Argument for a twin primes theorem.
Landscapes, panoramas and horizons of the Eratosthenes sieve.

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Abstract We explore three ways on the twin primes problem. We start with the intermediate sets produced by Eratosthenes sieve implementation. Properties related to the proportions of integers eliminated during process on one hand and the distances generated between integers on the other hand allow twice deducing the infinity of prime numbers and twin prime numbers. In the former case, the analysis of the proportions also allows getting an asymptotic evaluation similar to Hardy-Littlewood formula, but without fully valid proof. In the latter case, the analysis of the spacings between remaining integers yields a replica of Bertrand's postulate with approximate $2p_i$ spacing and the asymptotic evaluation of the maximum of the distances between pairs of numbers (of spacing 2), which is ranging around $\sum_i 2p_k$, enables to conclude to the divergence of twin prime numbers below abscissa p_i^2. Finally, an alternative method, that is readily generalizable to many Diophantine equations, is proposed as an invitation to new studies. Again, we infer the Euler product suggested by Hardy-Littlewood.

Argumentaire pour un théorème des nombres premiers jumeaux.
Crible d'Eratosthène. Crible du pgcd.

Résumé Nous étudions trois approches au problème des premiers jumeaux. Nous commençons par les ensembles intermédiaires produits par l'exécution du crible d'Eratosthène. Les propriétés liées aux proportions de nombres entiers éliminés d'une part et aux espacements générés entre nombres entiers d'autre part permettent par deux fois de déduire l'infinité des nombres premiers, puis des nombres premiers jumeaux. Dans le premier cas, l'analyse des proportions permet également d'obtenir une évaluation asymptotique identique à la formule d'Hardy-Littlewood, mais sans pleine et entière démonstration. Dans le second cas, l'analyse des espacements entre nombres restants permet d'obtenir une réplique du Postulat de Bertrand avec un espacement de l'ordre de $2p_i$ et l'évaluation asymptotique du maximum de la distance entre paires de nombres (d'écart 2), évaluation qui est équivalente à $\sum_i 2p_k$, permet de conclure à la divergence du cardinal des nombres premiers jumeaux en dessous de l'abscisse p_i^2. Enfin, une méthode alternative aisément généralisable à de nombreuses équations diophantines est proposée en guise d'invitation à d'autres études. Nous en déduisons à nouveau le produit d'Euler suggéré par Hardy-Littlewood.

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Paradox
see [1]
- no prime number is even except one
- no prime number is even except two

Summary

1. Preamble.

Rarefaction of pairs of primes distant of a given value 2n is a simple process based on Eratosthenes sieve. We will establish our theorems thanks to arithmetical laws governing integers' depletions within natural numbers set N while implementing the said algorithm.
The article below has certainly nothing complicated for specialists of this topic.

To give more clarity and strength to the argument, we will apply it initially to the enumeration of the prime numbers, i.e. we will attempt to retrieve the prime number theorem (PNT).

Passing from the prime numbers' case to the twin prime numbers will simply consist of replacing a given law of scarcity $(p_i\text{-}1)/p_i$ by another $(p_i\text{-}2)/p_i$, p_i being the i^{th} odd prime number. Although the Hardy-Littlewood formula is deduced, no proof is given (nor for the PNT). Only, the infinitely of twin prime numbers is deduced (following similar work on prime numbers).

We downgrade and overrate quantities of solutions in both cases which framing settles asymptotically converging upper and lower boundaries (tending towards infinity).

The study is also upgraded after that for our two groups of objects, prime numbers and twin prime numbers, by evaluating the distances between elements, the guiding threads being now, more or less, the two expressions $2p_i$ and $\sum_i 2p_k$ respectively.

However, more than the results in the p_i to p_i^2 range that allow us to conclude upon the stated problem, we focus attention, when running Eratosthenes algorithm, on the existence of recursive formulas' systems to evaluate asymptotically in the p_i to p_i+p_i# interval, p_i# = 2.3.5.7.11…p_i denoting the primorial of p_i, the integers' populations with given spacing $\Delta = 2j$ (populations of pseudo-primes on the one hand, populations of pseudo-twin primes or relative primes on the other hand), knowing less than j/2+1 initial staffs.
(The terms "pseudo" and "spacing" will be defined very soon in the present article).

Thus, the interest of this article has also become over the course of the different versions, this aspect having taken more and more importance with respect to the initial purpose, facing apparent absence of such a corpus elsewhere, that of an in-depth study of the Eratosthenes sieve.

The reader would have been disappointed with the lack of challenge if he had already found here all the statements demonstrated. On the contrary, and fortunately, he will still be able to exercise all of his insight facing high walls of difficulties, especially in order to appropriate himself the said recursive systems. There is a time for discovery and another for the domination of a subject.

2. An expeditious demonstration.

For the reader who does not have time, here is an appetizier for his immediate satisfaction.

Proposition 1

There is an infinity of twin prime numbers.

Proof

Let us apply the Eratosthenes algorithm up to step p_i. Then, beyond p_i, the intervals of size #p_i, the primordial of p_i, contain each $\prod(p_i\text{-}2)$ pairs of 2-gap numbers. This answers the question of the existence of pairs (not necessary primes). As the algorithm begins with the removal of the smallest dividers, the first pair is a pair of twin primes (you can challenge anyone to find a counter-example). Let us consider p_j the largest number of this pair. Let us continue the depletion algorithm up to p_j. Beyond p_j, the intervals of size #p_j each contain $\prod(p_j\text{-}2)$ pairs of 2-gap numbers, the first of which is a pair of twin prime numbers which is different of the first pair. So we get a second coveted pair. The argument applies to infinity by recurrence.

3. Terminology.

Gap and spacing :

Notions related to the distance between objects in this study can lead to pernicious confusion.
Precise terminology is therefore required to avoid it. We will have to manipulate either isolated integers or pairs of integers. We will call "gap" the distance within a pair of numbers and we will call "spacing" the distance between the studied features which are either isolated numbers or pairs of numbers.
Thus, for the pair of twin prime numbers (11, 13) considered as one object, the gap is 2, while for the two pairs of integers (11, 13) and (15, 17) considered as two objects, the spacing is 4 and the gap is 2.

Writing convention :

The expression « If(a,b,c) » means : If the condition a is true then the expression evaluates to b, otherwise the expression evaluates to c.

4. Fundamental theorems.

In addition to the PNT, two supplementary results will be useful and are presented below (theorem 1 and generalization of the Mertens theorem) to which we add the calculation of an integral.

4.1. Three theorems.

Theorem 1

Let us have r and s two coprime numbers.
There is then a permutation between the two sequences of numbers (0, 1, 2, ..., s-1) and (0, r, 2r, ..., (s-1).r) modulo s.

Proof

The second series' step is constant modulo s (and is equal to r modulo s). The integers r and s being coprime, none of the integers r up to (s-1).r can be zero modulo s (as they do not include any factor equal to s). Integers (0, r, 2r, ..., (s-1).r) modulo s are thus distinct and therefore a permutation of (0, 1, 2, ..., s-1).

Illustration

We will focus, later on, on couples of coprime integers $r = 2.3.5.7.11...p_i = p_i\#$ and $s = p_{i+1}$, p_{i+1} being the prime number next to p_i and we give below some examples :

Table 1

p_i	$r = 2\ldots p_i$	$s = p_{i+1}$	$2\ldots p_i$ mod p_{i+1}	Sequences
2	2	3	2	(0, 2, 1)
3	6	5	1	(0, 1, 2, 3, 4)
5	30	7	2	(0, 2, 4, 6, 1, 3, 5)
7	210	11	1	(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)
11	2310	13	9	(0, 9, 5, 1, 10, 6, 2, 11, 7, 3, 12, 8, 4)

Theorem 2 (Prime Number Theorem)

According to the PNT, the cardinal $\pi(x)$ of prime numbers less or equal to x is equivalent, when the real x tends towards $+\infty$, to the quotient of x to its neperian logarithm.
Hence :

$$\pi(x) \sim \frac{x}{\ln(x)} , \quad x \to +\infty \qquad (1)$$

Theorem 3 (Mertens theorem)

The third Mertens theorem gives the Euler product associated to (1-1/p).
We have, γ being the Mascheroni constant ($\approx$ 0,5772156649), the following result :

$$\prod_{p \leq x,\ x \to +\infty} (1-1/p) \equiv e^{-\gamma}/\ln(x) \qquad (2)$$

The prime number theorem, proved independently by Hadamard and Vallée Poussin, is one of the fundamentals of number theory [2]. Mertens theorem relative to the product of Euler of 1-1/p is addressed in [5]. We will use a corollary of it that we prove below.

Other useful results are sufficiently known not to be included in the list of the above theorems :
- convergence conditions of $\Pi_p(1-1/p^s)$ and $\Pi_p(1-1/p^s+c/p^{s'>s})$,
- ratio $i.\ln(p_i)/p_i$ tending towards 1 as i increases (from the prime number theorem).
- …

Subsequently, we will use either the sign #(E) or π(E) to refer to the cardinal of a set (E)

4.2. Generalization of Mertens theorem.

Corollary

Let us have a > 1 an integer, then :

$$\prod_{a < p \leq x,\ x \to +\infty} (1-a/p) \equiv c_a.e^{-a\gamma}/\ln^a(x),\ c_a \text{ a constant} > 0 \qquad (3)$$

The a = 2 case is the one useful to us :

$$\prod_{2 < p \le x,\ x \to +\infty} (1-2/p) \equiv c_2.e^{-2\gamma}/\ln^2(x),\ c_2 > 0 \qquad (4)$$

<u>Proof</u>

Let us have a positive integer a. Let us have p an element of the set of prime numbers P, set that we divide in two parts (the first one possibly void) : $p \le a$ and $p > a$.

We get, using the Newton binomial formula, c_i being integers :

$$(1-1/p)^a = 1+c_1/p+c_2/p^2+\ldots+c_a/p^a$$

Of course, we have

$$c_1 = -a$$

Let us write

$$ma = \prod_{p \le a} (1-1/p)$$

Here, ma = 1 if the set $p \le a$ is void.
Then, using Mertens theorem :

$$\prod_{p \le x,\ x \to +\infty} (1-1/p) \equiv e^{-\gamma}/\ln(x) \qquad (5)$$

we get :

$$e^{-a\gamma}/\ln^a(x) \equiv \prod_{p \le a} (1-1/p)^a . \prod_{\substack{a < p \le x,\\ x \to \infty}} (1-1/p)^a = ma^a. \prod_{\substack{a < p \le x,\\ x \to \infty}} (1-a/p+c_2/p^2+\ldots+c_a/p^a) \qquad (6)$$

Let us write then for $a \ne p$ (that is for $a < p$)

$$1-a/p+c_2/p^2+\ldots+c_a/p^a = (1-a/p).(1+(c_2/p^2+\ldots+c_a/p^a)/(1-a/p))$$

Hence, using the second and third terms of relation (6)

$$\prod_{a < p \le x,\ x \to \infty} (1-a/p) . \prod_{a < p \le x,\ x \to \infty}(1+(c_2/p^2+\ldots+c_a/p^a)/(1-a/p)) \equiv ma^{-a}.e^{-a\gamma}/\ln^a(x)$$

Let us have s a real number. It is well known that $\sum_n 1/n^s$ converge towards a non-null constant (strictly greater than 1) when $s > 1$. It is the same with $\Pi_p(1-1/p^s)$ as $\zeta(s) = \sum_n 1/n^s = \Pi_p(1-1/p^s)^{-1}$ for $Re(s) > 1$.

We have, for $1 < a < p$, the Taylor series expansion $1/(1-a/p) = 1+a/p+m_2/p^2+m_3/p^3\ldots$
Then $(c_2/p^2+\ldots+c_a/p^a)/(1-a/p) = (c_2/p^2+\ldots+c_a/p^a).(1+a/p+m_2/p^2+m_3/p^3\ldots) = c_2/p^2+r_2/p^3$ + higher order terms…
Thus, $\Pi_{p\to\infty}\ (1+(c_2/p^2+\ldots+c_a/p^a)/(1-a/p)) \equiv \Pi_{p\to\infty}\ (1+c_2/p^2+r_2/p^3+\ldots)$ and this last product converge as $1-1/p^{2-\varepsilon} < 1+c_2/p^2+r_2/p^3+\ldots < 1+1/p^{1-\varepsilon}$ for any coefficients $c_2, r_2,\ldots$ when p is large enough and with $0 < \varepsilon$ an infinitesimal.
The product is thus a non-null constant. We multiply the inverse of this constant by ma^{-a} and note the new constant c_a ($c_a > 0$).
Thus :

$$\prod_{a < p \le x,\ x \to \infty} (1-a/p) \equiv c_a.e^{-a\gamma}/\ln^a(x) \qquad (7)$$

Let us note that this result remains valid for a non-integer a, but this result is not useful here.

4.3. Logarithm weighted sums.

We focus here on the asymptotic value of the prime number sum $\Sigma\, p_i^n/\ln^m(p_i)$ ($n\ge0$, $m\ne0$).
We use $\pi(x) \to x/\ln(x)$, when $x \to +\infty$ written as :

$$\pi(x) = (1+o(1)).x/\ln(x) \qquad (8)$$

The $\pi(x)$ expression is a step function. Its derivative is 1 at the $x = p_i$ abscissas, 0 otherwise.
Thus :

$$\int_2^y (\pi(t))'.v(t).dt = \sum_{p_i \le y} v(p_i) \qquad (9)$$

Partial derivation gives :

$$\int_2^y u'(t).v(t).dt = u(y).v(y) - \int_2^y u(t).v'(t).dt \qquad (10)$$

Let us have $u(t) = \pi(t)$ and $v(t) = t^n/\ln^m(t)$.
Then :

$$v'(t) = t^{n-1}.(n-m/\ln(t))/\ln^m(t) \qquad (11)$$

and thus asymptotically :

$$v'(t) = (1+o(1)).n.t^{n-1}/\ln^m(t) \qquad (12)$$

hence asymptotically :

$$v/v'(t) = (1+o(1)).t/n \qquad (13)$$

So, asymptotically, derivation consists in multiplication by n/t and therefore integration consists in multiplication by t/n if $n \neq 0$.
Then :

$$\sum_{p_i \leq y} p_i^n/\ln^m(p_i) = \pi(y).y^n/\ln^m(y) - \int_2^y \pi(t).t^{n-1}.(n-m/\ln(t))/\ln^m(t).dt$$

Thus :

$$\sum_{p_i \leq y} p_i^n/\ln^m(p_i) = (1+o(1)).y/\ln(y).y^n/\ln^m(y) - \int_2^y (1+o(1)).t/\ln(t).t^{n-1}.n.(1+o(1))/\ln^m(t).dt$$

and :

$$\sum_{p_i \leq y} p_i^n/\ln^m(p_i) = (1+o(1)).(y^{n+1}/\ln^{m+1}(y) - n.\int_2^y t^n/\ln^{m+1}(t).dt\ )$$

Yet the integration is "porous" asymptotically to the logarithm as we have seen by relationship (13), so that $\int t^n/\ln^{m+1}(t).dt \approx 1/\ln^{m+1}(y).\int t^n.dt$.
Then :

$$\sum_{p_i \leq y} p_i^n/\ln^m(p_i) = (1+o(1)).y^{n+1}/\ln^{m+1}(y).(1-n/(n+1)). \qquad (14)$$

Finally :

$$\sum_{p_i \leq y} p_i^n/\ln^m(p_i) = (1+o(1)).(1/(n+1)).y^{n+1}/\ln^{m+1}(y) \qquad (15)$$

and :

$$\lim_{y \to +\infty} \frac{\sum_{p_i \leq y} p_i^n/\ln^m(p_i)}{y^{n+1}/\ln^{m+1}(y)} = 1/(n+1) \qquad (16)$$

This relationship shows easily true numerically (for n positive or zero) and converges much faster as n increases.
Later on, we will need the derived relationships :

$$\sum_{p_i \leq y} 1/\ln(p_i) \to y/\ln^2(y) \qquad (17)$$

$$\sum_{p_i \leq y} p_i/\ln(p_i) \to (1/2).y^2/\ln^2(y) \qquad (18)$$

$$\sum_{p_i \leq y} 1/\ln^2(p_i) \to y/\ln^3(y) \qquad (19)$$

5. Eratosthenes sieve.

5.1. Depletion algorithm.

This sieve of antique origin is described again. It is simply to phase out multiples of prime numbers starting with the smallest one.
So we get numbers without small divisors. To describe them, we adopt the following term.

Definition 1

The integers remaining after removal of the multiples of p_i are called of Eratosthenes numbers of rank i, in shortcut Eras_pseudo_prime(i) and give an infinite list of numbers Eras(i). As a shortcut, we will also use the term pseudo-primes without specifying the rank i. The "s" of Eras(i) comes from "starting" list or list of cycle 1.

We also chose to write $p_0 = 2$, $p_1 = 3$, etc.

The process is carried out here between 2 and the integer N. So we have initially N-1 integers. We take off the multiples of 2 :

Table 2

Step 0 : Era(0) list – Retrieval of multiples of 2 (except 2)

Entry	Cycle 1	Cycle 2	Cycle 3	Cycle 4	Cycle 5	Cycle 6	Cycle 7	Cycle 8	Cycle 9	Cycle 10	Cycle 11	Cycle 12	Cycle 13	Cycle 14	Cycle 15	Cycle 16	Cycle 17	Cycle 18	Cycle 19	Cycle 20	Cycle 21	Cycle 22	Cycle 23	Cycle 24	Cycle 25	Cycle 26	Cycle 27	Cycle 28	Cycle 29	Cycle 30	Cycle 31	Cycle 32	Cycle 33	Cycle 34
2	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49	51	53	55	57	59	61	63	65	67	69

In a cycle, as stated in the table above, of length 2, is missing 1 element (#A0 = 1) compared to the previous step (that is, the even number), hence proportion $\#RE_0 = 1/2$ of integers.

Step 1 : Era(1) list – Retrieval of multiples of 3 (except 3)

Entry		Cycle 1			Cycle 2			Cycle 3			Cycle 4			Cycle 5			Cycle 6			Cycle 7			Cycle 8			Cycle 9			Cycle 10			Cycle 11		
2	3	5	7		11	13		17	19		23	25		29	31		35	37		41	43		47	49		53	55		59	61		65	67	

In a cycle of length 2*3, is missing 1 element ($\#A_1 = 1$) compared to the previous step, hence $\#RE_1 = (2-1)/(2.3) = 1/6$ of integers.

Step 2 : Era(2) list – Retrieval of multiples of 5 (except 5)

Entry			Cycle 1															Cycle 2															…	
2	3	5	7		11	13		17	19		23			29	31			37		41	43		47	49		53			59	61			67	

Cycle 3													Cycle 4															Cycle 5						
71	73		77	79		83			89	91			97		101	103		107	109		113			119	121			127		131	133		137	139

								Cycle 6															Cycle 7											
	143			149	151			157		161	163		167	169		173			179	181			187		191	193		197	199		203			209

			Cycle 8															Cycle 9																
211			217		221	223		227	229		233			239	241			247		251	253		257	259		263			269	271			277	

In a cycle of length 2*3*5, is missing 2 elements (25 and 35 in the first cycle, $\#A_2 = 2$) compared to the previous step, hence $\#RE_2 = (2-1).(3-1)/(2.3.5) = 1/15$ of integers.

Step 3 : Era(3) list – Retrieval of multiples of 7 (except 7)

Entry				Cycle 1																														
2	3	5	7		11	13		17	19		23			29	31			37		41	43		47			53			59	61			67	

Cycle 1																																		
71	73			79		83			89					97		101	103		107	109		113				121			127		131			137	139

Cycle 1																																		
	143			149	151			157			163		167	169		173			179	181			187		191	193		197	199					209

Cycle 1				Cycle 2																														
211					221	223		227	229		233			239	241			247		251	253		257			263			269	271			277	

In a cycle of length 2*3*5*7, is missing 8 elements (49, 77, 91, 119, 133, 161, 203 and 217 in the first cycle, $\#A_3 = 8$) compared to the previous step, hence $\#RE_3 = (2-1).(3-1).(5-1)/(2.3.5.7) = 4/105$ of integers.

We observe a "rho" type process : we have a first part of numbers, we will call the "entry" part, which has a non-repetitive structure and parts that we call "cycles" with repetitive patterns. The amplitudes of these patterns are equal to $2.3.5\ldots p_i$, p_i being the last prime number whose multiples were removed (the number p_i being kept). Thus the numbers of the cycle n+1 are those of the cycle n by adding $2.3.5\ldots p_i$.
Cycle 1 starts at p_i+2 (except at step 0, where one must choose $p_i+1 = 3$).

We evaluate now disappearing quantities at each step.
At step 0, we have $\#A_0 = 1$ erasing. At step 1, $\#A_1 = 1$.

Theorem 4

The number of erasures $\#A_{i+1}$ and the proportion of depletion $\#RE_{i+1}$ in a cycle at step i+1 are given recursively to cardinals in a cycle at stage i ($p_0 = 2$):

$$\#A_{i+1} = \#A_i.(p_i-1) = \prod_{k=0}^{i} (p_k-1) \qquad (20)$$

and

$$\#RE_{i+1} = \#RE_i.(p_i-1)/p_{i+1} = (1/p_{i+1}).\prod_{k=0}^{i} ((p_k-1)/p_k) \qquad (21)$$

where $\#A_0 = 1$.

Proof

Let us get this proof choosing a representative example. A cycle 1 at step i+1 is built from a cycle 1 at step i by $2.3.5\ldots p_i$ add-ons. Thus :

Table 3

7		37	67	97	127	157	187	~~217~~	217 = 7.31
11		41	71	101	131	~~161~~	191	221	49 = 7.7
13		43	73	103	~~133~~	163	193	223	133 = 7.19
17	=>	47	~~77~~	107	137	167	197	227	77 = 7.11
19		~~49~~	79	109	139	169	199	229	49 = 7.7
23		53	83	113	143	173	~~203~~	233	203 = 7.29
31		61	~~91~~	121	151	181	211	241	91 = 7.13

As $2\ldots p_i$ mod p_{i+1} is a non-null integer coprime to p_{i+1} (here 2.3.5 mod 7 = 2), each previous line contains, according to theorem 1, only one single number 0 modulo p_{i+1} (the one who disappears) and so 1 among p_{i+1} numbers (here the proportion of 1 among 7). We illustrate this by restoring the above table modulo p_{i+1} ($p_{i+1} = 7$) :

7		2	4	6	1	3	5	~~0~~	7 = 0 mod 7
11		6	1	3	5	~~0~~	2	4	11 = 4 mod 7
13		1	3	5	~~0~~	2	4	6	13 = 6 mod 7
17	=>	5	~~0~~	2	4	6	1	3	17 = 3 mod 7
19		~~0~~	2	4	6	1	3	5	19 = 5 mod 7
23		4	6	1	3	5	~~0~~	2	23 = 2 mod 7
31		5	~~0~~	2	4	6	1	3	31 = 3 mod 7

Hence the result.

Theorem 5

Let us consider the integers' set 1 up to N.
Let us state :

$$\pi s(c,N) = M - (1/c) \sum_{k=0}^{+\infty} \#RE_k.MC_k \qquad (22)$$

$$M = N-1 \qquad (23)$$

$$M_0 = N-(2+1)+1 = N-2 \qquad (24)$$
$$M_k = N-(p_k+2)+1 = N-p_k-1 \quad k \geq 1 \qquad (25)$$
$$MC_k = \text{if}(M_k < 0, 0, M_k) \qquad (26)$$

and

$$\#RE_0 = 1/2 \qquad (27)$$

$$\#RE_i = \#A_i.\prod_{k=0}^{i}(1/p_k) = (1/p_i).\prod_{k=0}^{i-1}(p_k-1/p_k) \quad i \geq 1 \qquad (28)$$

Then, the cardinal of prime numbers is minored at abscissa p_i, starting at some rank i (which can be i = 1), by:

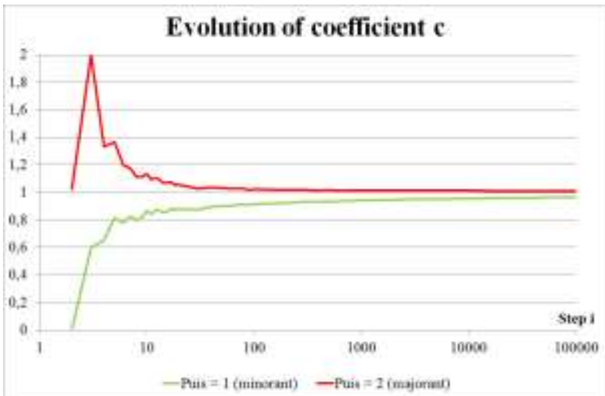
$$\pi s(c = 1,N) \qquad (29)$$

Proof

It is the simple transcription of the erasing by the sieve of Eratosthenes using depletion ratios.
The cardinal's diminution in the cycles at step i+1 is regulated by theorem 4.
These withdrawals begin in the first cycle never before p_i+2 except for stage 0 (in $p_i+1 = 2+1 = 3$). This therefore causes an excess on population enumeration when counting is anticipated to this boundary.
Moreover, as one cannot subtract to a set elements that are not within it, when M_i becomes negative, this term and all those who follow are null (thus relation 26).
Hence the result.

We give below the value of c which enables matching the prime numbers' cardinal giving approached numerical computation. We expect that this value tends to 1. This is what is effectively observed when a calculation is done near the origin as shown in the graph below :

Graph 1



To mark-up the cardinal, the following alternative choice, where the boundary is taken near p_i^2 instead of p_i,

$$MC_0 = \text{if}(N-4 < 0, 0, N-4) \qquad (30)$$
$$MC_i = \text{if}(N-p_i^2-1 < 0, 0, N-p_i^2-1) \qquad (31)$$

shows a faster convergence (above).

Theorem 6

Let us have using the same features :

$$\pi s(1, +\infty) = \lim_{N \to +\infty} M - \sum_{k=0}^{+\infty} \#RE_k.MC_k \qquad (32)$$

Then, the choice of the abscissa indexed by p_i gives a reduction (**minoration**) of the prime numbers cardinal and indexing by p_i^2 will give a mark-up (**majoration**).

Proof

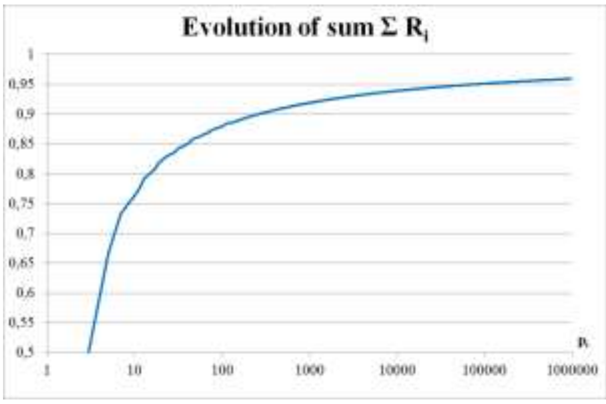
For p_i, it is immediate as multiples of p_i are after p_i.
For pi², it is because of the (a priori) existence of prime numbers between p_i and p_i^2. These numbers not being withdrawn from the cardinal during subtractions the calculation gives an excess of numbers taken into account.

Moreover, as prime numbers have a density 0 in N, we have the relationship:

$$\sum_{i=0}^{+\infty} \#RE_i = 1/2+1/6+1/15+4/105+8/385+\ldots = 1 \qquad (33)$$

We call $\#RE_i$ the depletion coefficients of the Eratosthenes Sieve (ES) and will give the proof of equality to 1 further on. In the meantime, we illustrate this point by the graph below :

Graph 2



Theorem 7

The cardinal of the prime numbers, inferior to x, diverges.

Proof

Let us go back to relation 33 and do our calculations ignoring the unit amount and write instead :

$$\sum_{i=0}^{+\infty} \#RE_i. = 1-\varepsilon \qquad (34)$$

As we cannot subtract to a set only elements it contains, we have necessarily in the previous relationship $\varepsilon \geq 0$.
Then we get using the relationship 32 :

$$\pi s(1, +\infty) = \lim_{N \to +\infty} (\varepsilon + \sum_{i=0}^{+\infty}\#RE_i).M - \sum_{i=0}^{+\infty}\#RE_i.M_i \qquad (35)$$

So that :

$$\pi s(1, +\infty) = \lim_{N \to +\infty} \varepsilon.M + \sum_{i=0}^{+\infty} \#RE_i.(M-M_i) \qquad (36)$$

Thus :

$$\pi s(1, +\infty) = \lim_{N \to +\infty} \varepsilon.M+\#RE_0+ \sum_{i=1}^{+\infty} \#RE_i.p_i \qquad (37)$$

Then developing $\#RE_i$:

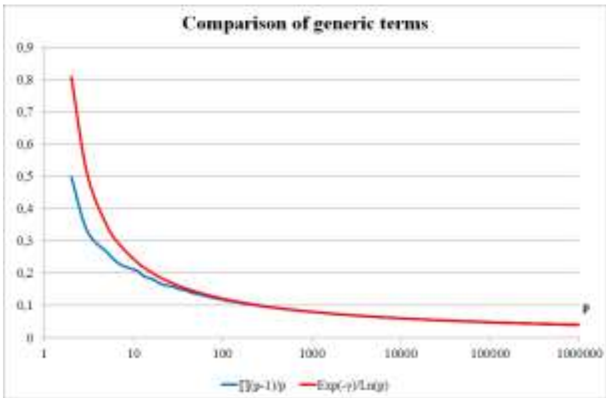
$$\pi s(1, +\infty) = \lim_{N \to +\infty} \varepsilon.M+\#RE_0+ \sum_{i=1}^{+\infty} \prod_{k=0}^{i-1}(p_k-1)/p_k \qquad (38)$$

Hence :

$$\pi s(1, +\infty) = \varepsilon.M+1/2+(2-1)/2+(2-1).(3-1)/(2.3)+(2-1).(3-1).(5-1)/(2.3.5)+ (2-1).(3-1).(5-1).(7-1)/(2.3.5.7)+ (2-1).(3-1).(5-1).(7-1).(11-1)/(2.3.5.7.11)+\ldots \qquad (39)$$

According to theorem 3 (Mertens theorem), the previous generic term $\#RE_i.p_i$ tends towards $e^{-\gamma}/\ln(p_i)$ when i tends towards infinity.

