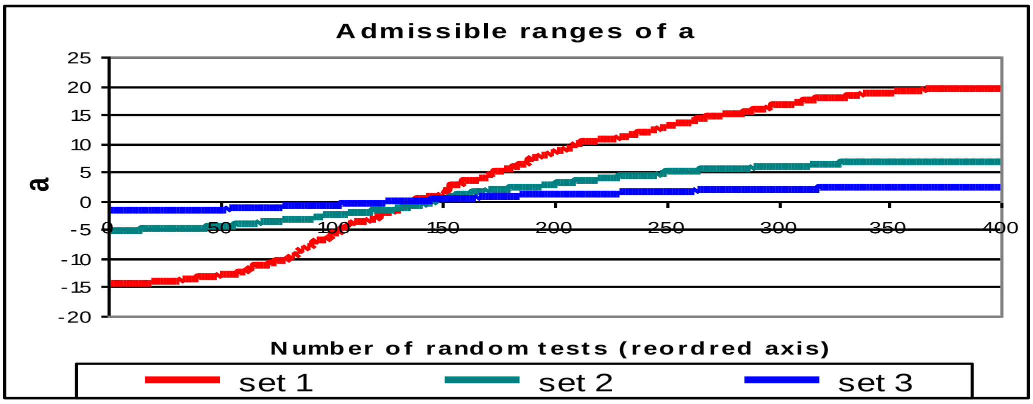
<u>Interpretation of the energy a = 0 level : Density of the universe</u>

When the universe is finite, density, that is. the ratio of the positions m (2, 3, 4...) actually occupied in respect to all possible positions (2, 3, 4,..., ∞) is null and the parameter is either equal to 0 (non-degenerated system including at least an infinite mass) or in "average" equal to 0 (degenerated system without infinite mass). Conversely, for bodies density not null (infinite enumerable set of bodies for example), the value of the parameter is no more equal to 0.

Note :
The average notion is placed between parentheses above. Indeed, when we are looking for the "density" of the universe in the absence of infinite masses by random tests, the distribution of parameter a, surrounding 0, does not give an average zero value. However, we can find this 0 value by comparing (superposing) the distribution curves. For example, let us consider the sets : Set 1: {102/100, 106/100}, set 2: {12/10, 16/10}, set 3: {3, 7}. Supplementing with mass 1, which is always implied, we built in fact the following sets :

Sets	Elements	Intervals between masses
1	{1, 102/100, 106/100}	I = 0,02 and 2I
2	{1, 12/10, 16/10}	I = 0,2 and 2I
3	{1, 3, 7}	I = 2 and 2I

The ratio of masses (here factor 2) remains from one set to another. The curves intersect then around a single point to a near 0 y-axis value, the accuracy depending on the number of tests carried out in these random experiments.



Let us note, finally, that we have used non-integer masses in order to get further distinct curves. By the way, going back to the 24-bodies problem, we observe effectively an intersection at 0 y-value as the sets are, in this earlier case, in constant ratios of the masses (factors equal to 1).

Note : In this last case, as we handled non-integer numbers, we didn't consider the $(-1)^{m-1}$ signs in front of the expressions. However, the fundamental spirit of the exercise remains unchanged, as for random b, the signs of cosine and sine are random and $(-1)^{m-1}$ will intervene in a marginal way on the numerical values but neither on the basics or on the conclusions.

Solutions enumeration : partial or total deformation of solutions locus

Before investigate the infinite universe, we address the (a,b) couples counting issue.
For one body, the solutions enumeration is relatively trivial. We search solutions beyond a single body by trying to infer new solutions by "continuity".

Let us have, for the initial couple (a,b), the equations :

$$1-2^{-a}.\cos(b.\ln(2)) = 0 \text{ and } 2^{-a}.\sin(b.\ln(2)) = 0$$

We wish to go by "continuity" to the couple (a',b') for which :

$$1-2^{-a'}.\cos(b'.\ln(2))+3^{-a'}.\cos(b'.\ln(3)) = 0 \text{ and } 2^{-a'}.\sin(b'.\ln(2))-3^{-a'}.\sin(b'.\ln(3)) = 0$$

If an ongoing mechanism allows passing from one to another and from this to the first one, we may deduct a bijection between these couples and assert these two as enumerable sets (as the first one is enumerable).