Graph 3



So that, cte1 and cte2 being strictly positive constants :

$$\pi s(1, +\infty) = \varepsilon.M + cte1 + cte2.\sum_{i=1}^{+\infty} e^{-\gamma}/\ln(p_i) \qquad (40)$$

Using relation (16), we have then :

$$\pi s(1, +\infty) = \lim_{x \to +\infty} \varepsilon.M + cte1 + cte2.e^{-\gamma}.x/\ln^2(x). \qquad (41)$$

The previous sum tending towards infinity, the initial gap between the two above curves is negligible.

The last term contains no (negative) linear component likely to compensate at infinity the linear term ε.M. Yet π(1) increases (according to the PNT) like x/ln (x), and therefore contains no linear term, meaning that :

$$\varepsilon = 0 \qquad (42)$$

Moreover the last term in the previous relation diverges, so cte1 is negligible in front of infinity, thus leaving only the said last term (cte = cte2.$e^{-\gamma}$ > 0) :

$$\pi s(1, +\infty) = \lim_{x \to +\infty} cte.x/\ln^2(x) \qquad (43)$$

This expression means effectively that the cardinal of prime numbers tends to infinity.

Fundamental note

The final result for πs(1+∞), relationship 39, shows as a sum of fractions less than 1. This comes from the fact that we use M-M_i in the intermediate calculation. It is essential to note here that nevertheless we do not handle fractions of units. If it were so, our estimate would be false, because we would have to take all these fractions as zeros to form the reduction (since a integer shows up in full, not as a part of it, otherwise it may not show in general), which would amount to an overall reduction equal to 0. In fact, upon calculations, we handle M on one hand and #RE_i.M_i (cf. relation 32) on the other. The latter are of numbers effectively greater than 1 up to a certain rank (before becoming negative) and are counted as such. When the choice of rounding is done, it necessarily leads to an increase in the reduction and we therefore preferred to plot the graph with a more pessimistic view by not rounding (i.e. we count all positive #RE_i.M_i that are afterwards subtracted to M).
Incidentally, rounding integers or not, the results of the calculations vary very little.

Theorem 8

The cardinal of the prime numbers, less than x, diverges as x/ln(x).

Argument

We wrote above a result of Hadamard and De la Valley-Poussin. There is no need to prove it again.
Now, relationship 43 does not resemble to the PNT. But we will show next why. Appearances are deceiving, because we have not yet considered close nature of the x axis. To do this, let us first look at the alternative choice M_i = N-p_i^2-1 (which leads, as we have seen above, to a more fast convergence towards the expected value).
With this choice, knowing that ε = 0, we get:

$$\pi c(1, +\infty) = 3.\#RE_0 + \sum_{i=1}^{+\infty} \#RE_i.p_i^2 \qquad (44)$$

Then developing $\#RE_i$:

$$\pi c(1, +\infty) = 3.\#RE_0 + \sum_{i=1}^{+\infty} p_i.\prod_{k=0}^{i-1}(p_k-1)/p_k \qquad (45)$$

Thus :

$$\pi c(1, +\infty) = cte1' + cte2'. \sum_{i=1}^{+\infty} e^{-\gamma}.p_i/\ln(p_i) \qquad (46)$$

Then using relation (16), we get :

$$\pi c(1, +\infty) = \lim_{x \to +\infty} cte2'.(e^{-\gamma}/2).x^2/\ln^2(x). \qquad (47)$$

The expression is different from $\pi s(1, +\infty)$, but the result is the same, namely a divergence to infinity.

Now let us look at the axis in two expressions $\pi s(1, +\infty)$ and $\pi c(1, +\infty)$. In the first expression, the measurements are made with sampling at $p_{i+1}-p_i \approx \ln(p_i)$ distances. In the second, the distances are now $p_{i+1}^2-p_i^2 \approx p_i.\ln(p_i)$.. We can deduce backwards what it would be when index i does guide the calculation.
To do this, we sketch the following table:

Table 4

M_i ($i \geq 1$)	$M_i = N-p_i^2-1$	$M_i = N-p_i-1$	$M_i = N-i-1$ ($i \approx p_i/\ln(p_i)$)
Interval between measures	$p_i.\ln(p_i)$	$\ln(p_i)$	1
Ratio1 deduced :	$p_i^2/(p_i.\ln(p_i)) = p_i/\ln(p_i)$	$p_i/\ln(p_i)$	$p_i/\ln(p_i)$
Corresponding sum	$\Sigma\, cte2'.(e^{-\gamma}/2).p_i/\ln(p_i)$	$\Sigma\, e^{-\gamma}/\ln(p_i)$	$\Sigma\, 1$
Limit	$cte''.x^2/\ln^2(x)$	$cte'.x/\ln^2(x)$	$cte.x/\ln(x)$
Ratio2 deduced (taking $x \equiv p_i \equiv p$)	$p/\ln(p)/(p^2/\ln^2(p)) = \ln(p)/p$	$1/\ln(p)/(p/\ln^2(p)) = \ln(p)/p$	$1/(p/\ln(p)) = \ln(p)/p$

First, it should be noted that this is a less accurate result than the prime number theorem. The goal is not to demonstrate again this theorem (namely the multiplicative constant is equal to 1) but only to establish the consistency of the results (i.e. there is effectively a multiplicative constant), which will then entirely meet our ambition here.

Ratios 1 and 2 remain well respectively constants from one column to another.
So to $M_i = N-p_i/\ln(p_i)-1 \to N-i-1$ matches up the expression :

$$\pi(1, +\infty) = \lim_{x \to +\infty} cte.x/\ln(x). \qquad (48)$$

and in the same time (i being the p_i index) :

$$\pi(1, +\infty) = \sum_{i}^{+\infty} 1 \qquad (49)$$

The penultimate expression is actually the PNT by taking cte = 1, while the last is trivial (which does not reduce in any way his great interest here as the result is obvious).

Important note :

We stress here that the expressions (43), (48) and (49) are equivalent: they would give the same result, namely the same numerical value if it happened to be finite. Not so here, they give simply by three times the value $+\infty$. Going from one expression to the other and aligning the data on the same curve (at least approximately) here amounts to a simple elongation (or contraction) of the abscissas.

To conclude here, we came up with the asymptotic evaluation of a set of density 0 within the set of natural integers N by subtraction of elements that do not belong to the set. Subtracted quantities are based on a recurrent series $\#RE_i$, where the sum $\Sigma\ \#RE_i$ is 1. This has enabled us to confirm the infinity of prime numbers and rediscover meanwhile its expected asymptotic growth in x/ln(x).

Thereafter, for the twin prime numbers enumeration, we will redo an identical construction to state a similar conclusion by simply replacing p_i-1 by p_i-2. The precedent study is also essential in the fact that it has helped to define the nature of the x-axis support of the prime numbers count.
This support axis' determination will also be indispensable and readdressed for the twin prime numbers count.

Before that, we propose to discover the structure of the spacings between integers generated by the Eratosthenes sieve.

5.2. Landscaping of spacings between pseudo primes.

This paragraph is essential to the preparation of paragraph 6.4.
Our study is focusing here on an interval of size $\#p_i$, the primorial of p_i, the aim being to find usable results in the interval $[p_i, p_i^2]$ in the said paragraph.

5.2.1. Panoramas of populations.

The pseudo-primes are here those of the Eras(i) list remaining when running the Eratosthenes algorithm.
Thus we briefly analyse spacings in the cycle 1 at step i. This is done in a very different way compared to what we will do and see in chapter 6 for twin prime numbers. What we do here, is to list the distances of an element to the previous one and this one only.
We start by counting them for steps 1 up to 9.

Table 5

Steps i	1	2	3	4	5	6	7	8	9
p_i	3	5	7	11	13	17	19	23	29
Cycle 1 size	6	30	210	2310	30030	510510	9699690	223092870	6469693230
Spacings ΔP	Number of ΔP spacings = #SP(j,i)								
2	1	3	15	135	1485	22275	378675	7952175	214708725
4	1	3	15	135	1485	22275	378675	7952175	214708725
6		2	14	142	1690	26630	470630	10169950	280323050
8			2	28	394	6812	128810	2918020	83120450
10			2	30	438	7734	148530	3401790	97648950
12				8	188	4096	90124	2255792	68713708
14				2	58	1406	33206	871318	27403082
16					12	432	12372	362376	12199404
18					8	376	12424	396872	14123368
20					0	24	1440	61560	2594160
22					2	78	2622	88614	3324402
24						20	1136	48868	2100872
26						2	142	7682	386554
28							72	5664	324792
30							20	2164	154220
32							0	72	10128
34							2	198	15942
36								56	7228
38								2	570
40								12	1464
42									272
44									12
46									2
Numbers of spacings $\sum_j$ #SP(j,i)	2	8	48	480	5760	92160	1658880	36495360	1021870080

Steps i	1	2	3	4	5	6	7	8	9
Ratio $\sum_j$ #SP(j,i) / $\sum_j$ #SP(j,i-1)		4	6	10	12	16	18	22	28
Average spacings Δ_{mean}	3	3,75	4,375	4,8125	5,2135	5,5394	5,8471	6,1129	6,3312
c = $\Delta_{mean}/\ln(p_i)$ $\to e^{\gamma}$	2,7307	2,3300	2,2483	2,0070	2,0326	1,9552	1,9858	1,9496	1,8802
$\Delta_{max}/\Delta_{mean}/i$ $\to 2e^{-\gamma}$	1,3333	0,8000	0,7619	0,7273	0,8440	0,7823	0,8307	0,8179	0,8073

By construction, adding the spacings between numbers, we find the overall magnitude of the cycle 1. So 1*2+1*4 = 6, 3*2+3*4+2*6 = 30, 15*2+15*4+14*6+2*8+2*10 = 210, etc.

Theorem 9

The number of spacings at step i (for column i, j = 1 to j max) is equal to the product of the p_k-1, k = 1 to i.

$$\sum_j \#SP(j,i) = \prod_i (p_k-1) \qquad (50)$$

Proof

It is simply a repeat of theorem 4.

The average spacing $\Delta m(i) = \prod_i p_k/(p_k-1)$ is immediately deduced and tends towards $e^{\gamma}.\ln(p_i)$ where $e^{\gamma} \approx 1,781$.
If the maximum spacing is in the order of magnitude of $2p_i$, the ratio $\Delta max/\Delta m$ tends towards $2e^{-\gamma}.p_i/\ln(p_i)$, hence $2e^{-\gamma}.i$, meaning, it is increasing linearly with i ($2e^{-\gamma} \approx 1,123$).

The distances of 2 and 4 generated by the Eratosthenes sieve will be examined in the next chapter. We will see that they have actually same cardinal and increase by a p_i-2 ratio (table 26 page 44). We get here the same counts as in the next chapter due to the fact that these two small spacings, the configurations related to the enumerations are in all points identical.
For other quantities appearing in the table (spacings > 4), their anticipation is more complex and we will remain mainly in a conjectural domain of analysis.

Let us address first how quantities do increase when the step is incremented.

Table 6

Steps i	1	2	3	4	5	6
Supplementary prime numbers	3	5	7	11	13	17
Cycle 1 size	6	30	210	2310	30030	510510
Spacings ΔP	Ratios #RP(j,i) = number of spacings at rank i/number of spacings at rank i-1					
2		3	5	9	11	15
4		3	5	9	11	15
6			7	10,14	11,90	15,76
8				14	14,07	17,29
10				15	14,60	17,66
12					23,50	21,79
14					29	24,24
16						36
18						47
20						+∞
22						39

Lemma 1

We have (when #RP(j,i) exists) :

$$\#RP(j,i) \geq p_i-2 \qquad (51)$$

and

$$\lim_{i \to +\infty} \#RP(j,i) \to p_i\text{-}2 \qquad (52)$$

Proof

The two points result from the fact that Eratosthenes algorithm generates in the cycle 1 (and the following) gradually larger spacings at the level of a same x-coordinate. This creates a gradual saturation of small spaces (starting with the smallest one), left spaces that will gradually fit in the "mainstream", i.e. in the base proportion allocated by the depletion process when two numbers are taken into account at the same time (and not just one), proportion which is p_i-2 as we will prove, in chapter 6 (theorem 12).

Lemma 2

The spacings' cardinals are even, except for the first two of them (corresponding to 2 and 4).

Proof

Indeed, one of the dividers to each of the $n_1, n_2, \ldots, n_k$ constituting the vacant spacing between two numbers is in the set $\{3, 5, \ldots, p_i\}$ and similarly so also for $2.3.5\ldots p_i\text{-}n_1$, $2.3.5\ldots p_i\text{-}n_2$, .., $2.3.5\ldots p_i\text{-}n_k$. However n_1-2 and n_k+2 having no divisors throughout $\{3, 5, \ldots, p_i\}$, it will be the same for $2.3.5\ldots p_i\text{-}(n_1\text{-}2)$ and $2.3.5\ldots p_i\text{-}(n_k\text{+}2)$.
So spacings come in pairs.

For spacings 2 and 4, the cardinal is odd due to the fact that the elements are centred and self-symmetrical.

5.2.2. Horizons on the iterative enumeration of populations.

Lemma 3

There is a constant c_j such that the number of spacings on the j line compared to the total number of spacings at stage i is greater than $c_j/\ln(p_i)$.

$$\#SP(j,i)/\textstyle\sum_j \#SP(j,i) \geq c_j/\ln(p_i) \qquad (53)$$

Proof

Let us note ij the stage i from which on #SP(j,i) begins to exist (becomes different from 0). According to the relationship (51), $\#RP(j,i) \geq p_i\text{-}2$. From there, according to (50), for i>ij, the progression of the $\#SP(j,i)/\sum_j \#SP(j,i)$ ratio is faster than that of the product $\prod_{i>ij} (p_k\text{-}2)/(p_k\text{-}1)$. So, for all i, we have $\#SP(j,i)/\sum_j \#SP(j,i) \geq .\prod_{i<=ij} 1/(p_k\text{-}1).\prod_{i>ij} (p_k\text{-}2)/(p_k\text{-}1) = c_{j'}.\prod_{i>ij} (p_k\text{-}2)/(p_k\text{-}1) = c_{j''}.\prod_{i} (p_k\text{-}2)/(p_k\text{-}1)$. The latter product tends asymptotically (with i) towards c $c/\ln(p_i)$ according to Mertens theorem generalization (relationship (3)).
Hence the result.

Conjecture 1

The populations #SP(j,i) are expressed by a system of iterative relations (on some given j line) from a certain rank i on.

Examples

Let us give a few examples before explaining how to get these iterative relationships.

Table 7

j	Formulas
1	$\#SP(1,1) = 1$ $\#SP(1,i) = (p_i\text{-}2).\#SP(1,i\text{-}1)$
2	$\#SP(2,1) = 1$ $\#SP(2,i) = (p_i\text{-}2).\#SP(2,i\text{-}1)$
3	$x1(2) = 2$ $x1(i) = (p_{i-1}\text{-}3).x1(i\text{-}1)$ $\#SP(3,1) = 0$ $\#SP(3,i) = (p_i\text{-}2).\#SP(3,i\text{-}1)+x1(i)$
4	$x1(3) = 2$ $x1(i) = (p_{i-2}\text{-}4).x1(i\text{-}1)$ $x2(2) = 0$ $x2(i) = (p_{i-1}\text{-}3).x2(i\text{-}1)+x1(i)$ $\#SP(4,1) = 0$ $\#SP(4,i) = (p_i\text{-}2).\#SP(4,i\text{-}1)+x2(i)$

j	Formulas
5	$x1(4) = 4$ $x1(i) = (p_{i-2}-4).x1(i-1)$ $x2(3) = 2$ $x2(i) = (p_{i-1}-3).x2(i-1)+x1(i)$ $\#SP(5,2) = 0$ $\#SP(5,i) = (p_i-2).\#SP(5,i-1)+x2(i)$
6	$x1(5) = 12$ $x1(i) = (p_{i-3}-5).x1(i-1)$ $x2(4) = 8$ $x2(i) = (p_{i-2}-4).x2(i-1)+x1(i)$ $x3(3) = 0$ $x3(i) = (p_{i-1}-3).x3(i-1)+x2(i)$ $\#SP(6,2) = 0$ $\#SP(6,i) = (p_i-2).\#SP(6,i-1)+x3(i)$
7	$x1(6) = 36$ $x1(i) = (p_{i-3}-5).x1(i-1)$ $x2(5) = 20$ $x2(i) = (p_{i-2}-4).x2(i-1)+x1(i)$ $x3(4) = 2$ $x3(i) = (p_{i-1}-3).x3(i-1)+x2(i)$ $\#SP(7,3) = 0$ $\#SP(7,i) = (p_i-2).\#SP(7,i-1)+x3(i)$
8	$x1(6) = 24$ $x1(i) = (p_{i-4}-6).x1(i-1)$ $x2(5) = 12$ $x2(i) = (p_{i-3}-5).x2(i-1)+x1(i)$ $x3(4) = 0$ $x3(i) = (p_{i-2}-4).x3(i-1)+x2(i)$ $x4(3) = 0$ $x4(i) = (p_{i-1}-3).x4(i-1)+x3(i)$ $\#SP(8,2) = 0$ $\#SP(8,i) = (p_i-2).\#SP(8,i-1)+x4(i)$
9	$x1(7) = 144$ $x1(i) = (p_{i-4}-6).x1(i-1)$ $x2(6) = 120$ $x2(i) = (p_{i-3}-5).x2(i-1)+x1(i)$ $x3(5) = 8$ $x3(i) = (p_{i-2}-4).x3(i-1)+x2(i)$ $x4(4) = 0$ $x4(i) = (p_{i-1}-3).x4(i-1)+x3(i)$ $\#SP(9,3) = 0$ $\#SP(9,i) = (p_i-2).\#SP(9,i-1)+x4(i)$
10	$x1(8) = 240$ $x1(i) = (p_{i-5}-7).x1(i-1)$ $x2(7) = 336$ $x2(i) = (p_{i-4}-6).x2(i-1)+x1(i)$ $x3(6) = 24$ $x3(i) = (p_{i-3}-5).x3(i-1)+x2(i)$ $x4(5) = 0$ $x4(i) = (p_{i-2}-4).x4(i-1)+x3(i)$ $x5(4) = 0$ $x5(i) = (p_{i-1}-3).x5(i-1)+x4(i)$ $\#SP(10,3) = 0$ $\#SP(10,i) = (p_i-2).\#SP(10,i-1)+x5(i)$

j	Formulas
11	x1(9) = 1152 x1(i) = (p_{i-5}-7).x1(i-1) x2(8) = 1728 x2(i) = (p_{i-4}-6).x2(i-1)+x1(i) x3(7) = 372 x3(i) = (p_{i-3}-5).x3(i-1)+x2(i) x4(6) = 28 x4(i) = (p_{i-2}-4).x4(i-1)+x3(i) x5(5) = 2 x5(i) = (p_{i-1}-3).x5(i-1)+x4(i) #SP(11,4) = 0 #SP(11,i) = (p_i-2).#SP(11,i-1)+x5(i)
12	x1(9) = 2880 x1(i) = (p_{i-6}-8).x1(i-1) x2(8) = 1800 x2(i) = (p_{i-5}-7).x2(i-1)+x1(i) x3(7) = 216 x3(i) = (p_{i-4}-6).x3(i-1)+x2(i) x4(6) = 20 x4(i) = (p_{i-3}-5).x4(i-1)+x3(i) x5(5) = 0 x5(i) = (p_{i-2}-4).x5(i-1)+x4(i) x6(4) = 0 x6(i) = (p_{i-1}-3).x6(i-1)+x5(i) #SP(12,3) = 0 #SP(12,i) = (p_i-2).#SP(12,i-1)+x6(i)
13	x1(9) = 2580 x1(i) = (p_{i-6}-8).x1(i-1) x2(8) = 1186 x2(i) = (p_{i-5}-7).x2(i-1)+x1(i) x3(7) = 50 x3(i) = (p_{i-4}-6).x3(i-1)+x2(i) x4(6) = 2 x4(i) = (p_{i-3}-5).x4(i-1)+x3(i) x5(5) = 0 x5(i) = (p_{i-2}-4).x5(i-1)+x4(i) x6(4) = 0 x6(i) = (p_{i-1}-3).x6(i-1)+x5(i) #SP(13,3) = 0 #SP(13,i) = (p_i-2).#SP(13,i-1)+x6(i)

To evaluate the initial values xi(...), numbering int((j+2)/2) including possibly some 0's of the j line, it suffices to know at most the int((j+2)/2) first non-zero values of #SP(j,i). This is done by extracting successively from the later the remnants of Euclidian divisions by p_{i-k}-(k+2).

For example, for the line j = 6, we have to use the int((6+2)/2) = 4 first values at most (some of which are therefore possibly 0) corresponding below to the part of the table double framed. Performing the 4 successive Euclidian divisions, like the calculations shown in the last column below, we observe systematically the appearance of values equal to 0 to the right of the double frame.

Table 8

p_i	5	7	11	13	17	19	23	…
Line j	0	0	8	188	4096	90124	2255792	…
Euclidian division 1	0	0	8	100	1276	20492	363188 = 2255792-90124*(23-2)	…
Euclidian division 2	0	0	8	36	276	2628	35316 = 363188-20492*(19-3)	…
Euclidian division 3	0	0	8	12	24	144	1152 = 35316 -2628*(17-4)	…
Euclidian division 4	0	0	8	12	0	0	0 = 1152-144*(13-5)	…

Each Euclidian division allows the determination of a new initial value that is added on a diagonal. This is here (0, 0, 8, 12, (0)), values which are used afterwards to build the numerical table asymptotically :

Table 9

p_i	5	7	11	13	17	19	23	29	31	37	41	43	…
	0	0	8	188	4096	90124	2255792	68713708	2206209208	83462164156	3474628537016	151047124809308	…
		0	8	100	1276	20492	363188	7807324	213511676	6244841876	219604134932	8587354791652	…
			8	36	276	2628	35316	543564	10521252	266514948	7279511148	242397664236	…
				12	24	144	1152	13824	193536	3483648	83607552	2173796352	…
					0	0	0	0	0	0	0	0	…

The last line, usually omitted in the following text, is implied.

General expression of recursive systems

The general writing of the recursive relationships' system is as follows

Table 10

x(j,i-int(j/2))	x(j,i-int(j/2)+1)	…	x(j,i)	x(j,i+1)	…
	x(j-1,i-int(j/2)+1)	…	x(j-1,i)	x(j-1,i+1)	…
		…	…	…	…
			x(j-int(j/2),i)	x(j-int(j/2),i+1)	…
				0	…

with

$$x(k,i) = (p_{i-(k-1)}-2-k-1).x(k,i-1)+x(k-1,i) \quad (54)$$

and

$$\#SP(j,i) = x(j,i) \quad (55)$$

Numerical examples

The values below have been checked up to rank i = 9. Beyond that, the values are speculative.
In the tables below, the values of #SP(j,i) in parentheses allow us to establish the constants xi(r) necessary to apply the iterative formulas.

Table 11

i	p_i	#SP(1,i)	#SP(2,i)	#SP(3,i)	#SP(4,i)	#SP(5,i)	#SP(6,i)
1	3	(1)	(1)				
2	5	3	3	(2)			
3	7	15	15	14	(2)	(2)	
4	11	135	135	142	(28)	(30)	(8)
5	13	1485	1485	1690	394	438	(188)
6	17	22275	22275	26630	6812	7734	4096
7	19	378675	378675	470630	128810	148530	90124
8	23	7952175	7952175	10169950	2918020	3401790	2255792
9	29	214708725	214708725	280323050	83120450	97648950	68713708
10	31	6226553025	6226553025	8278462850	2524575200	2985436650	2206209208
11	37	217929355875	217929355875	293920842950	91589444450	108861586050	83462164156
12	41	8499244879125	8499244879125	11604850743850	3682730287600	4396116829650	3474628537016
13	43	348469040044125	348469040044125	481192519512250	155231331960250	186022750845750	151047124809308

i	p_i	#SP(7,i)	#SP(8,i)	#SP(9,i)	#SP(10,i)	#SP(11,i)	#SP(12,i)
1	3						
2	5						
3	7						
4	11	(2)					
5	13	(58)	(12)	(8)		(2)	
6	17	(1406)	(432)	(376)	(24)	(78)	(20)
7	19	33206	12372	(12424)	(1440)	(2622)	(1136)
8	23	871318	362376	396872	(61560)	(88614)	(48868)
9	29	27403082	12199404	14123368	2594160	3325554	(2100872)
10	31	903350042	423955224	512670088	106604280	126803610	88345892
11	37	34861119734	16996070868	21218333416	4814320320	5463271134	4075111904
12	41	1475437583074	741616123248	949982718776	230780018520	253219805154	199176739444
13	43	65082209263162	33583362918924	43986950258888	11319407188560	12098327744322	9949934146072

In view of the (conjectured) regularity of the iterative formulas, the anticipation of these constants xi(r) would completely solve the problem of counting. This could not be achieved here.

We can however specify the location of the first non-zero element on the j-line of the population table 5 :
For j such as $p_{i-2}+1 \leq j \leq p_{i-1}$, this first element is necessarily at position i (generally) or beyond.
For $j = p_{i-1}$, $i \geq 1$, moreover, the population, therefore the value of this first element, is systematically equal to 2 except for $j = p_0 = 2$ with initialization to 1.
We note the notable exception of the case of column $p_i = 23$ where we find a non-zero number beyond the $j = p_{i-1}$ line (in $j = p_{i-1}+1$). We think it unique but we are hardly able to prove it.

Let us now compare the initial values (of Table 7), at the point where we were able to determine them, to the data of the population table (Table 5). These initial values are inscribed in red font below within the said population table (except zeroes) :

Table 12

Steps i	1	2	3	4	5	6	7	8	9	…
p_i / $\Delta P = 2j$	3	5	7	11	13	17	19	23	29	…
2	**1**	3	15	135	1485	22275	378675	7952175	214708725	…
4	**1**	3	15	135	1485	22275	378675	7952175	214708725	…
6		**2**	14	142	1690	26630	470630	10169950	280323050	…
8			**2**	28	394	6812	128810	2918020	83120450	…
10			**2**	26+**4**	438	7734	148530	3401790	97648950	…
12				**8**	176+**12**	4096	90124	2255792	68713708	…
14				**2**	38+**20**	1370+**36**	33206	871318	27403082	…
16					**12**	408+**24**	12372	362376	12199404	…
18					**8**	256+**120**	12280+**144**	396872	14123368	…
20					**0**	**24**	1104+**336**	61320+**240**	2594160	…
22					**2**	50+**28**	2250+**372**	86886+**1728**	3323250+**1152**	…
24						**20**	920+**216**	47068+**1800**	2097992+**2880**	…
26						**2**	92+**50**	6496+**1186**	383974+**2580**	…
28							**72**	4536+**1128**	320664+**4128**	…
30							**20**	1380+**804**	150632+**3588**	…
32							**0**	**72**	7056+**3072**	…
34							**2**	136+**62**	13260+**2682**	…
36								**56**	5488+**1740**	…
38								**2**	196+**374**	…
40								**12**	1176+**288**	…
42									**272**	…
44									**12**	…
46									**2**	…
…										…

We find the initial values (the values in blue font of table 8 for example for 2j = 12) by making a horizontal reading of the table.
The populations close to the maximum of $\Delta P = 2j$ are equal to the initial values and gradually only a portion of it is to be taken into account (as initial values).

<u>Malleability of systems</u>

Finally, and this applies to the other formulas of the same type that we will find in this article, it should be noted the malleability of these iterative formulas. Indeed, we can swap the order of the c_k in the $(p_{i-k}-c_k)$ expressions at leisure while finding exactly the same #SP(j,i) by simply adjusting the initial conditions xk(r).
We wrote a specific article on this subject "Invariance in a triangular system of recursive equations and unitriangular matrixes" [7].
Needless is to say that the ascending order of k (and c_k) is the obvious one and is the one that has been retained here. Besides, giving concrete meaning to the initial coefficients in the context of an arbitrary order is not obvious.

The example for j = 13 is given below.

13	$x1(9) = 2580$ $x1(i) = (p_{i-6}-8).x1(i-1)$ $x2(8) = 1186$ $x2(i) = (p_{i-5}-7).x2(i-1)+x1(i)$ $x3(7) = 50$ $x3(i) = (p_{i-4}-6).x3(i-1)+x2(i)$ $x4(6) = 2$ $x4(i) = (p_{i-3}-5).x4(i-1)+x3(i)$ $x5(5) = 0$ $x5(i) = (p_{i-2}-4).x5(i-1)+x4(i)$ $x6(4) = 0$ $x6(i) = (p_{i-1}-3).x6(i-1)+x5(i)$ $\#SP(13,3) = 0$ $\#SP(13,i) = (p_i-2).\#SP(13,i-1)+x6(i)$	$x1(9) = 5052$ $x1(i) = (p_{i-6}-6).x1(i-1)$ $x2(8) = 1236$ $x2(i) = (p_{i-5}-8).x2(i-1)+x1(i)$ $x3(7) = 58$ $x3(i) = (p_{i-4}-3).x3(i-1)+x2(i)$ $x4(6) = 2$ $x4(i) = (p_{i-3}-5).x4(i-1)+x3(i)$ $x5(5) = 0$ $x5(i) = (p_{i-2}-2).x5(i-1)+x4(i)$ $x6(4) = 0$ $x6(i) = (p_{i-1}-4).x6(i-1)+x5(i)$ $\#SP(13,3) = 0$ $\#SP(13,i) = (p_i-7).\#SP(13,i-1)+x6(i)$	$x1(9) = 7972$ $x1(i) = (p_{i-6}-4).x1(i-1)$ $x2(8) = 1732$ $x2(i) = (p_{i-5}-2).x2(i-1)+x1(i)$ $x3(7) = 64$ $x3(i) = (p_{i-4}-8).x3(i-1)+x2(i)$ $x4(6) = 2$ $x4(i) = (p_{i-3}-7).x4(i-1)+x3(i)$ $x5(5) = 0$ $x5(i) = (p_{i-2}-3).x5(i-1)+x4(i)$ $x6(4) = 0$ $x6(i) = (p_{i-1}-5).x6(i-1)+x5(i)$ $\#SP(13,3) = 0$ $\#SP(13,i) = (p_i-6).\#SP(13,i-1)+x6(i)$

Asymptotic behaviour

The resulting numerical values follow. The numbers in parentheses are obtained from the initial xk(r) conditions to be adjusted, and then remain the same, regardless of the permutation adopted.

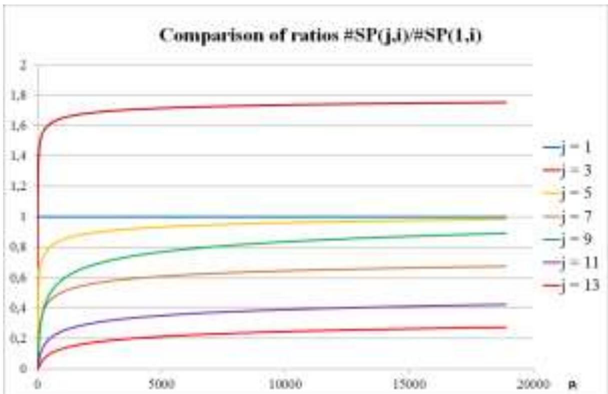
i	p_i	#SP(13,i)
3	7	(0)
4	11	(0)
5	13	(0)
6	17	(2)
7	19	(142)
8	23	(7682)
9	29	(386554)
10	31	18296026
11	37	917779870
12	41	47868405830
13	43	2523638720330
14	47	140310923994850
15	53	8521044521043950
16	59	562884816841615450
17	61	38006808659692941250
18	67	2776584409210071637450
19	71	212874333408720904370450
20	73	16674778854319869359926850
21	79	1401166023549229397548238150
22	83	122977907658913527789701081950
23	89	11502780841555360481825175525050
24	97	1165580304713859247287339606190850
25	101	122562697582639018843161308883447850
26	103	13112754736781472886415720803648313050
27	107	1453351921671783646083875844678718429850
28	109	163783729028421214171254691350900881085650
29	113	19090760983054610636273575350824582551981450
30	127	2490149754971161047278626232764643094100655650
31	131	334505343704327817752693163631815861180635580650
32	137	46908332520608833556782238732974003111934848052050
33	139	6667536123998885033362185956972826564358181730380950
34	149	1013808433029832410059335901349067128255376332794388050
35	151	156097440634286284953011093147817174505594521931801749050
36	157	24959101448658489141113826564592082779615686887185872886850
37	163	4138772764364345164239583547725371204294664920449705998153550
38	167	702574529619742553207414136546661390692757889239462591720406350
39	173	123435023346526582185411291027417474707510191562352469863110298050
40	179	22419312647708222993713153535618153671640710866274320769140117641950

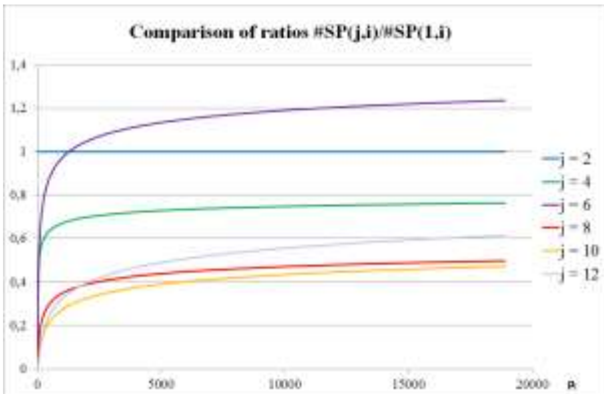
i	p_i	#SP(13,i)
41	181	4115527016222756202180561899505189728379359941505888984112546544826950
42	191	7964144635185480040370773815182866961101005843829614500368536492323 43950
43	193	15566915824107987901841789382499099344693770282316711404169761554484081 8850
44	197	31042285222743476450457585552199407740672342683825137478447501405306409875450
45	199	62507759284496049948158550688289604117724033624615342978720328437996608009 93550

A little further on, we get :

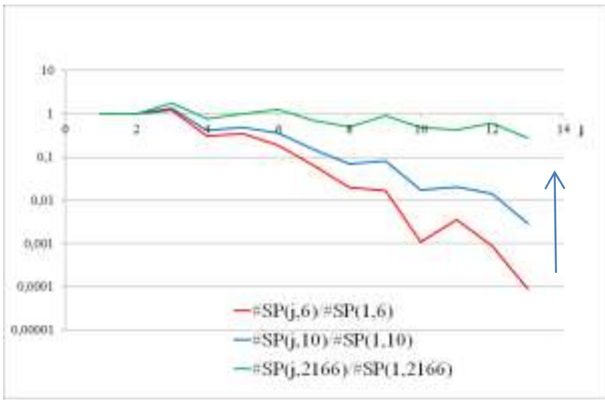
i	p_i	#SP(13,i)
2150	18911	1938680047241827632475991866668894895601809458679053758397700794963257127667657938170963161034999827080601695081488030074306363182318899190762384977449853638969484065158759687673444951218376411559015800312507753999825488008365509518587894118696987946774426270025531730673185054671412636928871750857835492203151583641890474450574468108588441952261417735979401926347070258810655297209828700255614512463003376247574239291662590142009704169677312422524466238624646568070757854651400261158537523266881841625242203927544237410664039267915019210312685892101996105225937438025974216361163742388117109623044206398424809232728692536212624174070118564378367252317163045715484814741314628367296992247513143248257626761216118199429490079744922379155473066957830910412970573738244175400349273675329980696181508057382964378010674818104875925972467498010382457232693477911754665901716971554457606288629184299615155423133976197758911748166413347758206755089236075027480089424157409911122957407806877524992373193309328254878629516541730690544238523308742381128483272774092211388655594353537383207001733620196248372113680839369167871908472225041705199803286680724795762153507986444351953778690494658881647117448286266216281127262482665365953145994081660878262296976908269043475422119641472865017205389578646472908753828673735073762949030257933245365847325501807447603478259108719443772830611893302182385586906238150431003367343997735450243478327134737411914140503793999208649696794629140309504360284766990422507492368703107507726531879714887978840850819274495358158747214912681554107010586703293391440905348878623250297383620658238848178336266030866381889991782608757069704870188425003399035295584469533794671293213707182951691157864730904917051046407901420889732356259017807218205625667561731264691922370357903511425343939204004795476190554277321510266278549034915796827605805216020319097418422832664392621596073155568181654712801033162231115507638457980957143053205830495849077721873743559994554517100795934577123145789701231987168996860870854944816472111558131292694498952391289508778253808205166499835259476305743426055713421024560454695425847609511123089934451846032783048435718215939104435655161657291771717442696628291297850666977613247559745375744392824126053728408259984313606540438981344595147359899041299107831922208077150765461544716108360537681686424566903496078329082959141839260669511785550945693618770964853346679196925120157558507227416609625745371430839885785554085527450376462811427485200139050978360648116714281254757743417259151197356105038426264340376902267806071342950694089416703930240004683573914299436787965555220602258340824569181345989427393139957759929421695793302431195094483132555374307281473037560109305418121089274227432678665957042901355662894773674990828822687652737907155315724100292340424033066598239364129574916456309265434431907418717246282889940955573283412057187535203756873957295978011309019960000376021490799450759953782765445872328903772470888647184186626839988937022132958370519772039692651309856133533928072511232643332287956444341853846697035923754872090121813778983328010390731325466677145101700750459733680219051946357783234470634722817701051211894629925434584783781557574955488330051928200099775884622870494095403432830791636476930784233522576759295126690522248468907343830652488165032166196972708799290456245913334656296346220201030887211770210280194437077538684444308154611067828294607005479485069641680014484319225616841030354476199267925150634071703927364446822485976560935278438610106166391166295632178989255544606500394442754201743394230593792460964023094644207798979469635981667383625253411420264538507757948511704337309991333435441971101204764716745446400170353365512454958256515972843165622290488910015766996243435447439996005329046137027379105225280900431352123521251196832344157813849798138161747201588693427864904981110714113081335055464846079692212530956110766319504424643381194016235840073542062744180969358848750995970436167952909097672688930085982143008930898648297244030846127458551849673172053324393488505046037207176378672218260711208878077323376750473167865309051991720373048050142014280389533358498307036565181739291392769920786735458174112205586498919927024124260360892592143409321745382572936032771792643122342329365469350864167201386746117073306170600222524100038895245514995925752659217083191345986421555585467959477589937475380287852878886925914576055369855667357657102776578173042480097799635751091887975872241571421370587958997650630834303919715659247627585397602392631061919232143259873156821944329003776792168216357335070400413601563964845364005545155711138131192414712870408469471111812255571985941141498248098412753528586805327998047089572168703397230561941682582916085243015237485697883380240645125727890691828364038644085975751437078755032991837762937420994074373083326037565184353294159378200866807542759470743472813896463040179661736289664539875997753628697204975410856130020082246453933762640990461275998588917082292921271013815936804616189087929223850293884396937181484209049146255905435746649123446288714520574402595343934470803195048008007061758922813755891021539804073257458548255858469496943071944913884840124453683393028798537382876866074515094072631701209471971275844297895400144824824366752731553345559224786713892822043830562955061957596110822413235751023217577562461295199178213090362443611941374822771880907953387268638239982446631091928496601263086849831700702534399900009795705931732709915904197968141217829160742085803666004811495902915844764574471190269414106923218908301771446378354521028835803974337138288871152070183251823680918158720318747600297913503237947060772215107023803980057327872475568574525954365482128802047888327350304433853886801647889331419983350820470557708739687406356366819638858071455557799345469000383442374644826896374490698582242910490126445211189389784717683773733850172073990534938519727626426399175139442425750989069836104068540200121445053180006229592626145511703798019593735164686049793779066001073765136742948116282484967091455335641469549969980365093273234430483979679889019417606929300971974579617014391476377558760059166938930237891300638158381496943712331203225992045761100215267221686852292226271885919371769985177851687145382102777831326333353460682827368169358535136287835509181388107006052378028787946421525990158103841711587809943747554040732986086009115658723509652075587874909546574381677587762554242246529385431327325725614090316494382380944214786141614739785446973473307901389380695059520827369366974457551784427073780763000419055481774421736097597775092894708339468810890814085055578733429345689850420776820935007406084307308505865487605267014694018883703661408956722793711085376794759467659820011163096021852528002511529456243342963622080866754268701810859663314469599617545870407225988158624697592020782956358638861975050485664111031153883508062811405437968326649842330153517315272629989036643997759020916837390217425268566271752006469429651270672188694411721933032211052510019609371855251408318078285305890539164363069518889642261660549003689315371751616594477608947077910937005574207742945558073269366854489287711872759392875115048554331670812427415218535149796527652915851073492403631413647185095510099552976333738464212651317526496966602586579104790443753616562802181822202268623089522460037472506820276130655161759901609804770790712448263366713550780302914912333376321094654623651050468796064117837892582230963509146004482203153209433395622400193465037911769281147538472935387647272590132664329073546208519780223030922226711816747120243406519288546217809199445103766514046861875196537741445865069420066544572156109068723439816407440391407202805825686440373648968860518802667101153236322329278304055292273654102602324807808479934070452893260068266931999008013242283162841774625329356001434095382822234621182710872880828160948601329708301108619957400678084721342197942156970568366481184300669248098613989023561113136962043938469975926672666859857521503179384305174103241496010995371218725923087633044761997760183320708799000190419315226439713376397536264740363917347717920837080651628859350

A similar study up to i = 2150 for all of the examples j = 1 to 13 allows us to draw the following curves :





In the early stages i, the comparative numbers of populations of 2j-spacings are in significantly different proportions, for example a ratio of more than 1 to 10000 between {i = 6 (p_i = 17), j = 13} and {i = 6, j = 1}. As i tends towards progressively towards infinity, values are shunned as a result of contributions supplemented by the increase in the number of equations in the recursive system (a new equation for each 2 added value to j). Thus, the ratio mentioned above drops to a ratio 1 to 3.66 (ratio close to its asymptotic value).



Starting from theorems 9 and 12, assuming that the order of magnitude of the #SP(j,i) is that of #SP(1,i) when i tends towards infinity, there would then exist a constant c such as $\prod_{i\to+\infty} (p_i-1) = \sum_j$ #SP(j,i→+∞) > c.j_{max}.#SP(1, i→+∞) = c.$j_{max}.\prod_{i\to+\infty} (p_i-2)$. Hence $j_{max} < (1/c).\prod_{i\to+\infty} (p_i-1)/(p_i-2)$ and, from Mertens's theorem, we might conclude that there is a constant c' such as $j_{max} < c'\ln(p_i)$.
The order of magnitude of the number of lines j in row i, (including non-zero values) would then be asymptotically in $\ln(p_i)$. This order of magnitude is much lower than what is observed really (j_{max} around of p_i in fact as we will see later on) as there are in fact intermediate values between the populations of the line j = 1 and those of the line j_{max}.

However, what we are looking to highlight here is the very strong de facto constraint on the maximum value of j for given i. It is difficult, and in actual fact impossible, to reconcile the growth in the populations generated by recursive formulas with a j_{max} that would regularly be beyond p_i (value given below). Indeed, any effective population (change from zero to a non-zero value) immediately triggers afterwards a steady increase of the said population on following ranks i and conversely any delay in apparition will have to be catched up without fail, the ratio $\prod_{i\to+\infty} (p_i-1)$ of the overall populations been forced at each stage i. Recursive links existence and coercion due to the relationship (52) is self-regulating the asymptotic increase of the maximum value of j (for given j).

5.2.3. Evolution of aggregated populations.

We give below the cumulative staffs that correspond to spacings greater than a given value.

Table 15

	Steps i	1	2	3	4	5	6	7	8	9
	p_i	3	5	7	11	13	17	19	23	29
j	Spacings ΔP	Aggregation of populations ΔPC = #SPC(j,i)								
1	≥ 2	2	8	48	480	5760	92160	1658880	36495360	1021870080
2	≥ 4	1	5	33	345	4275	69885	1280205	28543185	807161355
3	≥ 6	0	2	18	210	2790	47610	901530	20591010	592452630
4	≥ 8		0	4	68	1100	20980	430900	10421060	312129580
5	≥ 10			2	40	706	14168	302090	7503040	229009130
6	≥ 12				10	268	6434	153560	4101250	131360180
7	≥ 14				2	80	2338	63436	1845458	62646472
…	…				…	…	…	…	…	…

Because recursive formulas are linear, the aggregations follow the same types of relationships, the initial values obtained previously are simply added altogether (into table 7):

Table 16

j	Formulas
1	#SPC(1,1) = 2 #SPC(1,i) = (p_i-1).#SP(1,i-1)
2	x1(2) = 1 x1(i) = $(p_{i-1}-2)$.x1(i-1) #SP(2,1) = 1 #SP(2,i) = (p_i-1).#SP(2,i-1)+x1(i)
3	x1(2) = 2 x1(i) = $(p_{i-1}-2)$.x1(i-1) #SP(3,1) = 0 #SP(3,i) = (p_i-1).#SP(3,i-1)+x1(i)

j	Formulas
4	$x1(5) = 32$ $x1(i) = (p_{i-2}-3).x1(i-1)$ $x2(4) = 28$ $x2(i) = (p_{i-1}-2).x2(i-1)+x1(i)$ $\#SP(4,3) = 4$ $\#SP(4,i) = (p_i-1).\#SP(4,i-1)+x2(i)$
5	$x1(6) = 18$ $x1(i) = (p_{i-3}-4).x1(i-1)$ $x2(5) = 46$ $x2(i) = (p_{i-2}-3).x2(i-1)+x1(i)$ $x3(4) = 20$ $x3(i) = (p_{i-1}-2).x3(i-1)+x2(i)$ $\#SP(5,3) = 2$ $\#SP(5,i) = (p_i-1).\#SP(5,i-1)+x3(i)$
6	$x1(6) = 54$ $x1(i) = (p_{i-3}-4).x1(i-1)$ $x2(5) = 58$ $x2(i) = (p_{i-2}-3).x2(i-1)+x1(i)$ $x3(4) = 10$ $x3(i) = (p_{i-1}-2).x3(i-1)+x2(i)$ $\#SP(6,3) = 0$ $\#SP(6,i) = (p_i-1).\#SP(6,i-1)+x3(i)$
7	$x1(8) = 576$ $x1(i) = (p_{i-4}-5).x1(i-1)$ $x2(7) = 1062$ $x2(i) = (p_{i-3}-4).x2(i-1)+x1(i)$ $x3(6) = 442$ $x3(i) = (p_{i-2}-3).x3(i-1)+x2(i)$ $x4(5) = 56$ $x4(i) = (p_{i-1}-2).x4(i-1)+x3(i)$ $\#SP(7,4) = 2$ $\#SP(7,i) = (p_i-1).\#SP(7,i-1)+x4(i)$
…	...

The reader will also be able to build the systems of recursive equations corresponding to the aggregations like "spacings $\Delta P \leq 2j$ " instead of above resolved "spacings $\Delta P \geq 2j$ ".

Having failed on the anticipation of the initial values in the previous paragraph, the purpose of this paragraph was to find some way this here. However, for these two types of aggregations, there seems to be not more success possibility than before.

5.2.4. Cradle of the multiplicative factors.

The reader will find underneath the wise course in order to find a proof for the existence of recursive relationships. Indeed, the multiplier factors observed in these linear relationships do appear at once when we carry out successive sortings based on modulo $\#p_i/p_k$ aggregations where p_k is the decreasing list of the prime dividers of the primordial #pi. The evidence sought is therefore intimately linked to the proper understanding of these sortings. Below we describe this method and the properties of the relevant objects.

Method of sorting.

Starting from the integers over an interval $[x_0, x_0+p_0p_1p_2\ldots p_i[$, $(x_0 > p_i)$, we remove all multiples from $p_0 = 2$ to p_i. The remaining numbers are in quantity $(p_1-1)(p_2-1)\ldots(p_i-1)$ and are sorted according to the increasing values of spacing (to the preceding ones).

The numbers x of spacing 2 are then sorted according to the increasing values of x modulo $p_0p_1p_2\ldots p_i/p_i$. They appear in families with p_i-2 identical modulo values and are all 1 modulo 6 valued. The total amount of elements responds to a system of one recursive equation. For spacing 4, the routine is then analogous except that the elements are all 5 mod 6 valued. For these first two groups of families of cardinal p_i-2-0, the proof is that of the theorem 12 (and of the preliminary theorem 4).

The numbers x of spacing 6 = 4+2 are then sorted according to the increasing value of x modulo $p_0p_1p_2\ldots p_i/p_i$. Those that appear in families with p_i-2 identical modulo values are grouped apart. The others appear modulo $p_0p_1p_2\ldots p_i/p_{i-1}$ in families with p_{i-1}-2-1 identical modulo values and are grouped on their side. The set responds to a system of two recursive equations.

…

The numbers x of spacing 4+2.(j-2) are then sorted according to the increasing value of x modulo $p_0p_1p_2\ldots p_i/p_i$. Families with p_i-2 identical modulo values that appear are grouped apart when they exist. We then proceed with the same way modulo $p_0p_1p_2\ldots p_i/p_{i-k}$, k being gradually incremented while making groups of numbers showing p_{i-k}-2-k identical modulo values to the k-1 sequence.

We do this until the stock runs out. The number of sorting, at a given spacing, cannot exceed i. The resulting recursive system cannot have more than i equations.

Origin of the multiplicative ratio

The p_{i-k}-2-k identical modulo-values at the k+1 sequence answer to the following count. We operate modulo $p_0p_1p_2\ldots p_i/p_{i-k}$. In an interval of size $p_0p_1p_2\ldots p_i$, we initially come up exactly with p_{i-k} integers. For these trivially, being remotely equidistant, there are exactly 1 integer x that is multiple of p_{i-k} and 1 other among $x-2j+r.p_0p_1p_2\ldots p_i/p_{i-k}$, r = 0 to p_{i-k}-1, which is also multiple p_{i-k}, and this regardless of the value of j. This is trivial in contrast to the following feature : The elimination of the additional k integers is due to exactly 1 elimination for the sorted modulo $p_0p_1p_2\ldots p_i/p_i$ series, 1 elimination for the sorted modulo $p_0p_1p_2\ldots p_i/p_{i-1}$ series, …, 1 elimination for the sorted modulo $p_0p_1p_2\ldots p_i/p_{i-(k-1)}$ series, these k cases being all to be found in the 2j-spacing set of numbers (see examples below).

Symmetry property

In an interval $[x_0, x_0+p_0p_1p_2\ldots p_i[$, $x_0 > p_i$, subject to Eratosthenes sieve, there will remain, with the provision of an offset, the same quantities of integers as in the interval $]-p_0p_1p_2\ldots p_i/p_0, +p_0p_1p_2\ldots p_i/p_0[$ subject to the same algorithm provided you also remove p_0, p_1, p_2, …, p_i (and $-p_0$, $-p_1$, $-p_2$, …, $-p_i$). The result of the latter after sieving being perfectly symmetrical, there i therefore in the initial interval also a symmetry modulo $p_0p_1p_2\ldots p_i$ for an axis to be determined.
We will thus systematically find for any configuration, a concept that we will define below, a symmetrical configuration, unless it is its own symmetrical.

It should also be noted that the count properties observed for the part of the integers beyond p_i when running the Eratosthenes algorithm are the same as if one studies these numbers in an interval beginning at 0, provided that 2, 3, 5, 7, 11, ... p_i are removed too.

Supplementary remarks.

First of all, the conjecture is clear for in steps i = 1 to 9.

Using sufficient initial conditions, any population can be analysed in the form of recursive formulas, since an adjustment of one unit on the lower diagonal (table 10) changes each of the values vertically from the same unit exactly. The point here is to show that a finite number of initial values will suffice for the asymptotic assessment for a given 2j-spacing and that the multiplier factors are then appropriate.

Below, we give the population #ΔP evaluation as it stands for the spacing of ΔP = 14 for steps i = 1 to 9.

p_i	3	5	7	11	13	17	19	23	29
Line 1	0	0	0	2	58	1406	33206	871318	27403082
Line 2		0	0	2	36	536	9304	173992	3877496
Line 3			0	2	20	176	1800	25128	397656
Line 4				2	14	36	216	1728	20736
Line 5					14	8	0	0	0

As for calculation purposes, recursive formulas work perfectly provide the correct adjustment of the lower diagonal. However, it would be irrelevant to seek meaning in the numbers displayed when the multiplier factor of a line becomes negative as in line 5 for p_i = 17. Appropriate explanations are only to be sought up to line 4 and starting with non-zero population.

Numeric examples.

The underneath numerical examples are intended to give a clearer understanding of the sequences of integers that give rise to previous arguments.

The spacings are taken, as agreed in this article, between the number displayed and its previous one respecting the spacing ΔP. For example, for m = 11 some integer effectively inscribed in the table, the associated integer for ΔP = 4 will be 7 (not 15).
We start from step i = 1 using the sorting method.

Step 1 : $p_0p_1 = 6$.

We initially choose the interval [11, 17[, but any other interval modulo 6 of initial abscissa greater than $p_1 = 3$ could be chosen.

Table 17

Spacings ΔP	#ΔP	List of integers	Properties
2	1	13	1 mod 6/3
4	1	11	1 mod 6/3

This initiates Table 5.
Note here the way how to set the modulo condition in the form $p_0p_1\ldots p_i/p_i$.
Besides p_i-2 = 1 is indeed the cardinal of the elements for spacing 2 and 4 respectively.

Step 2 : $p_0p_1p_2 = 30$.

Table 18

Families	Spacings ΔP	#ΔP	List of integers	Properties	Configurations d = 30/5
1	2	3	13, 19, 31, (13)	13 mod 30/5 (1 mod 6)	d, 2d, 2d
2	4	3	11, 17, 23, (11)	11 mod 30/5 (5 mod 6)	d, d, 3d
3	6	2	37 29		

The numbers 13 and 11, of the preceding step, are generators of the families 1 and 2 through the property 1 mod 30/5 for the first one and 5 mod 30/5 for the second one. We have demonstrated, using the arguments of depletion developed in pages 8 and 44 and illustrated by Tables 3 and 27, that by going from stage i-1 to stage i, there are p_i candidates in which only p_i-2 are suitable and indeed here for family 1, only 25 and 37 are not suitable (the first to be multiple of 5, the second as 37-2 is multiple of 5) and the same for the family 2, where 29 and 35 are excluded (the first as 29-2 is multiple of 3 and the second to be multiple of 5).

Family 3 "recovers" the previously excluded numbers 29 and 37 (but not 25 and 35 that are multiple of a divider of 30). This is then, for these numbers, their final position because 29-6 and 37-6 are not multiple of a divider of 30. We do the p_{i-1}-2-1 count getting so value 0. Thus the elements of family 3 are neither in p_i-2 quantities nor in p_{i-1}-2-1 quantities. In some way, we can say that they are "self-generating" being not subject to any particular multiplier factor (which is a somewhat exaggerated word, since in fact they are only outside the previous classifications). So there is no possibility to attach them a property x modulo $p_0p_1p_2\ldots p_i/p_i$ or x modulo $p_0p_1p_2\ldots p_i/p_{i-1}$ (hence the empty box).

Here and later, we will call the spacing arrangement a "configuration". Any circular permutation of the spacings is the same configuration.
The configurations here are symmetrical to themselves, i.e. the symmetrical of {6, 12, 12} is {12, 12, 6}, the latter being identical by circular permutation to {6, 12, 12}.

Step 3 : $p_0p_1p_2p_3 = 210$.

Table 19

Spacings ΔP	#ΔP	List of integers	Properties	Configurations (d_1 = 210/7 = 30, d_2 = 210/5 = 42)	Miscellaneous d_3 = 210/3 = 70
2	15	13, 43, 73, 103, 193, (13) 139, 169, 199, 19, 109, (139) 181, 211, 31, 61, 151, (181)	13 mod 210/7 19 mod 210/7 31 (or 1) mod 210/7	$d_1, d_1, d_1, 3d_1, d_1$	139-13 = $3d_2$ 181-139 = d_2 210+13-181 = d_2
4	15	17, 47, 107, 167, 197, (17) 143, 173, 23, 83, 113, (143) 101, 131, 191, 41, 71, (101)	17 mod 210/7 23 mod 210/7 11 mod 210/7	$d_1, 2d_1, 2d_1, d_1, d_1$	
6	14	37, 67, 127, 157, 187, (37) 149, 179, 29, 59, 89, (149) 53, 137, (53) 79, 163, (79)	37 (or 7) mod 210/7 29 mod 210/7 11 mod 210/5 37 mod 210/5	$d_1, 2d_1, d_1, d_1, 2d_1$ $2d_2, 3d_2$	149-37 = 112 = d_2+d_3

Spacings ΔP	$\#\Delta P$	List of integers	Properties	Configurations ($d_1 = 210/7 = 30$, $d_2 = 210/5 = 42$)	Miscellaneous $d_3 = 210/3 = 70$
8	2	97 121			
10	2	209 11			

We have p_i-2 = 7-2 = 5, p_{i-1}-2-1 = 5-3 = 2 and p_{i-3}-2-3 is negative. Only searches according to the properties x modulo $p_0p_1p_2\ldots p_i/p_i$ = 210/7 and x modulo $p_0p_1p_2\ldots p_i/p_{i-1}$ = 210/5 with groupings by 5 and 2 therefore make sense.