Let us write thus for $(\varepsilon_1,\varepsilon_2)$ two infinitesimals :

$$1-2^{-a'}.\cos(b'.\ln(2))+\varepsilon_1 = 0 \text{ et } 2^{-a'}.\sin(b'.\ln(2))+\varepsilon_2 = 0$$

By infinitesimal increments, ε_1 must pass from 0 to $3^{-a'}.\cos(b'.\ln(3))$ and ε_2 from 0 to $3^{-a'}.\sin(b'.\ln(3))$ when (a,b) passes continuously to (a',b'). This process will fail many times even when the equation includes more bodies that are presented here. Trigonometric functions cosine and sine are bounded by [-1,1]. When these limits are reached before ε_i arrives at the desired integer values, the solution under investigation disappears. Conversely solutions will appear from nowhere when the boundary conditions occur. This shows the specificity of each of these types of problem even if a certain common framework exists.

Let us note also a second specificity of the one body problem among all the other finite-bodies cases by returning to the previous procedure where we have left it. We wrote $\varepsilon_1 = \varepsilon_3^{-a''}.\cos(b''.\ln(3))$ and $\varepsilon_2 = \varepsilon_4^{-a''}.\sin(b''.\ln(3))$. However a'' is close to 0 (since a = 0 in the one-body problem), which is equivalent to write for ε_1 : $\varepsilon_1 \approx \varepsilon_3^{\approx 0}.\cos(b'.\ln(3)) \approx 1.\cos(b'.\ln(3))$ which is in contradiction with ε_1 an infinitesimal. This difficulty cannot apparently be bypassed with mathematical rigour to go from the one-body problem to the two-body problem. We thus see misfits in obtaining solutions by "continuity", not only in a partial way but systematically (when we make the transition from one to two bodies).

Infinite universe problem

In this (unlikely) universe that we build here (to fit to Riemann and Dirichlet Eta series), twin bodies do not exist. The bodies fit a succession of integer masses from 2 to infinity without exception. Their population varies from small asteroids to planets, from baby stars to giant stars, and finally to black holes with gigantic masses (by the way in great majority).

Concerning the admissible range of a, our conclusion is devised on the 1 (fixed point), 2 or n (accumulation point when one or more of the bodies become hugely heavy) -bodies problems. As the infinite universe holds at least one body of infinite mass, parameter **a** must thus converge and reduce to the single, already known value, that is according to the billions of identified solutions (A. Odlyzko) :

$$a = 1/2$$

Thus we confirm : All zeroes of Riemann Zeta function have real value 1/2.

Density

The passage of value 0 of the parameter a, for all finite sets, to value 1/2, for the problem that interests us most particularly here (infinite enumerable set), can be interpreted by the density of the bodies, concept that we introduced at page 9. For body density not zero (enumerable or compact amount of bodies) the value of the parameter a is no more equal to 0.

It is easy to imagine exercises where we would vary density between a near value of 0 (by occupying a small fraction of all integer positions of masses m) and a very important value (by occupying many new positions in-between the integer values) thus gradually increasing the value of the parameter a from 0 to 1/2 and then beyond. This is wrong. It is necessary to take into account the paradoxes of the infinite sets and be wary of quick conclusions. The reader knows that we can explicit a bijection between enumerable sets, for example between N and 2N or N and αN, where α is a (real) constant. These sets have the "same number" of items. Their density is **identical**. The result is that the **parameter a and therefore the real part of zeros corresponding "L-functions" remains equal to 1/2 for masses describing αN*** (L as linear).

Extension to other infinite universes

Let us go now from enumerable universes to non-enumerable universes (compact universe with real numbers m).
Hence, we do not consider futher series but integrals. As in $(-1)^{m-1}$, the masses m are no more integers, we do write (-1) as $e^{i.\pi}$.
We therefore consider the following problem :

$$I = \int_m \frac{e^{i.\pi.(m-1)}}{m^s}.dm = 0 \qquad (29)$$

where we still have s =s a+i.b. Then, using also $e^{-i.\pi}.= -1$:

$$I = \int_m \frac{e^{i.(\pi.(m-1)-b.\ln(m))}}{m^a}.dm = - \int_m \frac{e^{i.(\pi.m-b.\ln(m))}}{m^a}.dm = 0 \qquad (30)$$