In the "properties" column at the top of Table 19, we collect the list of integers in Table 18. We find them on the previous spacing lines, filling them entirely for spacings 2 and 4 and partially for spacing 6, the largest spacing in the previous step. They are developed thanks to the property modulo $p_0p_1\ldots p_i/p_i$ in the new lists. Let us recall again here that it is established that any number present at the i-1 step generates p_i-2 numbers at step i in the same spacing line.

The numbers $\#\Delta P$ for ΔP = 2 and ΔP = 4 are given by a one-equation recursive system. Each number in the previous table (Table 18) is found in the upper lines of the property column and generates a batch of 7-2 = 5 numbers. The multiplier factor is equal to p_i-2 as expected. For example, in the series 13, 43, 73, 103, 133, 163 and 193 (numbers between 11 and 2.3.5.7+10), all of which are integers valued 13 mod 210/7, one and only one integer has as a divider a divisor of 210, namely 133 (divisible by 7) and in the series 13-2, 43-2, 73-2, 103-2, 133-2, 193-2, only one has as a divider a divisor of 210, i.e. 161 (divisible by 7), thus two exclusions. These two are reassigned to the lines below. Note that the generator in the property column does not necessarily re-enter itself the final list.

The regularity of the spacings is besides well respected (identical configurations).
This is not surprising, since considering the number x1 having as a divider d (= p_i) in the first sample (here 133 divisible by 7) and the corresponding number x2 in the second sample with divider d (here 49 still divisible by 7), we are searching then the number y1 such that y1-Δ has a divider d (here Δ = 2). Then (y1+(x2-x1)- Δ) is trivially a divider of t. This gives the relative positions of the two associated pairs and the corresponding regular spacings.

The population $\#\Delta P$ for ΔP = 6 is determined by a two-equation recursive system.
The same rule applies for a part of the solution numbers, a proportion that is perfectly identified in the population values given by the following two recursive equations, which are derived from the general formula where only the initial values are to be constituted by numerical approach:

p_i	3	5	7
Line 1	0	2	14
Line 2		2	4

The cumulative is 14, of which 10 are generated by two numbers (37 (or rather 7) mod 30 and 29 mod 30) in line 1 on the one hand and 4 generated by two numbers in line 2 on the other hand.

The rest of them originate from previous modulo 210/7 rejects. Modulo $p_0p_1p_2\ldots p_i/p_{i-1}$ (here 42), the two generators turn out to be 11 and 37 (or rather 53 mod 42 and 79 mod 42 looking for the head of list of integers among the modulo 210/7 rejects). With distances $p_0p_1p_2\ldots p_i/p_{i-1}$ within a set of size $p_0p_1p_2\ldots p_i$, we generally have p_{i-1} integers to look at initially. In the following table, we report these p_{i-1} numbers (here p_{i-1} = 5) on which we proceed with two types of elimination :

m	11	53	95	137	179
m-6	5	47	89	131	173
Elimination if divider of 210	yes (bas) (p_{i-1} = 5)		yes (top) (p_{i-1} = 5)		
Elimination if previously listed					yes (179 = 29 mod 210/7)

Similarly, in the following table for 37 (by making a circular permutation of the m-values to better compare configurations) :

m	121	163	205	37	79
m-6	115	157	199	31	73
Elimination if divider of 210	yes (bottom) (p_{i-1} = 5)		yes (top) (p_{i-1} = 5)		
Elimination if previously listed				yes (37 = 7 mod 210/7)	

Thus 2 initial values correspond to 2 configurations represented by the two tables. This gives a multiplier factor of p_{i-1}-2-1 = 2 here, the first two under the "standard" elimination of two units (since 6 does not contain the divider 5) and the last by the fact that in an interval of size 210 one has already been listed in a spaced list modulo 210/7.

Having only two configurations in total and knowing that there is a symmetrical to any type of positioning modulo $p_0p_1p_2…p_i$, we do check this point here. The relative spacings between the first type of eliminations are the same (value 84 mod 210) and the symmetry axis is the middle of both eliminations. For the elimination of the second type, the integer 179 can be seen as being contiguous to the left of 11 (distance -42) in the first table, while 37 is well contiguous to the right of 205 (distance of 42) in the second table.

The passage of 6-spacing solutions for the part modulo $p_0p_1p_2…p_i/p_{i-1}$ = 210/5 from step 2 to step 3 is given below. It is made modulo $(p_0p_1p_2…p_i/p_i)/p_{i-1}$ = (210/7)/5 = 6 (which is not particularly noteworthy for generalization as long as i is small) :

Values at step 2	mod 6		Values at step 3	mod 6
29	5	→	53, 137	5
37	1	→	79, 163	1

The other two pairs of numbers (97, 121) and (209, 221) find their place with spacings 8 and 10 respectively. They self-generate, overusing these term, as they are not subject to any particular multiplying factor. Indeed, if we evaluate p_{i-2}-2-2 at this stage, we get -1 which does not correspond to a possible property x modulo $p_0p_1p_2…p_i/p_{i-2}$ (hence the empty box for both spacings).

Step 4 : $p_0p_1p_2p_3p_4$ = 2310.

Table 20

Spacings ΔP	#ΔP	List of integers	Properties	Miscellaneous d_3 = 2310/5 = 462
2	135	223, 433, 643, 853, 1063, 1273, 1483, 1693, 2113,… 1609, 1819, 2029, 2239, 139, 349, 559, 769, 1189, … 2071, 2281, 181, 391, 601, 811, 1021, 1231, 1651, …	13, 43, 73, 103, 193 19, 109, 139, 169, 199 31, 61, 151, 181, 211 (or 1) mod 2310/11	1609-223 = $3d_3$ 2071-1609 = d_3
4	135	… … …	17, 47, 107, 167, 197, 23, 83, 113, 143, 173, 41, 71, 101, 191, 131 mod 2310/11	
6	142	239, 449, 659, 1079, 1289, 1499, 1709, 1919, 29,… 877, 1087, 1297, 1717, 1927, 2137, 37, 247, 667, 1537, 1747, 1957, 67, 277, 487, 697, 907, 1327, 547, 757, 967, 1387, 1597, 1807, 2017, 2227, 337… 767, 977, 1187, 1607, 1817, 2027, 2237, 137, 557 2153, 53, 263, 683, 893, 1103, 1313, 1523, 1943 1339, 1549, 1759, 2179, 79, 289, 499, 709, 1129 2263, 163, 373, 793, 1003, 1213, 1423, 1633, 2053 673, 1333, 1663, 1993 2059, 409, 739, 1069, 257, 1247, 1577, 1907 1643, 323, 653, 983	29, 59, 89, 149, 179 37, 67, $127_{gén}$, 157, 187 137 53 79 163 mod 2310/11 13 mod 2310/7 79 mod 2310/7 257 mod 2310/7 323 mod 2310/7	 2153-767 = $3d_3$ 2263-1339 = $2d_3$ 2059-673 = $3d_3$ 1643-257 = $3d_3$

Spacings ΔP	#ΔP	List of integers	Properties	Miscellaneous $d_3 = 2310/5 = 462$
8	28	97, 307, 727, 937, 1147, 1357, 1567, 1777, 1987 2011, 2221, 331, 541, 751, 961, 1171, 1381, 1591 457, 787,1117, 1447 871, 1201, 1531, 1861 919, 1399	97 mod 2310/11 121 mod 2310/11 127 mod 2310/7 211 mod 2310/7 457 et 13 mod 2310/5	
10	30	… … 149, 809,1139, 2129 1511, 2171, 191, 1181 587, 1973 347, 1733	11 mod 2310/11 209 mod 2310/11 149 mod 2310/7 191 mod 2310/7 125 mod 2310/5 347 mod 2310/5	
12	8	211, 2111, 13, 479, 521, 1801, 1843, 2309		
14	2	127, 2197		

We have p_i-2 = 11-2 = 9, p_{i-1}-2-1 = 7-3 = 4, p_{i-2}-2-2 = 5-2 2 = 1 and p_{i-3}-2-3 is negative. Only searches according to properties x modulo $p_0p_1p_2…p_i/p_i$ = 2310/11, x modulo $p_0p_1p_2…p_i/p_{i-1}$ = 2310/7 and x modulo $p_0p_1p_2…p_i/p_{i-2}$ = 2310/5 with groupings by 9, 4 and 1 therefore make sense.

The populations #ΔP for ΔP = 2 and ΔP = 4 are given by a one-equation recursive system with the multiplier factor equal to p_i-2 = 9.
The numbers generated by 13 are :

List of integers	Configurations $d_1 = 2310/11$
223, 433, 643, 853, 1063, 1273, 1483, 1693, (223)	d_1, d_1, d_1, d_1, d_1, d_1, d_1, $2d_1$, $2d_1$

The configuration is the same for all series of numbers generated by the list of 2-spacing.

Note the same configurations identity for the list corresponding to spacing 4 (with a different configuration from the previous one) and the same for spacing 6 for the part corresponding to the elements issued from property "modulo 2310/11", the later spacing being studying underneath.

The populations #ΔP for ΔP = 6 are determined by a two-equation recursive system based on the table below :

p_i	3	5	7	11
Line 1	0	2	14	142
Line 2		2	4	16

We have 142-16 = 126 = (11-2).14 standard solutions modulo 2310/11.

The configuration of the first list is as follows and extends in the standard way to the other elements of the table.

List of numbers	Configuration d = 210
239, 449, 659, 1079, 1289, 1499, 1709, 1919, 29, (239)	d, d, 2d, d, d, d, d, 2d, d

After ordering according to the pertinent configuration, the head terms are :

List of head numbers	Miscellaneous d = 2310/2/3/5
239, 569, 899, 1229, 1559 547, 877, 1207, 1537, 1867	537-239 = 877-569 = … = 4d

Spacing between these numbers is $p_0p_1p_2…p_i/p_i/p_{i-1}$, distance that spreads to all relevant lists.

Below we give their spacings for the numbers that follow the previous list. The configuration remains the same.

List of numbers	Configuration d = 210
767, 977, 1187, 1607, 1817, 2027, 2237, 137, 557, (767) 2153, 53, 263, 683, 893, 1103, 1313, 1523, 1943, (2153) 1339, 1549, 1759, 2179, 79, 289, 499, 709, 1129, (1339) 2263, 163, 373, 793, 1003, 1213, 1423, 1633, 2053, (2263)	d, d, 2d, d, d, d, d, 2d, d

The 16 remaining values are obtained modulo $p_0p_1p_2\ldots p_i/p_{i-1}$ (hence here modulo 2310/7 = 330). The four generators turn out then to be 13, 79, 257 and 323. In the following tables, we re-enact two types of elimination (always by making a circular swap of the m values to better compare configurations) :

m	13	343	673	1003	1333	1663	1993
m-6	7	337	667	997	1327	1657	1987
Elimination if divider of 2310	yes (bottom) (p_{i-1} = 7)	yes (top) (p_{i-1} = 7)					
Elimination if previously listed				yes (1003 = 163 mod 210)			

m	1399	1729	2059	79	409	739	1069
m-6	1393	1723	2053	73	403	733	1063
Elimination if divider of 2310	yes (bottom) (p_{i-1} = 7)	yes (top) (p_{i-1} = 7)					
Elimination if previously listed				yes (79 = 79 mod 210)			

m	257	587	917	1247	1577	1907	2237
m-6	251	581	911	1241	1571	1901	2231
Elimination if divider of 2310		yes (bottom) (p_{i-1} = 7)	yes (top) (p_{i-1} = 7)				
Elimination if previously listed							yes (2237 = 137 mod 210)

m	1643	1973	2303	323	653	983	1313
m-6	1637	1967	2297	317	647	977	1307
Elimination if divider of 2310		yes (bottom) (p_{i-1} = 7)	yes (top) (p_{i-1} = 7)				
Elimination if previously listed							yes (1313 = 53 mod 210)

The multiplier factor is here, as conjectured, p_{i-1}-2-1 = 4.

The passage of 6-spacing solutions for the part modulo $p_0p_1p_2\ldots p_i/p_{i-1}$ = 2310/7 from step 3 to step 4 is discussed below. It is implemented modulo $(p_0p_1p_2\ldots p_i/p_i)/p_{i-1}$ = (2310/11)/7 = 30 :

Values at step 3	mod 30		Values at step 4	mod 30	Configurations d = 2310/7
163	13	→	673, 1333, 1663, 1993	13	2d, d, d, 3d
79	19	→	2059, 409, 739, 1069,	19	2d, d, d, 3d
137	17	→	257, 1247, 1577, 1907	17	3d, d, d, 2d
53	23	→	1643, 323, 653, 983,	23	3d, d, d, 2d

Each configuration has its symmetrical. We have only encountered one configuration so far because only two spacings values were present, thus the symmetrical merges with the original, as illustrated in the example below :

	Configurations
Original	x, x, y, x, y, x, x, x, x
Symmetric	x, x, x, x, y, x, y, x, x
Shifting of 2 units of the symmetric	x, x, y, x, y, x, x, x, x

But this pattern is no longer applicable here.

The populations #ΔP for ΔP = 8 are determined by a three-equation recursive system based on the table below :

p_i	3	5	7	11
Line 1	0	0	2	28
Line 2		0	2	10
Line 3			2	2

We have 2.(11-2) front-line solutions (97 and 121 mod 2310/11), 2.(7-2-1) second-line solutions (127 and 221 mod 2310/7) and two other solutions complete the count (919 and 1399) as initial values.

The populations #ΔP for ΔP = 10 are determined by a four-equations recursive system according to the table below :

p_i	3	5	7	11
Line 1	0	0	2	30
Line 2		0	2	12
Line 3			2	4
Line 4				2

We will come back later on how to determine these populations and additional ones (for spacings ΔP = 12 and ΔP = 14), the next step being more expressive and richer by the amount of data available.

Step 5 : $p_0p_1p_2p_3p_4p_5$ = 30030.

Table 21

Spacings ΔP	#ΔP	List of integers	Properties
2	1485	…	List at step 4 mod 30030/13
4	1485	…	List at step 4 mod 30030/13
6	1690	… … … … …	List at step 4 mod 30030/13 17, 407, 1577, 2357, 173, 563, 953, 2123, 379, 1159, 2329, 2719, 613, 1783, 2173, 2563 mod 30030/11
8	394	… … … 409, 21859, 26149 8179, 29629, 3889	List at step 4 mod 30030/13 487, 691, 769, 877, 1081, 1657, 1861, 1969, 2047, 2251 mod 30030/11 409 mod 30030/7 3889 mod 30030/7
10	438	… … … … 4307, 8597, 21467 10313, 14603, 27473 15437, 19727, 2567 8573, 21443, 25733	List at step 4 mod 30030/13 149, 251, 797, 929 1031, 1343, 1397, 1709 1811, 1943, 2489, 2591 mod 30030/11 17 1733 2567 4283 mod 30030/7

Spacings ΔP	#ΔP	List of integers	Properties
12	188	… … … 26281, 13411, 21991 9899, 27059, 5609 22921, 10051, 18631 2129, 19289, 27869 10753, 27913, 2173 19991, 7121, 11411 2983, 20143, 24433 16631, 3761, 8051 1399, 1973, 6023, 8989 12889, 13283, 16759, 17153 21053, 24019, 28069, 28643	List at step 4 mod 30030/13 223, 541, 1271, 1319 1423, 1471, 2201, 2519 mod 30030/11 541 1319 1471 2129 2173 2831 2983 3761 mod 30030/7
14	58	… … … 877, 18037, 22327, 26617 12007, 29167, 3427, 7717 2521, 6421, 6931, 12343 14191, 14341, 15703, 15853 17701, 23113, 23623, 27523	List at step 4 mod 30030/13 307 2437 mod 30030/11 877 3427 mod 30030/7
16	12	17, 11147, 7277, 15047, 22637, 25997 4049, 7409, 14999 22769, 18899, 30029	
18	8	2201, 16691, 20921, 24281 5767, 9127, 13357, 27847	
20	0		
22	2	9461, 20591	

We have p_i-2 = 13-2 = 11, p_{i-1}-2-1 = 11-3 = 8, p_{i-2}-2-2 = 7-2 2 = 3 and p_{i-3}-2-3 = 0. Only searches according to properties x modulo $p_0p_1p_2\ldots p_i/p_i$ = 30030/13, x modulo $p_0p_1p_2\ldots p_i/p_{i-1}$ = 30030/11 and x modulo $p_0p_1p_2\ldots p_i/p_{i-2}$ = 30030/7 with groupings by 11, 8 and 3 therefore make sense.

The numbers #ΔP for ΔP = 2 and ΔP = 4 are given by a one-equation recursive system with the multiplier factor equal to p_i-2 = 11.

The numbers #ΔP for ΔP = 6 are determined by a two-equation recursive system based on the table below :

p_i	3	5	7	11	13
Line 1	0	2	14	142	1690
Line 2		2	4	16	128

We have 1690-128 = 1562 = 11.142 standard solutions modulo 30030/13. The other 128 are obtained modulo $p_0p_1p_2\ldots p_i/p_{i-1}$ (here modulo 30030/11 = 2730). The sixteen generators turn out then to be 17, 407, 1577, 2357, (all 17 mod 390), 173, 563, 953, 2123, (all 173 mod 390), 379, 1159, 2329, 2719, (all 379 mod 390), 613, 1783, 2173 and 2563 (all 223 mod 390).
In the following table, we review each of the 4 series. We still have two types of eliminations :

m	2747	5477	8207	10937	13667	16397	19127	21857	24587	27317	17
m-6	2741	5471	8201	10931	13661	16391	19121	21851	24581	27311	11
Elimination if divider of 30030								yes (top) (p_{i-1}=11)			yes (bottom) (p_{i-1}=11)
Elimination if previously listed										yes (27317 = 1907 mod 2310)	

m	173	2903	5633	8363	11093	13823	16553	19283	22013	24743	27473
m-6	167	2897	5627	8357	11087	13817	16547	19277	22007	24737	27467
Elimination if divider of 30030								yes (top) (p_{i-1}=11)			yes (bottom) (p_{i-1}=11)
Elimination if previously listed										yes (24743 = 1643 mod 2310)	

m	19489	22219	24949	27679	379	3109	5839	8569	11299	14029	16759
m-6	19483	22213	24943	27673	373	3103	5833	8563	11293	14023	16753
Elimination if divider of 30030								yes (top) (p_{i-1}=11)			yes (bottom) (p_{i-1}=11)
Elimination if previously listed									yes (11299 = 2059 mod 2310)		

m	613	3343	6073	8803	11533	14263	16993	19723	22453	25183	27913
m-6	607	3337	6067	8797	11527	14257	16987	19717	22447	25177	27907
Elimination if divider of 30030								yes (top) (p_{i-1}=11)			yes (bottom) (p_{i-1}=11)
Elimination if previously listed									yes (22453 = 1663 mod 2310)		

The multiplier factor here is p_{i-1}-2-1 = 8.
The reader attentive to the configurations of the relative positions of the eliminations, numbering 2, that of 17, 173, 407, 563, 953, 1577, 2123, 2357 (all 17 modulo 78) on one hand and that of 379, 613, 1159, 1783, 2173, 2329, 2563, 2719 (all 67 modulo 78) on the other hand, will also be able to see such circumstances at the previous step. These two configurations are again symmetrical to each other.
For the elimination by divider, the first divisible number (by p_{i-1} = 11) is systematically at the top (line m), the second systematically at the bottom (line m-6) for this example. For elimination by membership of a previously listed family, it is found within the same spacing line (here ΔP = 6).

m	Divisible par 11	…	…	
m-6	…	…	…	Divisible par 11
Elimination by divisor	Yes (p_{i-1} = 11)			Yes (p_{i-1} = 11)
Elimination by default		Yes or No	No or Yes	

The numbers #ΔP for ΔP = 8 are determined by a three-equation recursive system based on the table below :

p_i	3	5	7	11	13
Line 1	0	0	2	28	394
Line 2		0	2	10	86
Line 3			2	2	6

We have 394-86 = 308 = (13-2).28 standard solutions modulo 30030/13. We have 86-6 = 80 = (11-2-1).10 solutions modulo $p_0p_1p_2…p_i/p_{i-1}$ (here modulo 30030/11 = 2730). The ten generators are 487, 877, 1657, 1969, 2047 (all 19 modulo 78) and 691, 769, 1081, 1861, 2251 (all 67 modulo 78).
In the following table, we show the two symmetrical configurations that appear :

m	487	3217	5947	8677	11407	14137	16867	19597	22327	25057	27787
m-8	479	3209	5939	8669	11399	14129	16859	19589	22319	25049	27779
Elimination if divider of 30030					yes (top) (p_{i-1}=11)				yes (bottom) (p_{i-1}=11)		
Elimination if previously listed								yes (19597 = 1117 mod 2310)			

m	22531	25261	27991	691	3421	6151	8881	11611	14341	17071	19801
m-8	22523	25253	27983	683	3413	6143	8873	11603	14333	17063	19793
Elimination if divider of 30030					yes (top) (p_{i-1}=11)				yes (bottom) (p_{i-1}=11)		
Elimination if previously listed						Yes (6151 = 1151 mod 2310)					

The multiplier factor is equal to p_{i-1}-2-1 = 8. Again, the second type of elimination corresponds to a family with the same spacing (so ΔP = 8 here).

There remains 6 solutions of line 3 governed by a relationship modulo $p_0p_1p_2\ldots p_i/p_{i-2}$ (hence here modulo 30030/7 = 4290):

m	409	4699	8989	13279	17569	21859	26149
m-8	401	4691	8981	13271	17561	21851	26141
Elimination if divider of 30030			yes (bottom) (p_{i-2}=7)	yes (top) (p_{i-2}=7)			
Elimination if previously listed		yes (4699 = 1969 mod 2730)			yes (17569 = 1399 mod 2310)		

m	8179	12469	16759	21049	25339	29629	3889
m-8	8171	12461	16751	21041	25331	29621	3881
Elimination if divider of 30030			yes (bottom) (p_{i-2}=7)	yes (top) (p_{i-2}=7)			
Elimination if previously listed		yes (12469 = 919 mod 2310)			yes (25339 = 769 mod 2730)		

The multiplier factor is equal to p_{i-2}-2-2 = 3. These tables are the first showing two eliminations of the second type (previously listed). In this case, one comes from belonging to a family modulo $p_0p_1p_2\ldots p_i/p_i$ and the other to a family modulo $p_0p_1p_2\ldots p_i/p_{i-1}$ (and with ΔP = 8).

The passage of previous solutions of spacing 8 for the part modulo $p_0p_1p_2\ldots p_i/p_{i-2}$ = 30030/7 from step 4 to step 5 is discussed below. It is implemented modulo $(p_0p_1p_2\ldots p_i/p_i)/p_{i-2}$ = (30030/13)/7 = 330 (d = $(p_0p_1p_2\ldots p_i)/p_{i-2}$ = 4290) :

Values at step 4	mod 330		Values at step 5	mod 330	Configurations
1399	79	→	409, 21859, 26149, (409)	79	5d, d, d
919	259	→	8179, 29629, 3889, (8179)	259	5d, d, d

The spacing between integers is the same because the two types of configurations are symmetrical to each other.

The populations #ΔP for ΔP = 10 are determined by a four-equations recursive system according to the table below :

p_i	3	5	7	11	13
Line 1	0	0	2	30	438
Line 2		0	2	12	108
Line 3			2	4	12
Line 4				2	0

The multiplier factor of the last line being p_{i-3}-2-3 = 0 (for i = 5), the last line does not give any contribution to the one above. We only take up an explanation here for the third line.

The 12 solutions of line 3 are governed by a relationship modulo $p_0p_1p_2\ldots p_i/p_{i-2}$ (here modulo 30030/7 = 4290) and are generated according to 4 initial configurations and multiplier factor p_{i-2}-2-2 = 3 :

m	17	4307	8597	12887	17177	21467	25757
m-10	7	4297	8587	12877	17167	21457	25747
Elimination if divider of 30030	yes (bottom) (p_{i-2}=7)			yes (top) (p_{i-2}=7)			
Elimination if previously listed					yes (17177 = 797 mod 2730)		yes (25757 = 347 mod 2310)

m	6023	10313	14603	18893	23183	27473	1733
m-10	6013	10303	14593	18883	23173	27463	1723
Elimination if divider of 30030	yes (bottom) (p_{i-2}=7)			yes (top) (p_{i-2}=7)			
Elimination if previously listed					yes (23183 = 1343 mod 2730)		yes (1733 = 1733 mod 2310)

m	11147	15437	19727	24017	28307	2567	6857
m-10	11137	15427	19717	24007	28297	2557	6847
Elimination if divider of 30030	yes (bottom) (p_{i-2}=7)			yes (top) (p_{i-2}=7)			
Elimination if previously listed					yes (28307 = 587 mod 2310)		yes (6857 = 1397 mod 2730)

m	17153	21443	25733	30023	4283	8573	12863
m-10	17143	21433	25723	30013	4273	8563	12853
Elimination if divider of 30030	yes (bottom) (p_{i-2}=7)			yes (top) (p_{i-2}=7)			
Elimination if previously listed					yes (4283 = 1973 mod 2310)		yes (12863 = 1943 mod 2730)

Again, the two eliminations of the second type originate from the belonging to a family modulo $p_0p_1p_2\ldots p_i/p_i$ and the other to a family modulo $p_0p_1p_2\ldots p_i/p_{i-1}$ (and $\Delta P = 10$).

The passage of previous solutions of spacing 10 for the part modulo $p_0p_1p_2\ldots p_i/p_{i-2} = 30030/7$ from step 4 to step 5 is implemented modulo $(p_0p_1p_2\ldots p_i/p_i)/p_{i-2} = (30030/13)/7 = 330$:

Values at step 4	mod 330		Values at step 5	mod 330	Configurations
347	17	→	4307, 8597, 21467, (4307)	17	1
587	257	→	15437, 19727, 2567, (15437)	257	S1
1733	83	→	10313, 14603, 27473, (10313)	83	2
1973	323	→	21443, 25733, 8573, (21443)	323	S2

Regarding configurations, the ranking according to the unit digits is accidental and quite anecdotal. In fact, this ranking is done by taking into account the spacings between integers and responds to the following table (d = 4290 = 30030/7) and shows that all configurations merge because the symmetrical spacings are the same as the initial spacings:

Configurations	Spacings
1 et 2	d, 3d, 3d
S1 et S2	d, 3d, 3d

The populations $\#\Delta P$ for $\Delta P = 12$ are determined by a five-equation recursive system based on the table below :

p_i	3	5	7	11	13
Line 1	0	0	0	8	188
Line 2		0	0	8	100
Line 3			0	8	36
Line 4				8	12
Line 5					12

For the generation of the first three lines, the previous examples are sufficient. We have also seen that the fourth line is governed by a multiplier factor 0 in $p_i = 13$ without proper contribution. The last 12 numbers self-generate (and indeed modulo $p_0p_1p_2\ldots p_i/p_{i-j}$, j = 1 to 3, no relevant grouping does appear).

The numbers $\#\Delta P$ for $\Delta P = 14$ are determined by a five-equation recursive system based on the table below :

p_i	3	5	7	11	13
Line 1	0	0	0	2	58
Line 2		0	0	2	36
Line 3			0	2	20
Line 4				2	14
Line 5					14

The first two lines respond to the standard assessment operating modulo 30030/13 for elements corresponding to line 1 and modulo 30030/11 for the elements of line 2.
For the third-line generation, there are 20-14 = 6 solutions initialized by 2 configurations. However, we meet four solutions for each of them 877, 18037, 22327, 26617 and 3427, 7717, 12007, 29167 giving respectively 877 mod 30030/7 and 3427 mod 30030/7. This does not call into question the sorting method, because there are at least the three expected solutions, but we do not know here which of the two integers it is appropriate to add in the batch of 14 solutions of line 5 (line 4 contributing for none).
If alternatively, we choose to solve using modulo 30030/13 for the first line, and then modulo 30030/13/11 for the second, the sorting leads to the same sets. Proceeding modulo 30030/13/11/7 for the remaining 20 integers, we get the following results :

Numbers corresponding to lines 3 to 5	mod 30
2521, 6421, 6931, 14191, 14341, 17701	1
27523, 12343, 15703, 15853, 23113, 23623	13
877, 3427, 7717, 12007, 18037, 22327, 26617, 29167	7

This time, we can distinguish 6+6+8 = 6+14 integers. Of course, we can also imagine other combinations for these totals. But what is of interesting to us here is simply to find some form of consistency with respect to relevant populations.

For the population #ΔP including ΔP >14, there is no specific classification to consider at this stage.

Step 6 : $p_0p_1p_2p_3p_4p_5p_6$ = 510510.

We have p_i-2 = 17-2 = 15, p_{i-1}-2-1 = 13-3 = 10, p_{i-2}-2-2 = 11-4 = 7, p_{i-3}-2-3 = 7-5 = 2 and p_{i-4}-2-4 is negative. Only searches according to properties x modulo $p_0p_1p_2\ldots p_i/p_i$ = 510510/17, x modulo $p_0p_1p_2\ldots p_i/p_{i-1}$ = 510510/13, x modulo $p_0p_1p_2\ldots p_i/p_{i-2}$ = 510510/11 and x modulo $p_0p_1p_2\ldots p_i/p_{i-3}$ = 510510/7 with groupings by 15, 10, 7 and 2 therefore make sense.

We present this case only partially, the aim being only to confirm the concepts already exposed, limiting ourselves to the spacing ΔP = 12 and the groupings x modulo $p_0p_1p_2\ldots p_i/p_{i-3}$ = 510510/7 and x modulo $p_0p_1p_2\ldots p_i/p_{i-2}$ = 510510/11.

Table 22

Spacings ΔP	#ΔP	List of integers	Properties
12	4096	… … … … … … … … … … … … … … … … … … … … … … … 6619, 14269, 75499, 90499, 158329, 160129, 218809, 236359, 304189, 364669, 371269, 440149, 496253, 503903, 70373, 139253, 145853, 206333, 274163, 291713, 350393, 352193, 420023, 435023	List at step 5 mod 510510/17 223, 631, 809, 1213, 1861, 2369, 2573, 2951, 3251, 3359, 3761, 3793, 3901, 4201, 4481, 4783, 4813, 5293, 5939, 6521, 6751, 7363, 7471, 7771, 8353, 8663, 9509, 10091, 10223, 10391, 10499, 10903, 10933, 11213, 11923, 12361, 12469, 13961, 14503, 14611, 14911, 14981, 15493, 16033, 17159, 17231, 17531, 17569, 17639, 18181, 21101, 21643, 21713, 21751, 22051, 22123, 23249, 23789, 24301, 24371, 24671, 24779, 25321, 26813, 26921, 27359, 28069, 28349, 28379, 28783, 28891, 29059, 29191, 29773, 30619, 30929, 31511, 31811, 31919, 32531, 32761, 33343, 33989, 34469, 34499, 34801, 35081, 35381, 35489, 35521, 35923, 36031, 36331, 36709, 36913, 37421, 38069, 38473, 38651, 39059 mod 510510/13 1861, 4073, 5221, 5293, 5323, 7433, 7639, 7949, 8051, 8963, 11411, 11953, 13283, 14681, 18041, 18553, 19909, 21239, 25183, 26513, 27869, 28381, 31741, 33139, 34469, 35011, 37459, 38371, 38473, 38783, 38989, 41099, 41129, 41201, 42349, 44561 mod 510510/11 19, 2569, 6619, 12469, 14269, 17569, mod 510510/7 and 55373, 58673, 60473, 66323, 70373, 72923 mod 510510/7

The numbers #ΔP for ΔP = 12 are determined by a five-equation recursive system based on the table below :

p_i	3	5	7	11	13	17
Line 1	0	0	0	8	188	4096
Line 2		0	0	8	100	1276
Line 3			0	8	36	276
Line 4				8	12	24
Line 5					12	0
Line 6						12

The 0 figure in the last column is the result of a calculation and the 12 figure in the last line is an "adjustment factor". The 24 in line 4 corresponds to the 24 integers at the bottom of Table 22. This population has doubled compared to the previous step and modulo $p_0p_1p_2\ldots p_i/p_{i-3}$ (510510/7 = 72930), we actually have exactly 12 distinct values (given in the same table).

We begin by looking at the 12 solutions, which are 6619, 14269, 75499, 90499, 158329, 160129, 218809, 236359, 304189, 364669, 371269 and 440149, governed by a relationship $p_0p_1p_2\ldots p_i/p_{i-3}$ and find the multiplier factor p_{i-3}-2-3 = 2 expected (and we note that $p_0p_1p_2\ldots p_i/p_i$ = 510510/17 = 30030, $p_0p_1p_2\ldots p_i/p_{i-1}$ = 510510/13 = 39270, $p_0p_1p_2\ldots p_i/p_{i-2}$ = 510510/11 = 46410):

m	6619	79549	152479	225409	298339	371269	444199
m-12	6607	79537	152467	225397	298327	371257	444187
Elimination if divider of 510510			yes (bottom) (p_{i-3} = 7)				yes (top) (p_{i-3} = 7)
Elimination if previously listed		yes (79549 = 33139 mod 46410)		yes (225409 = 29059 mod 39270)	yes (298339 = 28069 mod 30030)		

m	160129	233059	305989	378919	451849	14269	87199
m-12	160117	233047	305977	378907	451837	14257	87187
Elimination if divider of 510510			yes (bottom) (p_{i-3} = 7)				yes (top) (p_{i-3} = 7)
Elimination if previously listed		yes (233047 = 36709 mod 39270)		yes (378907 = 7639 mod 46410)	yes (451849 = 1399 mod 30030)		

m	75499	148429	221359	294289	367219	440149	513079
m-12	75487	148417	221347	294277	367207	440137	513067
Elimination if divider of 510510			yes (bottom) (p_{i-3} = 7)				yes (top) (p_{i-3} = 7)
Elimination if previously listed		yes (148429 = 30619 mod 39270)		yes (294289 = 24019 mod 30030)	yes (367219 = 42349 mod 46410)		

m	236359	309289	382219	455149	17569	90499	163429
m-12	236347	309277	382207	455137	17557	90487	163417
Elimination if divider of 510510			yes (bottom) (p_{i-3} = 7)				yes (top) (p_{i-3} = 7)
Elimination if previously listed		yes (309289 = 8989 mod 30030)		yes (455149 = 37459 mod 46410)	yes (17569 = 17569 mod 39270)		

m	304189	377119	450049	12469	85399	158329	231259
m-12	304177	377107	450037	12457	85387	158317	231247
Elimination if divider of 510510			yes (bottom) (p_{i-3} = 7)				yes (top) (p_{i-3} = 7)
Elimination if previously listed		yes (377119 = 16759 mod 30030)		yes (12469 = 12469 mod 39270)	yes (85387 = 38989 mod 46410)		

m	364669	437599	19	72949	145879	218809	291739
m-12	364657	437587	7	72937	145867	218797	291727
Elimination if divider of 510510			yes (bottom) (p_{i-3} = 7)				yes (top) (p_{i-3} = 7)
Elimination if previously listed		yes (437599 = 19909 mod 46410)		yes (17177 = 12889 mod 30030)	yes (145879 = 28069 mod 39270)		

There is only one configuration here in the sense of eliminations location. However, we can distinguish subconfigurations for the second type of elimination. Each has exactly one modulo $p_0p_1p_2\dots p_i/p_i$ elimination, one modulo $p_0p_1p_2\dots p_i/p_{i-1}$ elimination and now one modulo $p_0p_1p_2\dots p_i/p_{i-2}$ elimination. In addition, the 6 possible permutations of these three subconfigurations are each present in equal proportions (i.e. once here).

For the other 12 solutions, 70373, 139253, 145853, 206333, 274163, 291713, 350393, 352193, 420023, 435023, 496253 and 503903, we anticipate (writing only one) the same behaviour for subconfigurations, although that configuration might be different from the previous one. As in the previous steps, to a given configuration corresponds a symmetrical, the axis of symmetry being the "middle" of the two eliminated entities of the first type (elimination by divider) and the table below is the symmetrical of the third of the previous list :

m	362093	435023	507953	70373	143303	216233	289163
m-12	362081	435011	507941	70361	143291	216221	289151
Elimination if divider of 510510			yes (bottom) (p_{i-3} = 7)				yes (top) (p_{i-3} = 7)
Elimination if previously listed	yes (362093 = 8663 mod 39270)				yes (143303 = 4073 mod 46410)	yes (216233 = 6023 mod 30030)	

The transition of 12-spacing solutions from step 5 (see Table 21) to step 6 is implemented modulo $p_0p_1p_2\dots p_i/p_i/p_{i-3}$ = (510510/17)/7 = 4290 :

Values at step 5	mod 4290		Values at step 6	mod 4290	Configurations d = 510510/7
1399	1399	→	14269, 160129	1399	2d, 5d
8989	409	→	90499, 236359	409	
12889	19	→	218809, 364669	19	
16759	3889	→	158329, 304189	3889	
24019	2569	→	440149, 75499	2569	
28069	2329	→	371269, 6619	2329	
1973	1973	→	503903, 139253	1973	2d, 5d
6023	1733	→	435023, 70373	1733	
13283	413	→	206333, 352193	413	
17153	4283	→	145853, 291713	4283	
21053	3893	→	274163, 420023	3893	
28643	2903	→	350393, 496253	2903	

If we then go back to Table 22, we find, in line 3 at this step, 276-24 = 252 integers that correspond to the 24+12 integers in the previous step. The transition of 12-spacing solutions from step 5 (see Table 21) to step 6 is made modulo $p_0p_1p_2\ldots p_i/p_i/p_{i-2}$ = (510510/17)/11 = 2730 :

Values at step 5	Mod 2730		Values at step 6	Mod 2730	Configurations
16631	251	→	412481, 505301, 41201, 87611, 134021, 180431, 366071	251	1
11411	491	→	475511, 57821, 104231, 150641, 197051, 243461, 429101	491	1
19991	881	→	183791, 276611, 323021, 369431, 415841, 462251, 137381	881	1
3761	1031	→	339551, 432371, 478781, 14681, 61091, 107501, 293141	1031	1
7121	1661	→	110861, 203681, 250091, 296501, 342911, 389321, 64451	1661	1
8051	2591	→	193691, 286511, 332921, 379331, 425741, 472151, 147281	2591	1
21991	151	→	363241, 38371, 84781, 131191, 177601, 224011, 316831	151	S1
22921	1081	→	446071, 121201, 167611, 214021, 260431, 306841, 399661	1081	S1
26281	1711	→	217381, 403021, 449431, 495841, 31741, 78151, 170971	1711	S1
10051	1861	→	373141, 48271, 94681, 141091, 187501, 233911, 326731	1861	S1
18631	2251	→	81421, 267061, 313471, 359881, 406291, 452701, 35011	2251	S1
13411	2491	→	144451, 330091, 376501, 422911, 469321, 5221, 98041	2491	S1
2983	253	→	456163, 502573, 84883, 131293, 177703, 224113, 409753	253	2
27913	613	→	25183, 71593, 164413, 210823, 257233, 303643, 489283	613	2
20143	1033	→	383233, 429643, 11953, 58363, 104773, 151183, 336823	1033	2
2173	2173	→	389833, 436243, 18553, 64963, 111373, 157783, 343423	2173	2
10753	2563	→	98113, 144523, 237343, 283753, 330163, 376573, 51703	2563	2
24433	2593	→	237373, 283783, 376603, 423013, 469423, 5323, 190963	2593	2
5609	149	→	319559, 505199, 41099, 87509, 133919, 226739, 273149	149	S2
19289	179	→	458819, 133949, 180359, 226769, 273179, 365999, 412409	179	S2
27869	569	→	167099, 352739, 399149, 445559, 491969, 74279, 120689	569	S2
9899	1709	→	173699, 359339, 405749, 452159, 498569, 80879, 127289	1709	S2
2129	2129	→	21239, 206879, 253289, 299699, 346109, 438929, 485339	2129	S2
27059	2489	→	100769, 286409, 332819, 379229, 425639, 7949, 54359	2489	S2
28643	1343	→	4073, 96893, 143303, 189713, 236123, 282533, 468173	1343	1
1973	1973	→	285893, 378713, 425123, 471533, 7433, 53843, 239483	1973	1
28069	769	→	271039, 456679, 503089, 38989, 85399, 131809, 224629	769	S1
1399	1399	→	42349, 227989, 274399, 320809, 367219, 413629, 506449	1399	S1
16759	379	→	404419, 450829, 33139, 79549, 125959, 172369, 358009	379	2
8989	799	→	251959, 298369, 391189, 437599, 484009, 19909, 205549	799	2
21053	1943	→	304973, 490613, 26513, 72923, 119333, 212153, 258563	1943	S2
13283	2363	→	152513, 338153, 384563, 430973, 477383, 59693, 106103	2363	S2
6023	563	→	38783, 85193, 131603, 270833, 317243, 410063, 502883	563	3
17153	773	→	473063, 8963, 55373, 194603, 241013, 333833, 426653	773	3
12889	1969	→	83869, 176689, 269509, 315919, 455149, 501559, 37459	1969	S3
24019	2179	→	7639, 100459, 193279, 239689, 378919, 425329, 471739	2179	S3

We can notice that the last 12 solutions "work" in exactly the same way as the other 24 integers viewed modulo $p_0p_1p_2\ldots p_i/p_i/p_{i-2}$.
Regarding configurations, the classification according to the unit digits of each integer is accidental and quite anecdotal. Moreover, the attentive reader will have already noticed that the latter changes between the groupings of the first 24 and

the 12 subsequent. In fact, this ranking is done by taking into account the spacings between integers and answers the pattern of the following table (d = 46410 = 510510/11) :

Configurations	Spacings
1	2d, d, d, d, d, 4d, d
S1	4d, d, d, d, d, 2d, d
2	d, 2d, d, d, d, 4d, d
S2	4d, d, d, d, 2d, d, d
3	d, d, 3d, d, 2d, 2d, d
S3	2d, 2d, d, 3d, d, d, d

Next steps :

Essentially, there are no fundamentally richer teachings to expect than that acquired at the already studied steps.

5.2.5. Maximal spacing.

Now let us have focus on vertical considerations.

Conjecture 2

The maximum spacing ΔP_{max}, between integers of the Eras(i) list at the i depletion stage, is inferior or equal to $2p_i$-2.

The purpose is to prove that for the series {y, y+2, …, y+2c, …, y+$2p_i$-2}, where y is odd, there is at least an integer c between 0 and p_i-1, such as y+2c ≠ 0 mod p_k for any k between 1 and i. This conjecture is thus written in a totally equivalent way in the following form :

$$\forall\ \mathrm{y} = 1 \bmod 2, \exists\ \mathrm{c} \in \{0, 1, 2, \dots \mathrm{p_i} - 1\} \setminus \gcd(\mathrm{y} + 2\mathrm{c}\,, 3.5 \dots \mathrm{p_i}) = 1 \qquad (56)$$

This innocuous statement, in our view, is one of the most fundamental of arithmetic.
It presents itself after many attempts at resolution as a real headache for its complete resolution. Nevertheless, the problem can be circumscribed in its broad outlines according to the theorems and remarks made below.

Example : i = 8, p_i = 23, y = 513

p_k \ 2c	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44
3	0	2	1	0	2	1	0	2	1	0	2	1	0	2	1	0	2	1	0	2	1	0	2
5	3	0	2	4	1	3	0	2	4	1	3	0	2	4	1	3	0	2	4	1	3	0	2
7	2	4	6	1	3	5	0	2	4	6	1	3	5	0	2	4	6	1	3	5	0	2	4
11	7	9	0	2	4	6	8	10	1	3	5	7	9	0	2	4	6	8	10	1	3	5	7
13	6	8	10	12	1	3	5	7	9	11	0	2	4	6	8	10	12	1	3	5	7	9	11
17	3	5	7	9	11	13	15	0	2	4	6	8	10	12	14	16	1	3	5	7	9	11	13
19	0	2	4	6	8	10	12	14	16	18	1	3	5	7	9	11	13	15	17	0	2	4	6
23	7	9	11	13	15	17	19	21	0	2	4	6	8	10	12	14	16	18	20	22	1	3	5
Numbers of zeroes	2	1	1	1	0	0	3	1	1	1	1	1	1	2	0	1	1	0	1	1	1	2	0

Here the solutions, we are looking for, are c = 4, c = 5, c = 14, c = 17, c = 22 (2c = 8, 2c = 10, 2c = 28, 2c = 34, 2c = 44). They are therefore usually not uncommon in the chosen interval, except that when p_i tends towards infinity, the amount of 0 per column in the double frame should tend on average, on a purely statistical basis, to 1/3+1/5+1/7+…+1/p_i and thus to infinity (with the same reasoning starting with 1/5 instead of 1/3, or 1/7, etc.) which would make seem highly unlikely the systematic existence of c as conjectured here.

Theorem 10

The maximum spacing ΔP_{max} is larger or equal to $2p_{i-1}$.

Proof

The solution $\Delta P = 2p_{i-1}$ is obtained constructively (see theorem 11 below) and hence always exists.

Theorem 11

There is always a pair giving spacing $2p_{i-1}$. One of the elements of the pair is centred in M_1 and the other one in M_2 = 2.3… p_i-M_1 and the couple (M_1, M_2) meets the equations' systems :

$$\begin{array}{ll} M_1 = 0 \bmod 2.3.5\ldots p_{i-2} & M_2 = 0 \bmod 2.3.5\ldots p_{i-2} \\ M_1 = -1 \bmod p_{i-1} & M_2 = 1 \bmod p_{i-1} \\ M_1 = 1 \bmod p_i & M_2 = -1 \bmod p_i \end{array} \qquad (57)$$

Proof

We can limit to the study of the case of M_1 as M_2 is the mere symmetrical of M_1 (i.e. $M_1+M_2 = 2.3.5\ldots p_i$) that we have identified in the previous theorem. Again let us use then theorem 1. As $2\ldots p_{i-2}$ and p_{i-1} are coprime, we have that $k.2.3.5\ldots p_{i-2} \bmod p_{i-1}$, k = 1 à $p_{i-1}.p_i$, generate p_i repetitions of p_{i-1} distinct numbers (0 up to p_{i-1}-1). Similarly, $k.2.3.5\ldots p_{i-2} \bmod p_i$, k = 1 to $p_{i-1}.p_i$, generate p_{i-1} repetitions of p_i distinct numbers (0 to p_i-1). The two lists, obtained by k incrementing, form pairs of numbers, which, under the Chinese theorem (or theorem 1), are all distinct. One of these pairs is therefore necessarily {$-1 \bmod p_{i-1}$, $1 \bmod p_i$} and moreover it is unique.

To get the value of M_1 (or of M_2), one just solves two Bachet-Bézout equations. As the cycles are repetitive to infinity, the solution is necessarily also in cycle 1. Such a pair of solutions therefore always exists.

Its construction is done in a standard way according to the example below (where M = M_1) (i = 6, M = 217140 = 2.3.5.7.11.k and k = 94) :

Table 23

M±(2k+1)	M±(2k+1)	3	5	7	11	13	17
M-13	217127						
M-11	217129				X		
M-9	217131	X					
M-7	217133			X			
M-5	217135		X				
M-3	217137	X					
M-1	217139					X	
M+1	217141						X
M+3	217143	X					
M+5	217145		X				
M+7	217147			X			
M+9	217149	X					
M+11	217151				X		
M+13	217153						

Developing in the table according to the allocation ($M+p_k$, p_k), as M has 2 to p_i as divisors, all the interstices $M+j.p_k$ are addressed (meaning for us here that they are emptied), then M+1 and M-1 places are affected by construction. We get this way the largest free space between numbers. In addition, we can now assess the spacing. It is based on 13 and in the general case on p_{i-1} and gives therefore a spacing of $2p_{i-1}$. Of course, the most obvious, looking at the example, would be actually to take $2(p_{i-2}+2)$ because the contributions of p_{i-1} and p_i are made in M-1 and M+1, but one must not forget small dividers that allow us (thanks again here to theorem 1) to match a "small" divider up to the positions $M-(p_{i-1}-1)$ and $M+(p_{i-1}-1)$ modulo p_{i-1}.

Any change to this construction gives an intermediate empty space. It is the only one that can reach a value of $2p_{i-1}$ spacing. The question is whether an adjacency to another empty space (of integers with small divisors) is possible to further increase the spacing. To do this, simply look at the lower and upper boundaries just adjacent to this space $M-p_{i-1}$ and $M+p_{i-1}$, which are odd numbers, and check if they have or not, one or the other, divisors between 3 and p_i. To do this, let us rewrite the equations, resulting for the first of these limits: $M = 2.3\ldots p_{i-2}.k$, $M = -1+k1.p_{i-1}$, $M = 1+k2.p_i$, $M-p_{i-1} = k3.p_j$ where k, k1, k2, k3 are strictly positive integers and $3 \le p_j \le p_i$, $1\le k \le p_{i-1}.p_i$.
We have three cases:
If $p_j \le p_{i-2}$ then $p_{i-1} = M-k3.p_j = k4.p_j-k3.p_j= (k4-k3).p_j$, for some integer k4, which is impossible.
If $p_j = p_{i-1}$ then $M = k3.p_j+p_{i-1} = (k3+1).p_{i-1} = -1+k1.p_{i-1}$, thus $(k1-k3-2).p_{i-1} = 1$, which is impossible.
If $p_j = p_i$ then $M = k3.p_i+p_{i-1} = 1+k2.p_i$, thus $(k2-k3).p_i = p_{i-1}-1$ which is still impossible because $p_i > p_{i-1}-1$.
The argumentation is the same for the upper limit.
The previous empty interval is therefore the largest possible which ends proof set-up.

We give in appendix 3 the entire list of the M_1 and M_2 for i = 2 to 50, as well as i = 100, 150, …, 500, 1000 and 1500, using online calculator Pari GP.

Nota :

The fact that it gives the biggest spacing in general stems from its construction which fills the spaces optimally. This filling in itself contains two advantages:
- The first one is its symmetry versus the horizontal axis, which systematically doubles the gain at each new step.

- The second one is the inheritance of the previous setup, namely, there can be only optimum progression without questioning the previous configuration. Any other configuration is dependent, at rank i, on random variation of neighbour spacings, the average value of which is $\Delta_{mean}(i) \rightarrow e^{\gamma}.\ln(p_i) \approx 1{,}781.\ln(p_i)$. This is to be compared with a undeniable increase of the spacing, for the optimum standard scheme given here, of $2(p_{i-1}-p_{i-2})$, an expression that tends towards $2.\ln(p_i)$ asymptotically. The difference between the two is not staggering, but with a systematic routine extending to infinity, this regular asymptotic growth is definitely to the advantage of said scheme. It is reasonable to think that the following example is quite anecdotal, perhaps even unique.

A unique (?) overboosted example

For the case i = 8, ΔP_{max} is effectively superior to $2p_{i-1}$. Let us first give the standard scheme.

Table 24

Distance to the first value	M = 193483290 = 2.3.5...17.k and k = 379	Series of odd integers	3	5	7	11	13	17	19	23
0	M-19	193483271								
2	M-17	193483273						X		
4	M-15	193483275	X	(X)						
6	M-13	193483277					X			
8	M-11	193483279				X				
10	M-9	193483281	X							
12	M-7	193483283			X					
14	M-5	193483285		X						
16	M-3	193483287	X							
18	M-1	193483289							X	
20	M+1	193483291								X
22	M+3	193483293	X							
24	M+5	193483295		X						
26	M+7	193483297			X					
28	M+9	193483299	X							
30	M+11	193483301				X				
32	M+13	193483303					X			
34	M+15	193483305	X	(X)						
36	M+17	193483307						X		
38	M+19	193483309								

The number of redundancies (more than one cross on a line) is equal here to 2 over 20.

The « high-vitamin » example underneath is such that $\Delta P_{max} = 2p_{i-1}+2 = 2p_i-6$. It shows 6 pairs of solutions.

Scheme 1 Integers at position 0	Scheme 2 (symmetric of scheme 1) Integers at position 0
20332471	202760359
24686821	198406009
36068191	187024639
65767861	157324969
97689751	125403079
140722741	82370089

The table below schematizes the solution with 20332471 in position 0, the other 5 solutions of scheme 1 being available by swapping the 3 crosses on the last three columns (p_k = 17, 19 and 23) from one line to another, this being possible because these crosses are alone in their respective column.

Distance from the first value								
	3	5	7	11	13	17	19	23
-2								
0								
2	X		(X)					
4		X						
6				X				
8	X							
10					X			
12								X
14	X	(X)						
16			X					
18							X	
20	X							
22						X		
24		X						
26	X							
28				X				
30			X					
32	X							
34		X						
36					X			
38	X							
40								
42								

The number of redundancies are equal to 2 over 21.

Note: This maximum spacing corresponds to some case where the two void borders are not made up of a single integer without small dividers but by a pair of numbers. In the next step, it can only increase by 2 (new spacing = 42) and will therefore be smaller than the $2p_{i-1}$ spacing of the standard scheme (i = 9, $2p_{i-1}$. = $2p_8$ = 46).

<u>Statement 1</u>

When $\Delta P_{max} > 2p_{i-1}$, we think that framing is systematically realized by a pair of numbers as above.
We would thus be in the case of another problem (that of pairs of numbers) in which these exceptions play no role neither predominant nor even notable. These pairs take revenge for their anonymity there by playing here troublemakers.

5.2.6. Minimal spacing.

We are talking of the spacing 2 and integers that in the cycle 1 are not exclusively primes, but specifically numbers with large divisors (which gap 2 and are so named twins).
The average density of large twin dividers in the cycle 1 is exactly $\prod((p_k-2)/p_k)$, k = 1 to i, at step i. Assuming a relatively uniform distribution in a large enough interval, as for example the interval p_i +2 to p_i^2 (as soon as 30 values are included for example), interval which contains by algorithmic construction only primes, we get a generative density of twin prime numbers of about $\prod((p_k-2)/p_k) \approx c_2.e^{-2\gamma}/\ln^2(p_i)$ using the generalization of the Mertens theorem, that is also some $c/\ln^2(p_i)$ upstream of the abscissa p_i. This will create progressively in the range 0 to p_i (which increases when i increases) a quantity $c.p_i/\ln^2(p_i)$ of twin prime numbers.

Note:
Even if the distribution of 2-spacings is not uniform, nothing does influence or reduces their evolution apart from the average ratio $(p_i-2)/p_i$. The twin numbers late to the call between p_i +2 to p_i^2 will come up more numerously later on, where those in advance will delay the arrival of followers. Asymptotically the average necessarily prevails over any other phenomenon.

Thus again:

<u>Statement 2</u>

The asymptotic evolution of the cardinal of twin prime numbers is $c.p_i/\ln^2(p_i)$, c a positive constant (to be determined).
So there is an infinite number of twin primes.

We already have a statement along the desired lines. Let us nevertheless develop further the topic, especially that of ratio $(p_k-2)/p_k$.

6. Eratosthenes crossed sieve.

We are just talking of the Eratosthenes sieve to which we add a special counter that we name signature.

6.1. Case of the twin prime numbers.

We start with the odd numbers (hence the x-axis scaling with a step of 2, fact which one must pay attention later on) and we gradually remove multiples of prime numbers seeking for couples of twin prime numbers (1 is not a prime number, hence the absence of 2 under the integer 3 in the following table) :

Tables 25

Step 0 : Initial list

Entrée		Cycle 1	Cycle 2	Cycle 3	Cycle 4	Cycle 5	Cycle 6	Cycle 7	Cycle 8	Cycle 9	Cycle 10	Cycle 11	Cycle 12	Cycle 13	Cycle 14	Cycle 15	Cycle 16	Cycle 1	Cycle 2	Cycle 3
1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41
		2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2

Step 1 : 3-Multiples withdrawal (except 3)

Entry			Cycle1			Cycle2			Cycle3			Cycle4			Cycle5			Etc.		
1	3	5	7		11	13		17	19		23	25		29	31		35	37		41
		2	2			2			2			2			2			2		

Step 2 : 5-Multiples withdrawal (except 5)

Entry				Cycle1															Cycle2															Etc.
1	3	5	7		11	13		17	19		23			29	31			37		41	43		47	49		53			59	61			67	
		2	2			2			2						2						2			2						2				

Step 3 : 7-Multiples withdrawal (except 7)

Entry					Cycle 1 (not entirely represented)																													
1	3	5	7		11	13		17	19		23			29	31			37		41	43		47			53			59	61			67	
		2	2			2			2						2						2									2				

Step 4 : 11-Multiples withdrawal (except 11)

Entry							Cycle 1 (not entirely represented)																											
1	3	5	7		11	13		17	19		23			29	31			37		41	43		47			53			59	61			67	
		2	2			2			2						2						2									2				

Step 5 : 13-Multiples withdrawal (except 13)

Entry								Cycle 1 (not entirely represented)																										
1	3	5	7		11	13		17	19		23			29	31			37		41	43		47			53			59	61			67	
		2	2			2			2						2						2									2				

Step 6 : 17-Multiples withdrawal (except 17)

Entry										Cycle 1 (not entirely represented)																								
1	3	5	7		11	13		17	19		23			29	31			37		41	43		47			53			59	61			67	
		2	2			2			2						2						2									2				

In the previous process, when a multiple is removed of a column, the 2 at the following column is removed also (if still there).
We call the last line of the tables (containing the figures 2) the signatures' line.

We observe a "rho" type process : we have a first part of numbers, we will call the "entry" part, which has a non-repetitive structure and parts that we call "cycles" with repetitive patterns. The amplitudes of these patterns are equal to $2.3.5 \ldots p_i$, with p_i being the last prime number whose multiples were removed (the integer p_i being retained). Thus, the integers of the cycle n+1 are those of the cycle n by adding the $2.3.5 \ldots p_i$ product and the signatures will repeat identically up to infinity.
Cycle 1 starts at p_i+4 (p_i+2+2n in the general case of a gap of 2n instead of 2 except for $p_0 = 2$ (at p_0+3)).