Then :

$$I = - \int e^{i.2.\pi.\mathrm{frac}(m/2)}.e^{f(m)}.dm = 0 \qquad (31)$$

where f(m) = -(a+ib)ln(m) and frac(m/2) is the fractional part of m/2.
Here $e^{i.2.\pi.\mathrm{frac}(m/2)}$ is a periodic function (of Δm = 2 period).
Let us write then :

$$J = - \int e^{f(m)}.dm = 0 \qquad (32)$$

If $e^{f(m)}$ was a positive crescent monotonic function, this periodic function would induce multiplicative monotonic (crescent or decreasing) c_t coefficients (t = 1, 2, 3, …) for I versus J at each period Δm. Hence :

$$I = J.(\sum c_t / \sum t)$$

For a limit sum ($\sum c_t$) different from 0, we would have then an equivalence between I = 0 and J = 0.

$$I = 0 \Leftrightarrow J = 0$$

If $e^{f(m)}$ was negative monotonic decreasing, the reasoning and conclusion would be analogous.
In our case, $e^{f(m)}$ has no monotonic properties. We have $e^{f(m)} = e^{-(a+ib).\ln(m)} = e^{-a}.(\cos(b).\ln(m)+i\sin(b.\ln(m)))$. For given a and b, we deal with an oscillating function. This finite function has no limit. Thus the corresponding sum ($\sum c_t$) has no fixed limit. The condition ($\sum c_t$) ≠ 0 disappears in this case and we get in an unconditional way :

$$I = 0 \Leftrightarrow J = 0$$

Hence, for I = 0, it suffices to solve :

$$J = \int_{m} e^{-(a+i.b).\ln(m)}.dm = 0$$

Let us do the change of variable u = Ln(m). Then, du = dm/m = e^{-u}.dm, so that :

$$J = \int_{u} e^{-(a-1+i.b).u}.du = -1/(a-1+i.b).[e^{-(a-1+i.b).u}] \qquad (33)$$

Let us go back to the integral boundaries. The upper boundary m → +∞ involves u → +∞. For J do not diverge, this imposes a-1+i.b → 0, that is a → 1 et b → 0. Using $e^x \approx 1+x$ when x → 0, we get by choosing for the lower boundary u → 0 (and the upper boundary u → +∞) :

$$J \approx -[u] \to -\infty$$

Thus, there are no solutions (a,b) for J = 0 and hence to I = 0.
It should be noted that an other lower boundary choice does not change much to the end of this story.

To resume the aforementioned L-functions considerations, let us pose the problem with a linear variable change m → α.m, where α is a positive real constant. We have then :

$$J' = \int_{m} \frac{1}{(\alpha.m)^s}.dm = 0 \qquad (34)$$

So that

$$J' = \alpha^{-s} \int_{m} \frac{1}{m^s}.dm = 0$$

The conclusions concerning the convergences (a → 1,b → 0) and the absence of zeros are therefore the same here as for equation (29).

Arguments résumé

Parameter a of Riemann hypothesis is constant because there exist an infinite mass in the "Riemann universe", which is sufficient to conclude positively.

Parameter a is a measure of the density of the observed universe. We can dress the table of the different situations related to this density :

Universe	Finite	Enumerable	Compact
Mean <<a>>	0	1/2	→ 1
b	Enumerable (Bijection with N)	Enumerable	Unique (→ 0)
Couples (a,b) of zeros	Degenerated	Non-degenerated	Inexistent

An additional extension

We can possibly be surprised here that we do not reach intermediate values for the parameter a. This is accomplished by considering the exercise under a slightly different angle.
Let us write first :

$$\sum_{\substack{m=f(t)\\t=1}}^{t=\infty} (-1)^{m-1}.m^{-a}.\cos(b.\ln(m)) = 0 \qquad \sum_{\substack{m=f(t)\\t=1}}^{t=\infty} (-1)^{m-1}.m^{-a}.\sin(b.\ln(m)) = 0$$

Here the sign $\sum(-1)^{m-1}$. will be replaced by $\int e^{i.\pi.(m-1)}$ to move from a enumerable universe to a non-enumerable universe.
So far, we have been interested in the linear cases :

It suffices to consider

$$f(t) = \alpha.t \qquad (35)$$

$$f(t) = \alpha.t^{\omega} \qquad (36)$$

to get all the desired real part <a>> (from 0^+ to infinity) by varying ω.

Thus the parameter a is, in this new perspective, a function of the density d (d = 0, d = 1/2 or d = 1) of the given universe and of the masses exponential growing ω at work in it.

$$<<a>> = d/\omega \qquad (37)$$

It should be noted that function f(t) can be of a more general form including sums, subtractions, products and/or divisions of several terms. It suffices then to retain only the predominant part at infinity $f(t) \equiv \alpha.t^{\omega}$.