We can provide a picture of the signatures, odd "survivors" of this process, i.e. numbers which retain 2 facing them on the last row of the said table :

Table 26

Step i	p_i	2.3…p_i	Entry	#(entry survivors)	Cycle1	#(Cycle1 survivors) = #(B_i)	#(B_i)/#(B_{i-1})	p_i-#B_i/#B_{i-1}
1	3	6	1-5	1	7-12	1		
2	5	30	1-7	2	9-38	3	3	2
3	7	210	1-9	2	11-220	15	5	2
4	11	2310	1-13	3	15-2324	135	9	2
5	13	30030	1-15	3	17-30046	1485	11	2
6	17	510510	1-19	4	21-510530	22275	15	2
7	19	9699690	1-21	4	23-9699712	378675	17	2

The reader must be attentive to the fact that when we are talking of a survivor, we are talking about a pair of integers : this one who has the gap 2 registered under its value and the previous one that makes the pair with it. We do not count numbers but pairs of numbers. We count signatures.

We observe that the number of signatures in the repetitive parts evolves according to the formula :

Theorem 12

The number of signatures per cycle is given recursively by:

$$\#(B_{i+1})/\#(B_i) = p_{i+1}-2 \qquad (58)$$

Proof

Relation (58) results from theorem 1. We need to get at stage i, the number of eliminations, i.e. multiples of p_i (or integers 0 modulo p_i) present in 1 cycle 1. A sequence (0, r, 2r,..., (s-1).r) modulo s, where r = 2.3... p_{i-1} and s = p_i are coprime, contains exactly a single 0. It is the same by adding a constant c to each of the terms of (0, r, 2r, …, (s-1).r), that is for (c, c+r, c+2r, …, c+(s-1).r) mod s. We will have then exactly for a pair of numbers p and q such as p-q = 2, two eliminations because 2 being coprime with p_i, the 0 within (c, c+r, c+2r, …, c+(s-1).r) mod s and the 0 within (2+c, 2+c+r, 2+c+2r, …, 2+c+(s-1).r) mod s are necessarily shifted.
We take also $B_0 = 1$ ($p_0 = 2$) which initiate in a coherent way the recursive sequence.

It follows immediately:

$$\#(B_i) = \prod_{3 \le p_k \le p_i} (p_k-2) \qquad (59)$$

Illustration

p-q = 2 and $p_i = 7$
At step 2 (withdrawal of multiples of 5), we have the {13, 19, 31} survivors, as the reader will find above. At the next step, the survivors of interest here are between 11 and 220 (i.e. 7+4+2.3.5.7-1) and are built from {13, 19, 31} modulo 30 (30 = 2.3.5).

We get the following tables :

Table 27

For p (in p-q = 2)

13		13	43	73	103	~~133~~	163	193		133 = 7.19
19	=>	19	~~49~~	79	109	139	169	199		49 = 7.7
31		31	61	~~91~~	121	151	181	211		91 = 7.13

For q (in p-q = 2) :

11		11	41	71	101	131	~~161~~	191		161 = 7.23
17	=>	17	47	~~77~~	107	137	167	197		77 = 7.11
29		29	59	89	~~119~~	149	179	209		119 = 7.17

Let us reconsider the two tables modulo 7 with 0 shifted by 2 in the second table.
We get :

For p (in p-q = 2)

13		6	1	3	5	$\cancel{0}$	2	4	133 = 0 mod 7
19	=>	5	$\cancel{0}$	2	4	6	1	3	49 = 0 mod 7
31		3	5	$\cancel{0}$	2	4	6	1	91 = 0 mod 7

For q (in p-q = 2) :

11		4	6	1	3	5	$\cancel{0}$	2	161 = 0 mod 7
17	=>	3	5	$\cancel{0}$	2	4	6	1	77 = 0 mod 7
29		1	3	5	$\cancel{0}$	2	4	6	119 = 0 mod 7

It is, a priori, impossible to predict where in each table the eliminations will occur (even at a stage as early as above). But, we have necessarily a permutation of (0, 1, …, p_i-1) in each line and therefore a unique elimination (in each line) as 2.3. 5…p_i-1 is prime with p_i.

The positions of the eliminations are shifted from one line to the other in each of the two illustrations. The order of presentation of congruencies is the same following a circular permutation (here the order is 0, 2, 4, 6, 1, 3, 5), but this is not helpful for what we are here concerned.

In addition, and this time it is required to our purpose, the eliminations positions (as the other non-zero congruencies) are shifted from the first table to the second one between two corresponding lines (lines of 13 and 11, lines of 19 and 17, lines of 31 and 29) as gap 2 is prime with p_i. Hence, we get elimination of exactly $2p_i$ solutions for the p_i examined situations.

We get a depletion of the number of "survivors" at step i which is expressed not heuristically, but by an arithmetic law. At every step i, we have p_i columns of which 2 are eliminated.
The depletion of the signatures is thus given by the ratio :

$$(p_i-2)/p_i \qquad (60)$$

We find easily the relationship (58) since $\#(B_i)/\#(B_{i-1})$ is equal to this ratio multiplied by p_i :

$$\frac{\#(B_i)}{\#(B_{i-1})} = \left(\frac{p_i-2}{p_i}\right).p_i = p_i-2$$

This non-zero ratio shows that there is never exhaustion of some potential candidates to the prime numbers twin in the cycles. But this would not suffice to get infinite twins. Twin prime numbers remain at infinity because the eliminations due to the Eratosthenes crossed sieve, when the steps are incremented, are regulated by a proportion that is quite enough close to 1. For an assessment of the lower bound of the twin prime numbers population, the key is indeed in the ratio $(p_i-2)/p_i$.

The goal underneath is only some numerical clarifications. We give quantities at the start of the routine showing the 'evidence' of the result.

<u>Table 28</u>

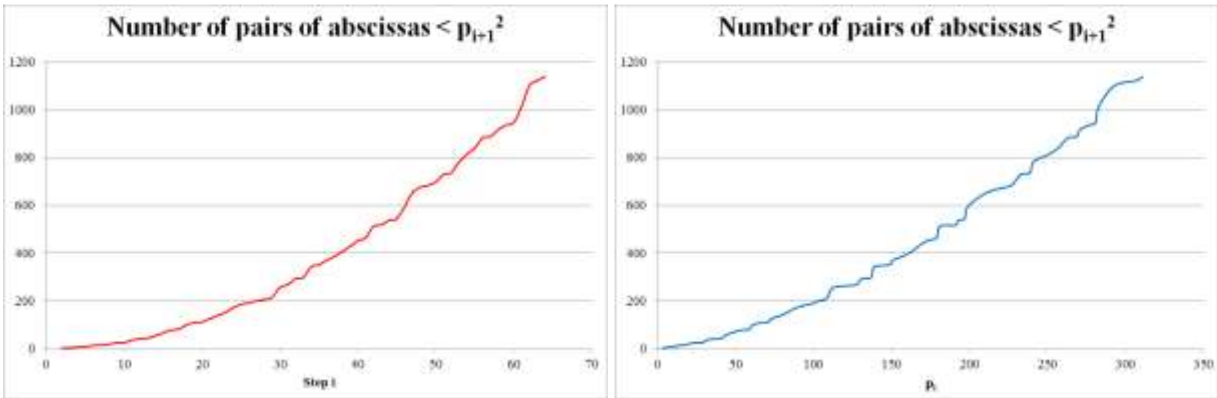
Step i	p_i	Rpi = $(2.3.5…p_i)/((3-2)(5-2)…(p_i-2))$	p_{i+1}^2/Rpi	Number of pairs present between p_i+4 and p_{i+1}^2	c/2 = p_{i+1}^2/Rpi/i²
1	3	6	4,17	3	4,17
2	5	10	4,90	4	1,23
3	7	14	8,64	7	0,96
4	11	17,11	9,88	8	0,62
5	13	20,22	14,29	14	0,57
6	17	22,92	15,75	15	0,44
7	19	25,61	20,65	18	0,42
8	23	28,05	29,98	25	0,47
9	29	30,13	31,89	26	0,39
10	31	32,21	42,50	36	0,43
11	37	34,05	49,37	42	0,41
12	41	35,80	51,65	44	0,36
13	43	37,54	58,84	54	0,35
14	47	39,21	71,64	66	0,37

Step i	p_i	Rpi = $(2.3.5\ldots p_i)/((3-2)(5-2)\ldots(p_i-2))$	p_{i+1}^2/Rpi	Number of pairs present between p_i+4 and p_{i+1}^2	c/2 = p_{i+1}^2/Rpi/i²
15	53	40,75	85,42	78	0,38
16	59	42,18	88,22	82	0,34
17	61	43,61	102,94	100	0,36
18	67	44,95	112,14	110	0,35
19	71	46,25	115,21	112	0,32

What means this table? At step i, we remove all the multiples of p_i. As p_{i+1} is prime, at the next step, the first withdrawal is necessarily beyond p_{i+1}^2. But the first pair is already present well below this abscissa. The abscissas ratio increases progressively (it may decrease a bit from time to time) and this phenomenon is irreversible.

The number of signatures is $\prod (p_k-2)$, k = 1 à i, $p_0 = 2$, in a cycle of size $2.3.5\ldots p_i$, hence statistically a distance between signatures of $2.\prod p_k/(p_k-2)$. In the $[p_i +4, p_{i+1}^2]$ interval, whose approximate size tends towards p_{i+1}^2, we therefore have $(p_{i+1}^2/2).\prod (p_k-2)/p_k \rightarrow (p_{i+1}^2/2).(c/\ln^2(p_i)) = (p_{i+1}^2/p_i^2).(c/2).p_i^2/\ln^2(p_i) \approx (c/2).i^2$ signatures.
The graphs below illustrate that :

Graphs 4 and 5



The growth of the number of pairs actually twin primes is parabolic versus to the current step (i.e. index i):

$$\#(\text{ number of twin prime pairs at step i}) \approx 0{,}34.i^2 \qquad (61)$$

Another way to find this result is to observe that, according to the relationship 59, the number of signatures in the cycle 1 at step i is given by $\#(B_i) = \Pi_{3 \leq pk \leq pi} (p_k-2)$. The size of the cycle 1 being $\Pi_{3 \leq pk \leq pi}\, p_k$, on average, the distance between the remaining signatures is so $\Pi_{3 \leq pk \leq pi}\, p_k/(p_k-2)$, expression that tends, according to the generalization of the Mertens theorem, towards $c.\ln^2(p_i)$ when i tends towards infinity with c some constant (of the order of 1,2). This means that within the cycle 1 between p_i+4 and p_i^2, there are on average $(1/c).(p_i^2-p_i-4)/\ln^2(p_i)$ pairs of numbers. However, these can be in this interval only (twin) prime numbers, since all the multiples of 3 up to p_i were removed. When p_i increases, p_i becomes negligible in front of p_i^2 and the order of magnitude of the expression is then $(1/c).p_i^2/\ln^2(p_i)$. As $p_i/\ln(p_i)$ tends towards i, when i tends to infinity the order of magnitude of quantities is c'i², c' tending towards a non-null constant.

6.2. Case of relative prime numbers.

We examined previously the case of the gap 2 for twin prime numbers. Let us look at the 2n gaps (relatives like cousins, etc.). We have compiled a table of a few cases to illustrate generality. Cycle 1 begins at $2n+p_i+2$. Table 29 gives the number of eliminations in cycle 1 (and in the following cycles) as the sequence increases, table 30 gives the number of survivors in the cycles.

<u>Tableau 29</u>

Gaps = 2n, with divisors of n only among	Examples / Sequence = p_i	3	5	7	11	13
		#(removals in cycle1) = $\#A_i$				
2	2, 4, 8, 16…	2	2	6	30	270
2 and 3	6, 12, 18, 24, 36, 48, 54…	1	4	12	60	540
2 and 5	10, 20, 40, 50…	2	1	8	40	360
2 and 7	14, 28, 56…	2	2	3	36	324
2 and 11	22, 44…	2	2	6	15	300
2 and 13	26, 52…	2	2	6	30	135
2, 3 and 5	30, 60…	1	2	16	80	720
2, 3 and 7	42…	1	4	6	72	648
2, 5 and 7	70…	2	1	4	48	432
2, 3, 5 and 7	210…	1	2	8	96	864
2, 3, 5, 7 and 11	2310…	1	2	8	48	960

<u>Tableau 30</u>

Gaps = 2n, with divisors of n only among	Examples / Sequence = p_i	3	5	7	11	13
		#(remainder in cycle1) = $\#B_i$				
2	2, 4, 8, 16…	1	3	15	135	1485
2 and 3	6, 12, 18, 24, 36, 48, 54…	2	6	30	270	2970
2 and 5	10, 20, 40, 50…	1	4	20	180	1980
2 and 7	14, 28, 56…	1	3	18	162	1782
2 and 11	22, 44…	1	3	15	150	1650
2 and 13	26, 52…	1	3	15	135	1620
2, 3 and 5	30, 60…	2	8	40	360	3960
2, 3 and 7	42…	2	6	36	324	3564
2, 5 and 7	70…	1	4	24	216	2376
2, 3, 5 and 7	210…	2	8	48	432	4752
2, 3, 5, 7 and 11	2310…	2	8	48	480	5280

<u>Lemma 4</u>

The number of remaining elements in one cycle is given recursively by :

$$\#B_i/\#B_{i-1} = \text{if}(p_i\backslash 2n,\ p_i\text{-}1,\ p_i\text{-}2) \qquad (62)$$

<u>Proof</u>

Let us go back to the proof of the theorem 4 page 8 showing the existence of a single element 0 modulo p_i with theorem 1. In the mechanism of withdrawal by the Eratosthenes crossed sieve, the two 0 modulo p_i, that match, can only be either shifted or aligned.
They are aligned if and only if p-q = 0 mod p_i, so if 2n = 0 mod p_i, or finally p_i divides 2n.
If there is a shifting, there are two eliminations (as shown above), otherwise if there is only one (as shown below).

<u>Illustration</u>

p-q = 10 and p_i = 5
At step 1 (removal of multiples of 3), there are remaining all the integers 5 modulo 6, the first cycle starting at 15 (that is 3 +2 +10).
At next step (removal of multiples of 5), the survivors that interest us are between 17 and 46 (that is 5 +2 +10 +2.3.5-1) and are built from {17} modulo 6. We have the tables :

For p (in p-q = 10)

17 => 17 23 29 ~~35~~ 41 35 = 5.7

For q (in p-q = 10) :

7	=>	7	13	19	~~25~~	31	25 = 5.5

Let us rewrite the two features modulo 5.
We get :

For p (in p-q = 10)

17	=>	2	3	4	~~0~~	1	35 = 0 mod 5

For q (in p-q = 10) :

7	=>	2	3	4	~~0~~	1	25 = 0 mod 5

The said alignment of values 0 modulo p_i is verified.

Theorem 13

The rarefaction of the number of elements in the cycles is the strongest when p-q = 2^m.

Proof

According to the previous lemma 4, the survivors ratio $\#(B_i)/\#(B_{i-1})$ is minimal (and equal to p_i-2) at each step since p_i never divides n as n is only multiple of 2 (and $p_i \geq 3$). Hence, we get minimum number of signatures and the result.

Thus, if there is an infinite number of twin prime numbers, there are an infinite number of relative prime numbers.

Lemma 5

The number of removals (or disappearances) is given by :

$$\#A_i = \#B_{i-1}.\text{if}(p_i\backslash 2n, 1, 2) \qquad (63)$$

Proof

It is a paraphrase of the topic concerning eliminations.

Lemma 6

The number of removals of in a cycle at step i+1 is given by the number of removals in a cycle at step i by:

$$\#A_{i+1} = \#A_i.\text{if}(p_i\backslash 2n, p_i\text{-}1, p_i\text{-}2).\text{if}(p_{i+1}\backslash 2n, 1, 2)/\text{if}(p_i\backslash 2n, 1, 2) \qquad (64)$$

Proof

We have $\#A_i = \#B_{i-1}.\text{if}(p_i\backslash 2n, 1, 2)$ and thus $\#A_{i+1} = \#B_i.\text{if}(p_{i+1}\backslash 2n, 1, 2)$.
As $\#B_i/\#B_{i-1} = \text{if}(p_i\backslash 2n, p_i\text{-}1, p_i\text{-}2)$, the result follows by simple application of proportions.
Besides, we take $\#A_0 = 1$ ($p_0 = 2$) to initiate in a coherent way the recursive sequence.

Lemma 7

For twin prime numbers, the number of removals in a cycle at step i+1 is given by:

$$\#AR_{i+1} = \prod_{3 \leq p_k \leq p_i} (p_k\text{-}2) \qquad (65)$$

Proof

We have n = 1 and then apply recursion $\#A_{i+1} = \#A_i.(p_i\text{-}2)$ since p_i does not divide n and we have besides have #AR1 = 1.

Lemma 8

For relative prime numbers, the number of removals in a cycle at step i+1 is given by:

$$\#A_i = \text{if}(p_i\backslash 2n, 1/2, 1).\prod_{\substack{p_k\backslash 2n \\ 3 \leq p_k < p_i}}(p_k\text{-}1)/(p_k\text{-}2).\#AR_i \qquad (66)$$

where $\#AR_i$ is the number of removals in a cycle for twin prime numbers (2n = 2), cardinal used as a reference.

Proof

This is mere application of lemma 6.

We can also write $p_k\backslash n$ and $p_i\backslash n$ instead of $p_k\backslash 2n$ and $p_i\backslash 2n$ since the formula is used for $i \geq 1$.

$$\#A_i = \text{if}(p_i\backslash n,\ 1/2,\ 1).\prod_{\substack{p_k\backslash n \\ 3 \leq p_k < p_i}} (p_k-1)/(p_k-2).\#AR_i \qquad (67)$$

The reader can observe that the determinant terms of the Euler product of Hardy and Littlewood formula that are $\Pi(p_k-1)/(p_k-2)$ for $p_k\backslash n$ show up here.
Then let us write :

$$\#HL_i = \prod_{\substack{p_k\backslash n \\ 3 \leq p_k < p_i}} (p_k-1)/(p_k-2) \qquad (68)$$

and

$$\#HL = \prod_{p_k\backslash n} (p_k-1)/(p_k-2) \qquad (69)$$

We get immediately :

$$\#A_i = \text{if}(p_i\backslash n,\ 1/2,\ 1).\#HL_i.\#AR_i \qquad (70)$$

6.3. Evaluation of relative prime numbers cardinals.

Theorem 14

The Eratosthenes crossed sieve gives the set of relative prime numbers by iteration to infinity.

Proof

The Eratosthenes crossed sieve gives at step i the whole set of relative prime numbers (i.e. distant of 2n fixed in advance) up to the abscissa p_i^2. When i growths to infinity, p_i tends to infinity as well as p_i^2.
Hence the result.

Thus, we can estimate the number of pairs from 0 to infinity by counting the of the signatures line' items from 0 to infinity.

6.3.1. Case of twin prime numbers.

The solutions are obtained by iterated subtractions of odd integers by the of Eratosthenes crossed sieve which is the only agent at work here.

We can evaluate this using lemma 7 or with tables 25 features :

At step 1, $p_i = 3$, the proportion of signatures (of odd integers, which is undertone starting now) disappearing after 3+4 is $\#A_1/p_1 = 2/3$.
At step 2, $p_i = 5$, the additional proportion of signatures disappearing after 5+4 is 2.(3-2)/(3.5).
At step 3, $p_i = 7$, the additional proportion of signatures disappearing after 7+4 is 2.(3-2).(5-2)/(3.5.7).
At step 4, $p_i = 11$, the additional proportion of signatures disappearing after 11+4 is 2.(3-2).(5-2).(7-2)/(3.5. 7.11).
At step 5, $p_i = 13$, the additional proportion of signatures disappearing after 13+4 is 2.(3-2).(5-2).(7-2)(11-2)/(3.5.7. 11.13).

Thus at step i, the additional proportion of signatures disappearing after p_i+4 is 2.(3-2).(5-2).(7-2)(11-2)(p_{i+1}-2)/(3.5.7. 11.13…p_i), so that :

$$\#RC_i = \#AR_i / \left(\prod_{p=3}^{p_i} p\right) = (2/p_i).\prod_{p=3}^{p_{i-1}} (p-2)/p \qquad (71)$$

This is the first of the depletion coefficients #RCi expressions of Eratosthenes crossed sieve (ECS).

6.3.2. Case of pairs of prime numbers distant of 2^m.

The process is the same as before and we give first two examples :

<u>Gaps of 4 :</u>

<u>Tables 31</u>

Step 0 : Initial list

1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35
		4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4

Step 1 : 3-Multiples withdrawal (except 3)

Entrée				Cycle1			Cycle2			Cycle3			Cycle4			Cycle5			Etc.		
1	3	5	7		11	13		17	19		23	25		29	31		35	37		41	43
		4	4		4			4			4			4			4			4	4

Step 2 : 5-Multiples withdrawal (except 5)

Entrée				Cycle1															Cycle2															Etc.
1	3	5	7		11	13		17	19		23			29	31			37		41	43		47	49		53			59	61			67	
		4	4		4			4			4									4			4			4								

<u>Gaps of 8 :</u>

<u>Tables 32</u>

Step 0 : Initial list

1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35
				8	8	8	8	8	8	8	8	8	8	8	8	8	8

Step 1 : 3-Multiples withdrawal (except 3)

Entrée						Cycle1			Cycle2			Cycle3			Cycle4			Cycle5			Etc.		
1	3	5	7		11	13		17	19		23	25		29	31		35	37		41	43		45
					8	8			8			8			8			8			8		

Step 2 : 5-Multiples withdrawal (except 5)

Entry					Cycle1															Cycle2															Etc.
7		11	13		17	19		23			29	31			37		41	43		47	49		53			59	61			67		71	73		
		8	8			8						8			8						8						8			8					

At step i, the number remaining in the cycle j is the same regardless of m in 2^m (here 1 at step 0, 1 at step 1 and 3 at step 2).

In the general case, we thus have :

At step 1, $p_i = 3$, the proportion of signatures disappearing <u>after</u> $3++2+2^m$ is 2/3.
At step 2, $p_i = 5$, the additional proportion of signatures disappearing after $5+2+2^m$ is 2.(3-2)/(3.5).
At step 3, $p_i = 7$, the additional proportion of signatures disappearing after $7+2+2^m$ is 2.(3-2).(5-2)/(3.5.7).
At step 4, $p_i = 11$, the additional proportion of signatures disappearing after $11+2+2^m$ is 2.(3-2).(5-2).(7-2)/(3.5. 7.11).
At step 5, $p_i = 13$, the additional proportion of signatures disappearing after $13+2+2^m$ is 2.(3-2).(5-2).(7-2)(11-2)/(3.5.7. 11.13).
…

Thus at step i, the additional proportion of signatures disappearing after p_i+2+2^m is 2.(3-2).(5-2).(7-2)(11-2)(p_{i-1}-2)/(3.5.7. 11.13…p_i), thus already :

$$\#RC_i = \#AR_i / \prod_{p=3}^{p_i} p = (2/p_i).\prod_{p=3}^{p_{i-1}} (p-2)/p \qquad (72)$$

6.3.3. Case of relative prime numbers.

On the previous model at step i, the additional proportion of signatures disappearing after p_i+2+2n is :

$$\#RC_i = \text{if}(p_i \backslash n, 1/2, 1).\#HL_i.\#AR_i / \left(\prod_{p=3}^{p_i} p\right) = \text{if}(p_i \backslash n, 1/2, 1).\ \#HL_i.(2/p_i).\prod_{p=3}^{p_{i-1}} (p-2)/p \qquad (73)$$

6.3.4. Formula of cardinals.

Let us repeat again that the disappearing proportions are imposed arithmetically. There is no margin incertitude over their total number when the **whole set of N up to the point at infinity** is taken into account.

Starting there, we can estimate the number of solutions for twin prime numbers and similarly for primes of gaps 2^m up to infinity by writing an infinite series that is built from the previous sieve.

Theorem 15

$$\pi(p-q = 2^m) = \lim_{N \to +\infty} M-(2/3).M_1-(2/(3.5)).M_2-(2.3/(3.5.7)).M_2-(2.3.5/(3.5.7.11)).M_3-...-RC_i.MC_i-\ldots \qquad (74)$$

where

$$M = (N-1-2^m)/2 \qquad (75)$$
$$M_i = (N-p_i-2-2^m)/2 \qquad (76)$$
$$MC_i = \text{if}((N-p_i-2-2^m)/2 < 0, 0, (N-p_i-2-2^m)/2) \qquad (77)$$

and $\#RC_i$ is defined above.

Proof.

We start from the odd numbers $3+2^m$ up to N, the infinite value being attributed to N in a second time. We have $M = (N-3-2^m)/2+1$ integers.
Then the numbers are removed following the proportions given in paragraph 6.3.1 starting at abscissa p_i+2^m, $M_i = (N-p_i-2^m)/2+1$.
The proportions bearing on the odd numbers, it is necessary to take a ratio 1/2 in the abscissa differences $N-(p_i+2+2^m)$. We define $M_i = (N-p_i-2-2^m)/2$. We then get the infinite sum giving the sought cardinal ($p_1 = 3$).

When such a numerical application is carried out, the series as in the case of the Eratosthenes sieve is not infinite. Specifically, the M_i coefficients must be taken equal to 0 when $(N-p_i-2-2^m)/2$ becomes negative and so for calculations we must retain the expression :

$$MC_i = \text{if}((N-p_i-2-2^m)/2 < 0, 0, (N-p_i-2-2^m)/2)$$

For our numerical applications, we then rewrite the relationship (74) as :

$$\pi(c) = \lim_{N \to +\infty} M-(1/c).((2/3).MC_1+(2/(3.5)).MC_2+(2.3/(3.5.7)).MC_2+(2.3.5/(3.5.7.11)).MC_3+...+\#RC_i.MC_i+\ldots \qquad (78)$$

When c = 1, then $\pi(c) = \pi(p-q = 2^m)$.

We then follow the evolution of the values of c that matches $\pi(c)$ to the actual number of relative prime numbers.

If $c \leq 1$, then the actual number of solutions is less than $\pi(1)$.
If $c \geq 1$, then the actual number of solutions is greater than $\pi(1)$.
The reader will understand that we use 1/c in the expression (78) not because we seek complication, but to match to the "$\leq$" sign a reduction and to "$\geq$" sign an increase.

Twin prime numbers example shows that the c number turns out to be greater than 1 (with rare exceptions) which means that the cardinal of twin prime numbers near the origin is greater than $\pi(1)$.
We can do a second evaluation by choosing a different category for reference by pretending that the first pair of twins can appear only starting from p_i^2, namely by choosing :

$$MC_i = \text{if}((N-(2+2n+p_i^2))/2 < 0, 0, (N-(2+2n+p_i^2))/2) \qquad (79)$$

This method should then give an underestimate of $\pi(1)$ reducing the cardinal of twin prime numbers near the origin as 'statistical' area of the first pair of twins range below $2+2n+p_i^2$. The numerical application confirms it.
It should be understood that these choices have a very relative importance, because the only point that interests us is the point to infinity for which c = 1 stands as the limit value every time. The choice of the x-axis has only effect than to stick

a little better to the real cardinals near origin.

For a gap 2n, the formula generalizes as :

Theorem 16

$$\pi(p\text{-}q = 2n) = \lim_{N \to +\infty} M - \sum_{i=1}^{+\infty} \#RC_i.MC_i \qquad (80)$$

where

$$M = (N-1-2n)/2 \qquad (81)$$
$$M_i = (N-p_i-2-2n)/2 \qquad (82)$$
$$MC_i = \text{if}((N-p_i-2-2n)/2 < 0, 0, (N-p_i-2-2n)/2) \qquad (83)$$

and

$$\#RC_i = \#A_i.\prod_{k=1}^{i}(1/p_k) \qquad (84)$$

Proof

We just use theorems 15 and 6 and the result follows immediately.

Numerical applications

For numeric applications, it suffices to use in the same way again,

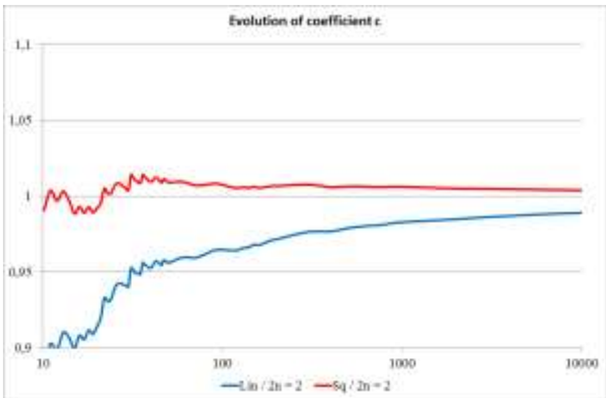
$$MC_i = \text{if}((N-p_i-2-2n)/2 < 0, 0, (N-p_i-2-2n)/2)$$

As well as the alternative choice :

$$MC_i = \text{if}((N-p_i^2-2-2n)/2 < 0, 0, (N-p_i^2-2-2n)/2) \qquad (85)$$

This gives for the coefficients c for 2n = 2 :

Graph 6



The first choice **reduces** the number of solutions, because generally the first number related after p_i+2+2n will appear only after a certain interval (it as the minimum x-coordinate of the first such number), while on the other hand, several cases could have occurred before abscissa p_i^2+2+2n, thus **raising** the number of solutions.

Theorems 15 and 16 formulas then give without much work interesting results by difference or division.

6.3.5. Common asymptotic branches.

Using difference, we get :

Theorem 17

The number of solutions of $\pi(p\text{-}q = 2^i)$ is either finite for all i, or infinite for all i.

Proof

$$\pi(p\text{-}q = 2^m) - \pi(p\text{-}q = 2) = \lim_{N \to +\infty} (2/3).(2^m-2)+(2/(3.5)).(2^m-2)+(2.3/(3.5.7)).(2^m-2)+(2.3.5/(3.5.7.11)).(2^m-2)+...$$

Thus N disappears in right-hand side by the subtraction operation and we can factor out the term 2^m-2.
In addition, as we cannot remove to a set more items that it contains, the sum

$$2/3+2/(3.5)+2.3/(3.5.7)+2.3.5/(3.5.7.11)+\ldots$$

is necessarily inferior or equal to 1.

Hence, after numerical verification that this sum is close to 1 (and in fact exactly equal to 1):

$$\begin{aligned}\pi(p-q = 2^m) - \pi(p-q = 2) &= (2^m-2).(2/3+2/(3.5)+2.3/(3.5.7)+2.3.5/(3.5.7.11)+2.3.5.9/(3.5.7.11.13)+...)\\ &\approx 2^m-2\end{aligned}$$

We infer that the difference of the number of solutions of $\pi(p-q = 2^i)$ and $\pi(p-q = 2^j)$ is finite.
Hence the result.

This can then be generalized.

Theorem 18

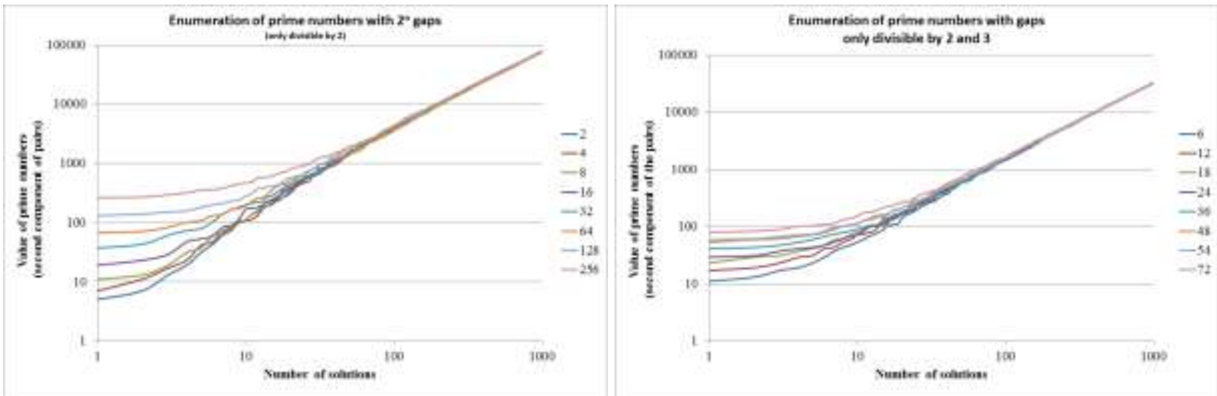
Let us have 2n and 2m with same dividers without exception. The numbers of solutions $\pi(p-q = 2n)$ and $\pi(p-q = 2m)$ are then either both finite or infinite.

Proof

Indeed, the infinite sum Σ # RC_i is less or equal to 1 as was point out in the previous paragraph.
We then resume the exercise with gaps of type 2n and 2m.
We have then $\pi(p-q = 2n) - \pi(p-q = 2m) = (2n-2m).(\Sigma \#RC_i) \leq (2n-2\ m)$.
The difference being finite, we infer the previous theorem.

Thus, if the number of solutions tends to infinity, the numbers of solutions are found on the same asymptote when dividers are all common.
This gives, for examples, the two following graphs:

Graphs 7 and 8



6.3.6. Implementation of a bijection between relative prime numbers with common asymptotic branches.

The whole chapter is carried over in appendix 4 to clarity to the mainstream article.

6.3.7. Hardy-Littlewood formula.

Theorem 19

The cardinal of relative prime numbers are in the ratio #HL of Hardy-Littlewood formula.

Proof

As Σ_i $\#RC_i$ = 1-ε, $\varepsilon \geq 0$, we get :

$$\pi(p\text{-}q = 2n) = \lim_{N \to +\infty} \varepsilon.M + \sum_{i=1}^{+\infty} \#RC_i.(M-M_i) \quad (86)$$

Using $M-M_i = (p_i+1)/2$, we write

$$\pi(p\text{-}q = 2n) = \lim_{N \to +\infty} \varepsilon.M+\sum_{i=2}^{+\infty} \text{if}(p_i \backslash n, 1/2, 1).\#HL_i.(p_i+1)/p_i.\prod_{p_k=3}^{p_{i-1}} (p_k-2)/p_k \quad (87)$$

Then for i the maximum index m of all divisors of n :

$$\pi(p\text{-}q = 2n) = \lim_{N \to +\infty} \varepsilon.M\text{-}cte1+\#HL.\sum_{i=m+1}^{+\infty} (p_i+1)/p_i.\prod_{p_k=3}^{p_{i-1}} (p_k-2)/p_k \quad (88)$$

where cte1 is a constant.

So that also :

$$\pi(p\text{-}q = 2n) = \lim_{N \to +\infty} \varepsilon.M\text{-}cte1+\#HL.\sum_{i=m+1}^{+\infty} \prod_{p_k=3}^{p_{i-1}} (p_k-2)/p_k+\#HL.\sum_{i=m+1}^{+\infty} (1/p_i).\prod_{p_k=3}^{p_{i-1}} (p_k-2)/p_k \quad (89)$$

Yet according to Mertens theorem corollary

$$\prod_{2 < p \leq x,\ x \to +\infty} (1\text{-}2/p) \equiv c_2.e^{-2\gamma}/\ln^2(x),\ c_2 > 0 \quad (90)$$

We get straightforward :

$$\pi(p\text{-}q = 2n) = \lim_{N \to +\infty} \varepsilon.M\text{-}cte1+\#HL.\sum_{i=m+1}^{+\infty} cte2/\ln^2(p_i)+\#HL.\sum_{i=m+1}^{+\infty} (1/p_i).cte2/\ln^2(p_i) \quad (91)$$

What we rewrite :

$$\pi(p\text{-}q = 2n) = \lim_{N \to +\infty} \varepsilon.M\text{-}cte1\text{-}cte3+\#HL.(\sum_{i=1}^{+\infty} cte2/\ln^2(p_i)+\sum_{i=1}^{+\infty} (1/p_i).cte2/\ln^2(p_i)\ ) \quad (92)$$

Neither the first sum, nor the second sum to the right of equality do contain a linear component that could compensate for the linear component $\varepsilon.M$. Being the only component of this type and knowing that the relative prime numbers are less dense than the prime numbers in N, we have necessarily $\varepsilon = 0$.
Moreover, the infinite sum $\Sigma\ 1/\ln^2(p_i)$ diverges, so cte1+cte3 is a non-significant term.
The remaining terms are thus :

$$\pi(p\text{-}q = 2n) = cte2.\#HL.(\sum_{i=1}^{+\infty} 1/\ln^2(p_i)+\sum_{i=1}^{+\infty} (1/p_i)/\ln^2(p_i)\ ) \quad (93)$$

We find there the same asymptotic proportions as those of Hardy-Littlewood formula.
Hence the theorem quoted above.

<u>Theorem 20</u>

There are an infinite number of relative prime numbers with given gap 2n.

<u>Proof</u>

The infinite sum $\Sigma\ 1/\ln^2(p_i)$ diverges as $\ln^2(p_i) < i$ from a certain rank on.
Let us have $u_i = 1/\ln^2(p_i)$ and $v_i = (1/p_i)/\ln^2(p_i)$. Then $v_i/u_i = 1/p_i \to 0$. The result is that $\Sigma v_i/\Sigma u_i \to 0$.
Thus the infinite sum $\Sigma\ (1/p_i)/\ln^2(p_i)$ is negligible towards the infinite sum $\Sigma\ 1/\ln^2(p_i)$.
So :

$$\pi(p\text{-}q = 2n) = cte''.\#HL.\sum_{i=1}^{+\infty} 1/\ln^2(p_i) \qquad (94)$$

Using relation (16), the previous expression will write as (cte' ≠ 0) :

$$\pi(p\text{-}q = 2n) = cte'.\#HL.\lim_{y \to +\infty} y/\ln^3(y) \qquad (95)$$

Hence the result, this expression tending towards infinity.

Argument

We can deduce again backwards as to the chapter on prime numbers, based on an analogy of table 4, what it would be when the index is i, and not p_i, which guide the initial calculation, thus redefining the abscissa axis support of the said calculation.
To do this, we design the following table :

Table 33

M_i (i ≥ 1)	$M_i = N\text{-}p_i^2\text{-}1$	$M_i = N\text{-}p_i\text{-}1$	$M_i = N\text{-}i\text{-}1$ ($i \approx p_i/\ln(p_i)$)
Interval between measures	$p_i.\ln(p_i)$	$\ln(p_i)$	1
Deduced ratio1	$p_i^2/(p_i.\ln(p_i)) = p_i/\ln(p_i)$	$p_i/\ln(p_i)$	$i \approx p_i/\ln(p_i)$
Corresponding sum	Σ cte1''.#HL.$p_i/\ln^2(p_i)$	Σ cte2''.#HL/$\ln^2(p_i)$	Σ cte3''.#HL/$\ln(p_i)$
Limit	cte1'.$x^2/\ln^3(x)$	cte2'.#HL.$x/\ln^3(x)$	cte3'.#HL.$x/\ln^2(x)$
Deduced ratio2 (taking $x \equiv p_i \equiv p$)	$p/\ln^2(p)/(p^2/\ln^3(p)) = \ln(p)/p$	$1/\ln^2(p)/(p/\ln^3(p)) = \ln(p)/p$	$1/\ln(p)/(p/\ln^2(p)) = \ln(p)/p$

Ratios 1 and 2 remain the same from one column to another.

The logarithm is a unit higher :

$$\pi(p\text{-}q = 2n) = cte.\#HL.\lim_{x \to +\infty} x/\ln^2(x) \qquad (96)$$

The usual Hardy-Littlewood formula is obtained by taking cte = 1.

Important note:

We repeat here the remark made for Eratosthenes sieve case. The end result for π(2n) comes in the form of a sum of fractions less than 1 in relationship 94. This comes from the fact that we manipulate $M\text{-}M_i$ in the intermediate calculation. It is essential to note here that, we handle not fractions of units because otherwise our estimate would be false. We would have to take all these fractions equal to 0, which would amount to a global reduction of 0. Instead, when the actual calculations are done, we handle M on one hand and $\#RC_i.M_i$ on the other hand in relationship 74. The first and the seconds are integers greater than 1 up to a certain rank. Rounding to integers or not, the results of the calculations again vary little here (meaning c is actually close to 1 when M is large).
Appendix 1 presents a calculation with rounding to integers and obtained coefficient c is very close to 1.

Theorem 21

There are an infinite number of relative prime integers with gap 2n.

Proof

This is an immediate result of the relationship 88.
Asymptotic progressions are in the Hardy-Littlewood $\#HL_i(2n)$ ratio, and thus if one of them is infinite, all of them are infinite.

To conclude, we have linked asymptotically equations arising from the Eratosthenes sieve to the PNT. This sieve with a slight modification (p_i-2 instead of p_i-1) gives a result similar to the PNT here with simply a factor in ln²() instead of ln(). For the same process, there is the same result : infinity in one case, infinity in the other. The remainder is calculation, useful however.

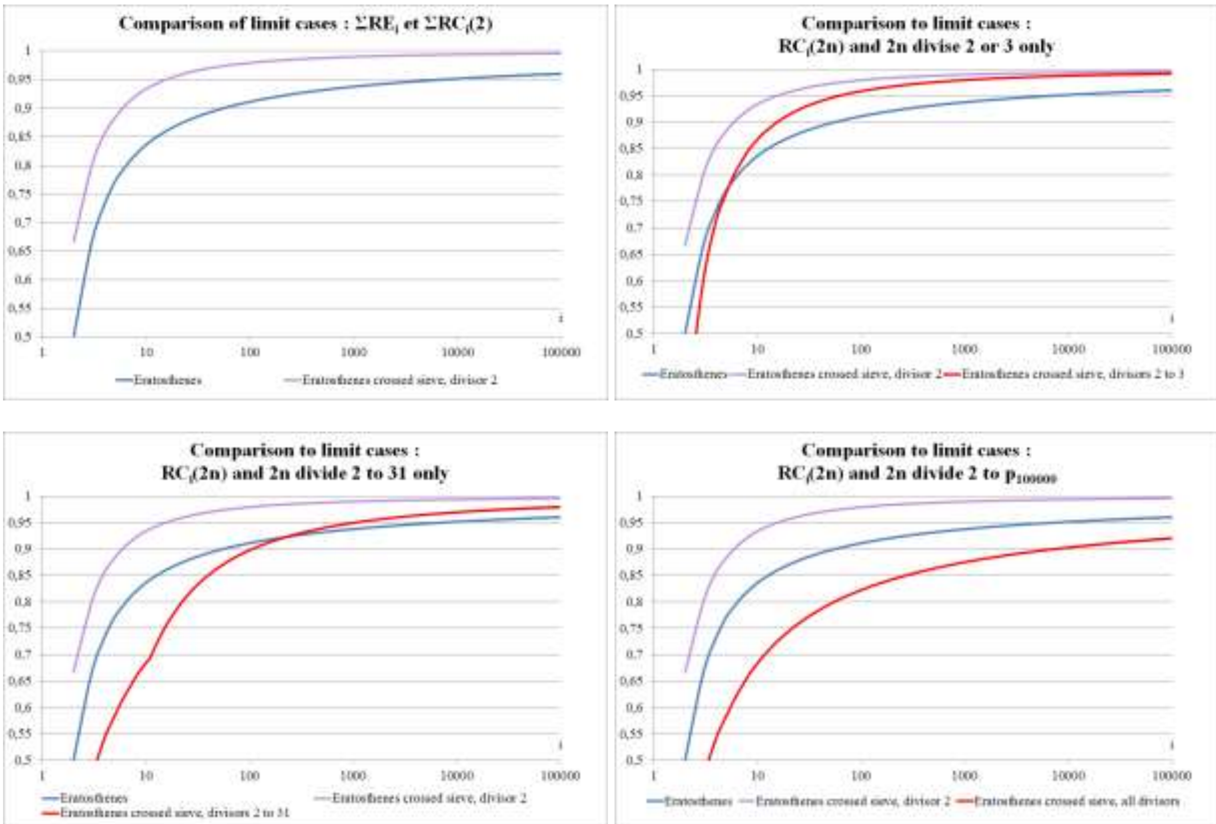
Note: We have not demonstrated the Hardy-Littlewood formula but simply retrieved the asymptotic proportions that are in it.

6.3.8. Comparative evolution of depletion coefficients.

The coefficients of depletion are at the heart of our study. Having the common property Σ_i #RC_i = 1, regardless of the choice of the gap p-q = 2n, in the same way as for Eratosthenes sieve (i.e. Σ_i #RE_i = 1), it is useful to take the time to compare their evolutions. To recognize different choices, we will use the notation #$RC_i(2n)$ for terms referring to the 2n gap.

There are two limit cases : The Σ_i #RE_i case of course and the Σ_i #$RC_i(2)$ case. The representative curves of all the others Σ_i#$RC_i(2n)$ cases are placed between these two limit cases from a certain rank i on (rank that can be as big as we want).
Thus, we have the following curves:

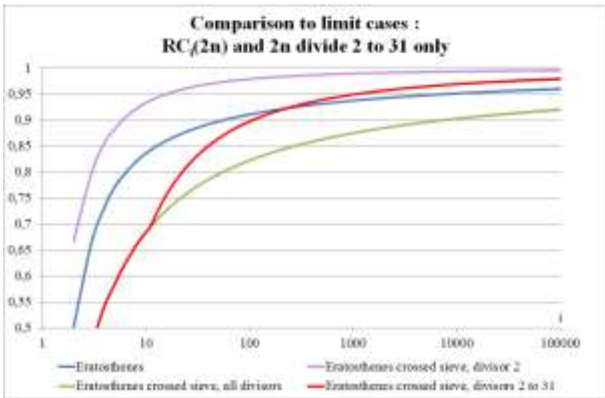
<u>Graphs 9, 10, 11 and 12</u>



The last curve is not an exception to limit cases that we have identified. Simply, the number of divisors is such that the red curve is still here below the blue curve at the stage i = 100000. It is necessary to extend the data very far to see these curves intersect and then the red curve going closer to the purple curve.
As contributions near the origin are finite, regardless of the chosen 2n value, these contributions are negligible before infinity and from a certain rank on the red curve will be much closer to the purple curve than from the blue curve, imposing then the result (i.e. a progression in $x/\ln^2(x)$).

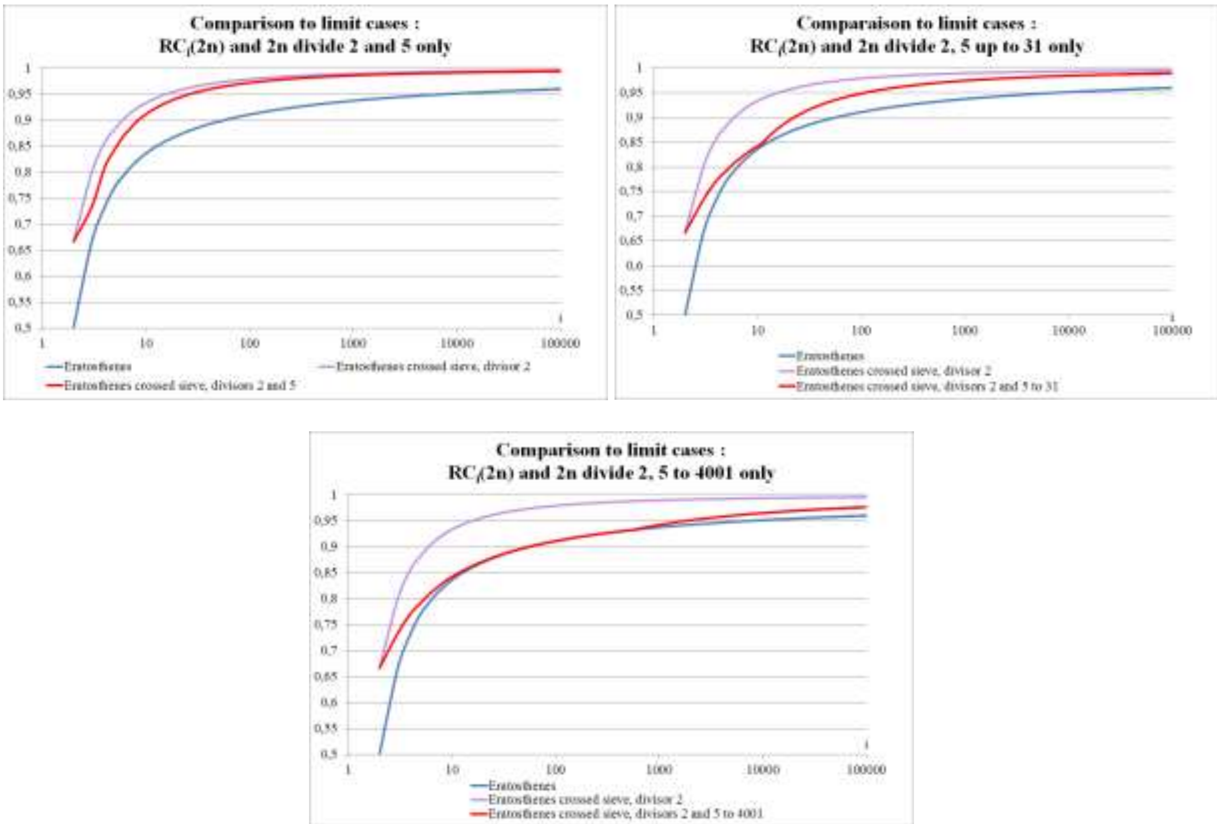
The green curve below, where 2n systematically contains all prime numbers up to a certain rank, is therefore a reference only up to a certain abscissa, any choice of n being necessarily finite. The red curve, corresponding to a gap where 2n systematically divides the prime numbers up to a certain rank p_i (here up to $p_i \leq 31$), goes along that same green curve up to the abscissa p_i (here $p_i = 31$) then going away above it.

Graph 13



Is a particularly interesting case where 3 is omitted in the list of the divisors of 2n, because it is no longer the previous limit curve (crossed Eratosthenes sieve green curve) that tangent partly the red curve but the curve blue (simple case of Eratosthenes sieve), and this starting when the chosen number of divisors becomes sufficient, tangential accompaniment being lost as soon as systematic dividers stop (here after p_{550} = 4001).

Graphs 14, 15 and 16



Of course, again, it is not because we can match Σ_i #$RC_i(2n)$ depletion curve, by a suitable choice, with Σ_i #RE_i upon as large range as we wish, that this changes anything on the overall behaviour of relative prime numbers at infinity.
Infinity is immeasurable, and regardless of the choice of n, the red curve will detach from the blue one to approach then the violet one. In other words, all the curves for p-q = 2n (and thus the depletion coefficients) are almost identical to those of p-q = 2 starting from a sufficiently large rank.
As the asymptotic contribution is the one that ensures the infinity of solutions, the conclusion is that p-q = 2n has either a finite number of solutions for any positive n or an infinite number of solutions for any positive n.

6.4. Landscaping of twin numbers spacings.

6.4.1. Generalities.

In this paragraph, we will establish the infinitely many twin primes in a relatively simple way. However this simplicity leads a strong underestimation of the asymptotic cardinal.

This paragraph follows paragraph 5.2 in which the spacings between primes in cycle 1 at step i of the Eratosthenes algorithm were analysed. It follows the said paragraph but is not its direct consequence. Thus we will see that the quasi-symmetry of the table 23's example for the sole prime numbers, if it possibly still exists for pairs of primes, is now no longer visible here.

The term landscaping is maintained here, but we use also architecture. We also note that previously there was no condition on primes and that so only one case was to be considered. On the contrary here, constraints are added to the integers which are objects of the study i.e. they are either twins, cousins, sexy, etc. This results in a special study for each of these cases, which cannot be done here exhaustively.

We will limit therefore often to the case of the architecture of the spacings between twin prime numbers (2n = 2). Specifically, we will study the architecture of the spacings between twin integers lacking small divisors, i.e. the twin integers of the Eratosthenes Eras(i) sets, hence the missing word "prime" in the paragraph's title. We list the spacings of an element to the previous and this one only. When we talk about element, we mean a pair of remaining numbers. The spacing is given by the distance between values in correspondence. For example, the spacing between the pair (3,5) and the pair (7,9) is equal to 9-5 = 7-3 = 4.

The study is done on an interval of size $\#p_i$. But the goal is to draw an interesting property that can be used over the interval $[p_i, p_i^2]$.

6.4.2. Basic idea.

The maximum spacing between integers in Eras(i) list is $2p_{i-1}$ (except for i = 8). Considering now pairs, assuming the best possible placement (positions values as relative primes), the occurrence of a maximum contingency appears a priori only once in doublet by forming an interval sum of the previous spaces, that is $\sum_i 2p_{k-1}$. We will check thereafter that the reality is somewhat different, especially that the maximum, although the order of magnitude is respected, can be larger and/or may be more numerous.

6.4.3. Panoramas od enumeration.

We start by enumerating spacings between twin numbers at steps 1 up to 7.

Table 34

Steps i	1	2	3	4	5	6	7
p_i	3	5	7	11	13	17	19
Cycle 1 sizes	6	30	210	2310	30030	510510	9699690
Spacings Δ	Quantity of spacings Δ in cycle 1						
6	1	1	3	21	189	2457	36855
12		2	8	56	504	6552	98280
18			2	22	238	3374	53690
24				6	96	1536	26208
30			2	22	270	4230	72378
36				4	60	1022	18776
42				4	84	1716	34812
48					20	474	10462
54						40	1968
60					12	380	9452
66					12	286	6322
72						64	2816
78						66	2620
84						12	632
90						24	1236
96						22	876
102							16
108						20	954
114							0
120							142
126							48
132							26
138							86
144							0
150							20

Steps i	1	2	3	4	5	6	7
Numbers of spacings	1	3	15	135	1485	22275	378675
Average spacings	6	10	14	17,11	20,22	22,92	25.61
c = $\Delta/\ln^2(pi)$	4,97	3,86	3,70	2,98	3,07	2,86	2,95

Other numerical data for cousins, sexy, etc. numbers are included in Appendix 5.

By construction, adding the spacings between integers, we find the overall magnitude of the cycle 1. So, using the values in the previous table, 1.6 = 6, 1.2+2.12 = 30, 3.6+8.12+2.18+2.30 = 210, etc.

The 6-spacings are in odd amounts, while others are in even-numbered quantities for the same reason as that given to the chapter of the spacings between prime numbers (lemma 2 page 15).

The number of spacings is equal to the number of signatures (here of value 2n = 2) and this one has already been evaluated in our study in table 26. It is equal to $\prod(p_k-2)$. The average spacing is thus equal to $2.\prod p_k/(p_k-2) \rightarrow c.\ln^2(p_i)$, the product bearing on i terms and c tending towards a constant, as i increases, according to the generalization of the Mertens theorem (c assessment is close to 2,4 around $p_i = 10007$).
Assuming a uniform random distribution, this average would be of the same order of magnitude in the interval p_i+2 to p_i^2-1 (as in the rest of the cycle 1), interval in which remain only prime numbers (twins of addition by construction). There is thus, when p_i becomes negligible in front of p_i^2, approximately $p_i^2/(c.\ln^2(p_i)) = (2/c).p_i^2/\ln^2(p_i^2))$ twin prime numbers in this interval, thus a growth proportional to $x/\ln^2(x)$.

Let us see then how quantities do increase when steps are incremented.

Table 35

Steps i	1	2	3	4	5	6	7
p_i	3	5	7	11	13	17	19
p_i-4		1	3	7	9	13	15
Cycle 1 sizes	6	30	210	2310	30030	510510	9699690
E(j) = Spacings Δ	#R(j,i) = quantity of spacings Δ at rank i/quantity of spacings at rank i-1						
6		1	3	7	9	13	15
12			4	7	9	13	15
18				11	10,82	14,18	15,91
24					16	16	17,06
30				11	12,27	15,67	17,11
36					15	17,03	18,37
42					21	20,43	20,29
48						23,7	22,07
54						∞	49,2
60						31,67	24,87
66						23,83	22,10
72							44
78							39,70
84							52,67
90							51,5
96							39,82
102							∞
108							47,7

Lemma 9

We have (when #R(j,i) exists) :

$$\#R(j,i) \geq p_i-4 \quad \text{and} \quad \#R(j,i) \underset{i \rightarrow +\infty}{\longrightarrow} p_i-4 \qquad (97)$$

Proof

For the second relationship, this ensues from Eratosthenes algorithm generating in the cycle 1 (and the followings) spacings E(j) growing necessarily at the level of a same x-coordinate. This creates a gradual saturation of small void spaces (starting with the smaller including 6 who is in this situation from the start), set of void spaces coming progressively in "standard" proportion, i.e. base proportion allocated by the depletion when two integers are taken into account simultaneously (and not one only), which is p_i-4. Indeed, recalling the lemma 1 (and theorem 12), we had 2 disappearances at each stage. But here these disappearances are matched (to a second element) and we have therefore 4 removals at each stage.

So we have in summary the three relationships:

$$\#S(j,i) = \prod^{i} p_k-2$$

$$\Delta(j).\#S(j,i) = \prod^{i} p_k$$

$$\#R(j,i) \geq p_i-4$$

The maximum value of Δ(j) = Δ(j,i) for which #S(j,i) is non-zero is highly conditioned for the condition #R(j,i) ≥ p_i-4 that acts as a counter-reaction : If at rank i we have a high value of Δ(j,i) max, then that is repeatedly carried over to the following ranks and especially at the expense of a new strong value of Δ(j+1,i) max. At page 127, appendix 11, we come up with simulations that show how difficult it is to "go through the roof."

Before resuming the study on columns, let us focus with lines. As we shall see, it would be relatively easy to deduce #S(j,i+1) from #S(j,i) data starting some rank i on provided one would have enough numerical values available beyond this rank i on a given j-line. Unfortunately, this is never the case. Indeed, the time required to get #S(j,i) populations is reasonable up to i = 9 (p_i = 29). It would take a month for i = 10 and probably several years for i = 11, etc. However, we will give the general principle of this assessment below from examples :

Conjecture 3

The #R(j,i) coefficients are expressed by a system of iterative relationships in j from a certain rank i on.

For the 2n = 2 case, the recurrence relationships are of similar structure (only coefficients changing) for j = 1 mod 2 and j+1 from a certain rank i on (for given j).

This is a complete reminder of the iterative relationships obtained in paragraph 5.2.2.
We give a number of examples as we did in the said paragraph :

Table 36

j	Δ	Formulas	Conditions
1	6	$\#S(1,i) = (p_i-4).\#S(1,i-1)$	i ≥ 2
2	12	$\#S(2,i) = (p_i-4).\#S(2,i-1)$	i ≥ 4
3	18	$\#S(3,i) = (p_i-4).\#S(3,i-1)+2^3.(p_{i-1}-6).(p_{i-2}-6)\ldots(p_3-6)$	i ≥ 4
4	24	$\#S(4,i) = (p_i-4).\#S(4,i-1)+ 2^5.3^2.(p_{i-1}-6).(p_{i-2}-6)\ldots(p_6-6)$	i ≥ 7

The recurrence applies for j = 4 at an earlier rank by replacing $(p_{i-1}-6).(p_{i-2}-6)\ldots(p_6-6)$ by 1. The values below have been checked up to rank i = 9. Beyond that rank, the values are speculative.
Let us note that the values of #S(j,i) in parentheses do not deduce from the iterative formulas.

i	p_i	#S(1,i)	#S(2,i)	#S(3,i)	#S(4,i)
1	3	(1)			
2	5	1	(2)		
3	7	3	(8)	(2)	
4	11	21	56	22	(6)
5	13	189	504	238	(96)
6	17	2457	6552	3374	1536
7	19	36855	98280	53690	26208
8	23	700245	1867320	1060150	539136
9	29	17506125	46683000	27184430	14178528
10	31	472665375	1260441000	749635250	398923200
11	37	15597957375	41594553000	25129354250	13567039200
12	41	577124422875	1538998461000	941919228250	514460232000
13	43	22507852492125	60020939979000	37159509136750	20500741404000

The writing of the iterative formulas for j = 3 and j = 4 was done in a concise form previously. It is equivalent to the following equation systems, namely 2 initial conditions and 2 linear equations (ax+b type):

Table 37

$j = 3, i \geq 4$	$x1(4) = 8$ $x1(i) = (p_{i-1}-6).x1(i-1)$ $\#S(3,3) = 2$ $\#S(3,i) = (p_i-4).\#S(3,i-1)+x1(i)$
$j = 4, i \geq 6$	$x1(6) = 288$ $x1(i) = (p_{i-1}-6).x1(i-1)$ $\#S(4,5) = 96$ $\#S(4,i) = (p_i-4).\#S(4,i-1)+x1(i)$

We can very well find the values of #S(j,i) up to $p_i = 29$ as previously calculated by replacing $p_{i-1}-6$ with p_{i-3}, the values of these numbers coinciding on a wide range :

p_i	3	5	7	11	13	17	19	23	29	31	37	41
$p_{i-1}-6$				1	5	7	11	13	17	23	25	31
p_{i-3}				3	5	7	11	13	17	19	23	29

The proposed formulas are therefore questionable, but what follows, reinforced by the similar formulas given earlier in Table 7, seems to prove us right for the choice we have made.

Beyond j = 4, a system of iterative relationships is much more practical of use than a unique concise relationship that is besides difficult to come forth with.
For j = 5, the coincidence of the results up to the rank i - 9 ($p_9 = 29$) can be expressed as follows :

Table 38

i	5	6	7	8	9	10	11	12	i
p_i	13	17	19	23	29	31	37	41	p_i
x1(i)			1008	9072	99792	1496880	31434480	722993040	$x1(i) = (p_{i-2}-8).x1(i-1)$
x2(i)		720	8928	125136	2227104	52720272	1349441280	42555672720	$x2(i) = (p_{i-1}-6).x2(i-1)+x1(i)$
#S(5,i)	270	4230	72378	1500318	39735054	1125566730	38493143370	1466801977410	$\#S(5,i) = (p_i-4).\#S(5,i-1)+x2(i)$

The ultimate case of the results we have been able to investigate, namely that of #S(7,i), appears easier to treat than that of #S(6,i) :

Table 39

i	4	5	6	7	8	9	10	11	12	i
p_i	11	13	17	19	23	29	31	37	41	p_i
x1(i)				768	2304	16128	145152	1886976	35852544	$x1(i) = (p_{i-3}-10).x1(i-1)$
x2(i)			288	2208	22176	260064	4046112	86855328	2033525088	$x2(i) = (p_{i-2}-8).x2(i-1)+x1(i)$
x3(i)		48	624	9072	140112	2641968	64811376	1707139728	54954856656	$x3(i) = (p_{i-1}-6).x3(i-1)+x2(i)$
#S(7,i)	4	84	1716	34812	801540	22680468	677184012	24054212124	944960705244	$\#S(7,i) = (p_i-4).\#S(7,i-1)+x3(i)$

Conjecture 4

The system of iterative relations (for the 2n = 2 case) involves int((j+1)/2) linear relations for int((j+1)/2) initial conditions at the j-line. The expression $(p_{i-k}-c_{i-k})$ within the linear relations follow reverse wise an incremental sequence {k = 0, k = 1, k = 2, …, k = m = int((j-1)/2)} with $\{c_i = 4, c_{i-1} = 6, c_{i-2} = 8, \ldots, c_{i-k} = 2k+4, \ldots, c_{i-m} = 2.\text{int}((j+3)/2)\}$ for the evaluation of #S(2m+1,i) and #S(2m+2,i).

This wholly recalls the series $\{c_i = 2, c_{i-1} = 3, c_{i-2} = 4, \ldots, c_{i-k} = k+2, \ldots, c_{i-n} = n+2\}$ that we met for isolate numbers spacings' populations calculation in paragraph 3.2.2.

We retrieve then effectively the couple of conditions given in relation (97).
Indeed, if we attribute to the iterations x1(i), x2(i), x3(i), …, $x_k(i)$, …, $x_n(i)$, #S(2n+1,i) the multiplying factors $p_{i-k}-(2k+4)$, then, whatever the initial values of $x_k(i)$ (in the previous example 768 for x1(7), 288 for x2(6), etc.), the ratio $x_{k-1}(i)/((p_{i-k}-(2k+4)).x_k(i-1))$ becomes negligible when i tends towards infinity because these multiplicative factors form a strictly increasing series $\{p_{i-n}-(2n+4), \ldots, p_{i-2}-8, p_{i-1}-6, p_i-4\}$, the distance between these latter values being at least 4.
This decrease of the contributions of $x_{k-1}(i)$ in $x_k(i) = (p_{i-k}-(2k+4)).x_k(i-1)+x_{k-1}(i)$ is shown underneath for the table 39's example :

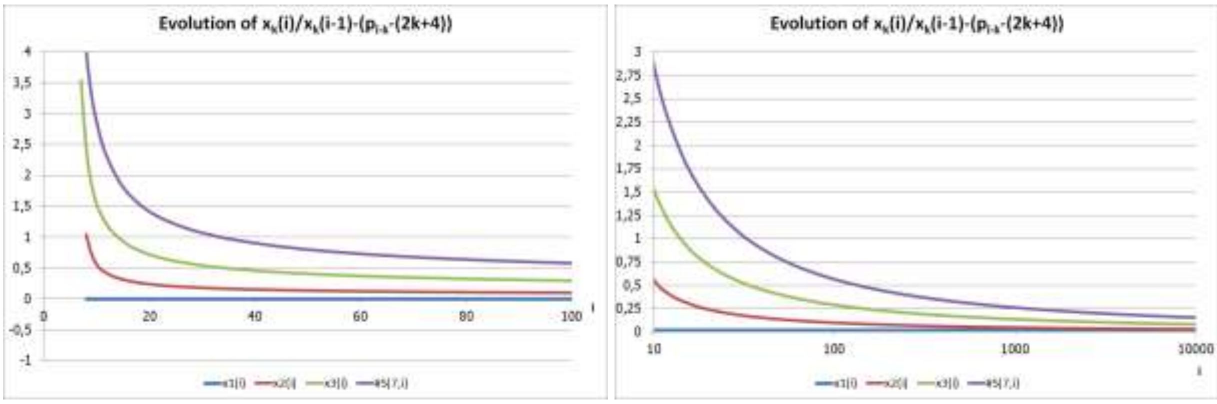
Table 40

i	5	6	7	8	9	10	11	12	13	14	15
p_i	13	17	19	23	29	31	37	41	43	47	53
x1(i)/x2(i)			0,34783	0,10390	0,06202	0,03587	0,02173	0,01763	0,01261	0,01021	0,00896
x2(i)/x3(i)		0,46154	0,24339	0,15827	0,09844	0,06243	0,05088	0,03700	0,03012	0,02642	0,02225
x3(i)/#S(7,i)	0,57143	0,36364	0,26060	0,17480	0,11649	0,09571	0,07097	0,05816	0,05106	0,04318	0,03564

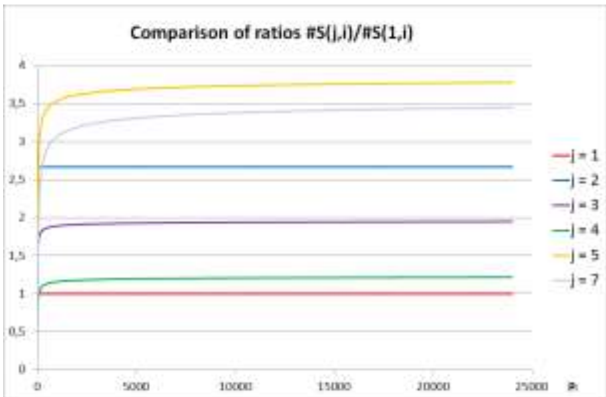
Therefore, we get then systematically $(x_k(i)-x_{k-1}(i))/x_k(i-1) \rightarrow p_{i-k}-(2k+4)$.

We show below, still for the table 39's example, the evolution of the values of $x_k(i)/x_k(i-1)-(p_{i-k}-(2k+4))$ versus i (p_{100} = 557, p_{10000} = 104743).

Graphics 17 and 18



As we did for #SP(j,i)/#SP(1,i) ratios at page 21, we can also have a look on the #S(j,i)/#S(1,i) ratios here. As before, we observe again, despite low (or not) initial values, an asymptotic catch-up of the said ratios with an order of magnitude of a unit.



On the basis of such a hypothesis, when i tends towards infinity, there is a constant c such as '$\prod_{i\rightarrow+\infty} (p_i-2) = \sum_j \#S(j,i\rightarrow+\infty) > c.j.\#S(1, i\rightarrow+\infty) = c.j.\prod_{i\rightarrow+\infty} (p_i-4)$. Hence $j < (1/c).\prod_{i\rightarrow+\infty} (p_i-2)/(p_i-4)$ and, using the generalization of the Mertens theorem, we conclude that there is a constant c' such as :

$$j < c'\ln^2(p_i) \qquad (98)$$

The order of magnitude of the number of lines j at sequence i is thus asymptotically in $\ln^2(p_i)$.

It should be noted, however, that in the absence of a proper proof, the specified general form is only a matter of assumption and coincidence.
Beyond this lack, the difficult part of this construction game is also the anticipation of the whole "random" part of the first values on a given j-line. As such, we give below the initial values that we have been able to determine. The reader will be able to compare this table to table 12. In particular, the first initial value is not systematically the first non-zero value of the j-line.

Table 41

	Steps i	1	2	3	4	5	6	7	8	9
	p_i	3	5	7	11	13	17	19	23	29
j	Δ(j)									
1	6	1	1	3	21	189	2457	36855	700245	17506125
2	12		2	8	56	504	6552	98280	1867320	46683000
3	18			2	14+8	238	3374	53690	1060150	27184430
4	24				6	96	1248+288	26208	539136	14178528
5	30			2	22	270	3510+720	71370+1008	1500318	39735054
6	36				4	60	1022	18776	356744+36720	10460840+36480
7	42				4	36+48	1428+288	34044+768	801540	22680468
8	48					20	474	10462	275040	8256720
9	54						40	1240+728	65712+3576	2472660+6540
10	60					12	240+140	8864+588	241720+1650	7359158+456608
11	66					12	286	6322	166526	5067262
12	72						64	2816	94492	3197558
13	78						66	2046+574	80828+2884	2844932+183268
14	84						12	632	25912+1044	1009376+17028
15	90						24	744+492	41856+1686	1548726+162342
16	96						22	876	27136	948278
17	102							16	704+3680	251328+13018
18	108						20	620+334	31536+790	1125562+68454
19	114								440	54546
20	120							142	7852	387506
	…							…	…	…

The initial values of the lines without values in red font could not be determined with certainty. We have at this point the following :

Lines j	1	2	3	4	5	6	7	8	9	10
Number of linear equations or initial values needed	1	1	2	2	3	3?	4	4?	≥4 (5?)	≥5

In Appendix 6, we present a number of cases beyond the 2n = 2 example.
The same remarks of caution must be taken into account there as well.

6.4.4. Generative process.

The existence of recursive relationships is linked to the same process observed in the case of pseudo-primes. It revolves around groupings modulo $\#p_i/p_k$ where p_k is the decreasing list of the primary dividers of the first $\#p_i$.
The implementation of the sorting, in any way analogous to the said case, is described below.

Method of sorting.

Starting from the pseudo-twin-primes covering an interval $[x_0, x_0+p_0p_1p_2\ldots p_i[$, $(x_0 > p_i)$, we have $(p_1-2)(p_2-2)\ldots(p_i-2)$ integers remaining. These are arranged according to the increasing values of the spacings (to the previous ones).

The integers x with 6-spacing are sorted according to the increasing values of x modulo $p_0p_1p_2\ldots p_i/p_i$. They appear in families of p_i-4 identical modulo values. The total amount of elements responds to a system to one recursive equation. For spacing 12, the routine is similar.

The integers with 18-spacing are sorted according to the increasing value of x modulo $p_0p_1p_2\ldots p_i/p_i$. Those who appear in families with p_i-4+pos identical modulo values, where pos is a positive or null cardinal, are gathered apart. The others appearing modulo $p_0p_1p_2\ldots p_i/p_{i-1}$ in families with p_{i-1}-6+pos identical modulo values, where pos is a positive or null cardinal, are ranked on their side. The set responds to a system with two recursive equations.

…

The integers x with 6j-spacing are sorted according to the increasing value of x modulo $p_0p_1p_2\ldots p_i/p_i$. Families with p_i-4+pos identical modulo values, where pos is a positive or null cardinal, are gathered apart when they exist. We then proceed in the same way modulo $p_0p_1p_2\ldots p_i/p_{i-k}$, k being gradually incremented, making groups of integers giving p_{i-k}-4-2k+pos identical modulo values, where pos is a positive or null cardinal, at sequence k+1.

We do this until the stock runs out. The number of sorting, at a given spacing, cannot exceed i. The resulting recursive system cannot have more than i equations.

Particular feature versus the pseudo-primes case.

The remarkable point is the existence of corrective factors for the cardinals of modulo-families. We noted this factor by "pos". This correction is always positive or null, in other words families are supernumerary. At least they are so initially. Indeed, the said factor will gradually evolve, possibly erratically, towards zero when step i increases. This is illustrated below by a few examples. Several values of coefficients pos (and therefore of the cardinal of families) are possible simultaneously for a given situation and these variability when occurring is transcribed below in the same box of our tables.

The first term of a line is not derived from a modulo grouping. It does not give rise to a multiplier factor. The arbitrary simulation of the “pos” factor (given in parentheses below) can therefore give a negative value. However, this negative value usually appears only on the first line of the lower diagonal.

$\Delta(1) = 6$

p_i	5	7	11	13	17	19	23
Factors	1*(5-4) = 1	1*(7-4) = 3	3*(11-4)= 21	21*(13-4) = 189	189*(17-4) = 2457	2457*(19-4) = 36855	36855*(23-4) = 700245

Pos	(0)	0	0	0	0	0	0

$\Delta(2) = 12$

p_i	5	7	11	13	17	19	23
Factors	1*(5-3)= 2	2*(7-3)= 8 0*(7-4)= 0	8*(11-4) = 56	56*(13-4) = 504	504*(17-4) = 6552	6552*(19-4) = 98280	98280*(23-4) = 1867320

Pos	(1)	1	0	0	0	0	0

$\Delta(3) = 18$

p_i	7	11	13	17	19	23
Factors	2*(7-6) = 2 0*(7-5) = 0 0*(7-4) = 0	2*(11-4) = 14	22*(13-4) = 198	238*(17-4) = 3094	3374*(19-4) = 50610	53690*(23-4) = 1020110
Factors		4*(7-5) = 8 0*(7-6) = 0	8*(11-6) = 40	40*(13-6) = 280	280*(17-6) = 3080	3080*(19-6) = 40040

Pos	(-2)	0	0	0	0	0
Pos		(1)	0	0	0	0

$\Delta(4) = 24$

p_i	11	13	17	19	23
Factors	3*(11-9) = 6 0*(11-8) = 0 … 0*(11-4) = 0	6*(13-3) = 60 0*(13-4) = 0	96*(17-4) = 1248	1536*(19-4) = 23040	26208*(23-4) = 497952
Factors		6*(11-5) = 36 0*(11-6) = 0	36*(13-5) = 288 0*(13-6) = 0	288*(17-6) = 3168	3168*(19-6) = 41184

Pos	(-5)	1 et 0	0	0	0
Pos		(1 et 0)	1	0	0

Δ(5) = 30

p_i	7	11	13	17	19	23
Factors	2*(7-6) = 2 0*(7-5) = 0 0*(7-4) = 0	2*(11-4) = 14	22*(13-4) = 198	270*(17-4) = 3510	4230*(19-4) = 63450	72378*(23-4) = 1375182
Factors		8*(7-6) = 8	8*(11-6) = 40	16*(13-5) = 128 64*(13-6) = 448	720*(17-6) = 7920	8928*(19-6) = 116064
Factors			16*(7-5) = 32 0*(7-6) = 0 … 0*(7-8) = 0	24*(11-7) = 96 16*(11-8) = 48	128*(13-7) = 768 48*(13-8) = 240	1008*(17-8) = 9072

Pos	(-2)	0	0	0	0	0
Pos		(0)	0	1 et 0	0	0
Pos			(3)	1 et 0	1 et 0	0

Δ(6) = 36

p_i	11	13	17	19	23
Factors	2*(11-9) = 4 0*(11-8) = 0 … 0*(11-4) = 0	4*(13-4) = 36	60*(17-3) = 840 0*(17-4) = 0	1022*(19-3) = 16352 0*(19-4) = 0	18776*(23-4) = 356744
Factors		4*(11-6) = 20	22*(13-6) = 154	182*(17-5) = 2184 0*(17-6) = 0	2424*(19-5) = 33936 0*(19-6) = 0
Factors		4*(7-6) = 4 0*(7-7) = 0 0*(7-8) = 0	2*(11-7) = 8 0*(11-8) = 0	16*(13-7) = 96 16*(13-8) = 80	240*(17-7) = 2400 0*(17-8) = 0
Factors			20*(7-6) = 20 … 0*(7-10) = 0	32*(11-9) = 64 0*(11-10) = 0	96*(13-9) = 384 0*(13-10) = 0

Pos	(-5)	0	1	1	0
Pos		(0)	0	1	1
Pos		(2)	1	1 et 0	1
Pos			(4)	1	1

Δ(7) = 42

p_i	11	13	17	19	23
Factors	2*(11-9) = 4 0*(11-8) = 0 … 0*(11-4) = 0	4*(13-4) = 36	84*(17-4) = 1092	1716*(19-4) = 25740	34812*(23-4) = 661428
Factors		4*(11-5) = 24 0*(11-6) = 0	8*(13-5) = 64 44*(13-6) = 308	624*(17-6) = 6864	9072*(19-6) = 117936
Factors		24*(7-6) = 24 0*(7-7) = 0 0*(7-8) = 0	12*(11-6) = 60 16*(11-7) = 64 0*(11-8) = 0	208*(13-7) = 1248 96*(13-8) = 480	2208*(17-8) = 19872
Factors			64*(7-5) = 128 0*(7-6) = 0 … 0*(7-10) = 0	160*(11-8) = 480 0*(11-9) = 0 0*(11-10) = 0	576*(13-9) = 2304 0*(13-10) = 0

Pos	(-5)	0	0	0	0
Pos		(1)	1 et 0	0	0
Pos		(2)	2 et 1	1	0
Pos			(5)	2	1

To obtain all the "pos" coefficients equal to 0 for Δ(6) = 36 and Δ(7) = 42, one would have to consider at least extending the calculations up to p_i = 29 which implies computing out of reach (one month of calculation for each of the objects + memory space problem on Pari GP).

6.4.5. Extrema research.

Let us now observe the maximum spacing by providing an array of values for steps 1 up to 10 to start with.

Table 42

Steps i	1	2	3	4	5	6	7	8	9	10
p_i (column guide divisor)	3	5	7	11	13	17	19	23	29	31
Em(i) = Max spacings	6	12	30	42	66	108	150	204	258	348
Sum(i) $= \sum_i 2p_k$	6	16	30	52	78	112	150	196	254	316
Diff = Em(i)-Sum(i)	0	-4	0	-10	-12	-4	0	8	4	32
Diff/Sum(i)	0,00%	-25,00%	0,00%	-19,23%	-15,38%	-3,57%	0,00%	4,08%	1,57%	10,13%

Let us recall that the maximum spacing between prime numbers at the i-stage is, according to hypothesis 2 (page 39) and theorem 11, equal to something like $2p_i$. Everything now goes, for the twin numbers without small divisors (Eras(i) effective divisors greater than p_i) remaining in step i, as if one has to take into account, for the order of magnitude of the maximum of the spacings Em(i), the sum of the $2p_k$, k = 1 to i.

We give, to visualize things, two tables corresponding to maximum spacings. We see the pairs of numbers up to their joint disappearances when one of them (of the pair) displays the guide divisor of the column. We see no obvious correlation to pass from one to the other. The difficulty lies in the fact that the maximum at step i does not inherit from the maximum at rank i-1. In addition, unlike the graphic evidence of the construction scheme of the maximum spacing in the case of the prime numbers (and its quasi-symmetry according to the table 23 example), there is no such thing here :

Tables 43 and 44

$p_i = 11$

		3	5	7	11
-2	1367	1367	1367	1367	1367
0	1369	1369	1369	1369	1369
2	1371				
4	1373	1373			
6	1375	1375			
8	1377				
10	1379	1379	1379		
12	1381	1381	1381		
14	1383				
16	1385	1385			
18	1387	1387			
20	1389				
22	1391	1391	1391		
24	1393	1393	1393		
26	1395				
28	1397	1397	1397	1397	
30	1399	1399	1399	1399	
32	1401				
34	1403	1403			
36	1405	1405			
38	1407				
40	1409	1409	1409	1409	1409
42	1411	1411	1411	1411	1411

$p_i = 13$

		3	5	7	11	13
-2	3851	3851	3851	3851	3851	3851
0	3853	3853	3853	3853	3853	3853
2	3855					
4	3857	3857	3857			
6	3859	3859	3859			
8	3861					
10	3863	3863				
12	3865	3865				
14	3867					
16	3869	3869	3869			
18	3871	3871	3871			
20	3873					
22	3875	3875				
24	3877	3877				
26	3879					
28	3881	3881	3881	3881		
30	3883	3883	3883	3883		
32	3885					
34	3887	3887	3887	3887	3887	
36	3889	3889	3889	3889	3889	
38	3891					
40	3893	3893				
42	3895	3895				
44	3897					
46	3899	3899	3899			
48	3901	3901	3901			
50	3903					
52	3905	3905				
54	3907	3907				
56	3909					
58	3911	3911	3911			
60	3913	3913	3913			
62	3915					
64	3917	3917	3917	3917	3917	3917
66	3919	3919	3919	3919	3919	3919

The maximum at step i depends on the best arrangement and is questioned at every new step.
In contrast, even though there may be several solutions, the maximum comes around a relatively fixed pattern. Constraints are limiting the possible variations of the maximum.

Let us look at a concrete example with case p_i = 17, which gives 20 solutions of maximum spacings 108 :

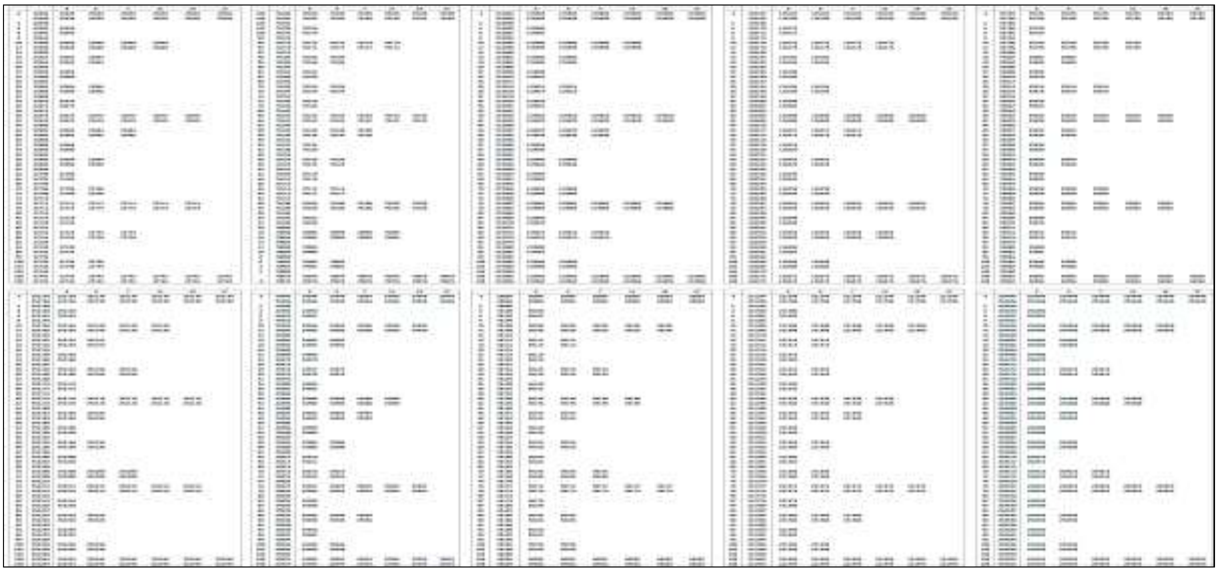
Table 45

List 1	List 2
(22634,22636) ; (22742,22744)	(487766,487768) ; (487874,487876)
(24944,24946) ; (25052,25054)	(485456,485458) ; (485564,485566)
(55784,55786) ; (55892,55894)	(454616,454618) ; (454724,454726)
(58094,58096) ; (58202,58204)	(452306,452308) ; (452414,452416)
(70076,70078) ; (70184,70186)	(440324,440326) ; (440432,440434)
(126164,126166) ; (126272,126274)	(384236,384238) ; (384344,384346)
(218984,218986) ; (219092,219094)	(291416,291418) ; (291524,291526)
(221294,221296) ; (221402,221404)	(289106,289108) ; (289214,289216)
(252134,252136) ; (252242,252244)	(258266,258268) ; (258374,258376)
(254444,254446) ; (254552,254554)	(255956,255958) ; (256064,256066)

The list 2 is symmetric of list 1 modulo 2.3…p_i (for example 22634+487876 is 510510).

We give below the evolution of the remaining pairs based on step i. The shadows formed by the surviving pairs are relatively similar views by far. They are identical for a pair and its symmetrical pair (this last is not represented). The reader will be able to clearly view the tables at appendix 7.

Table 46

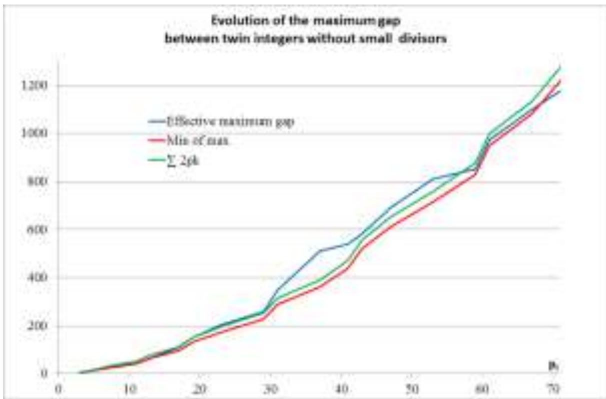


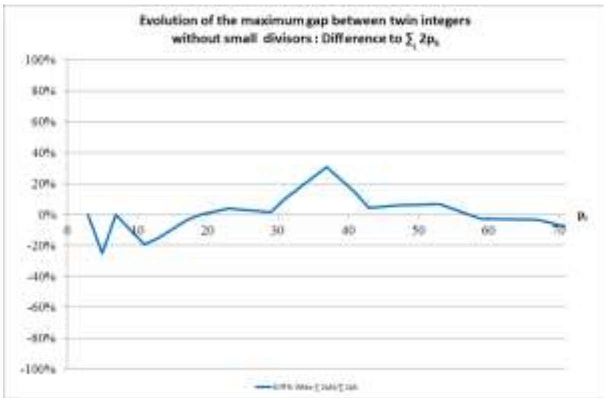
Continuing the routine, our table, initially table 42, is as follows. However our search for the maximum is not exhaustive below as a result of the excessive number of cases to be examined (even with computer means), hence the Ep(i) proposal pending some final value Em(i) :

Table 47

Steps i	11	12	13	14	15	16	17	18	19
p_i (guide divisor)	37	41	43	47	53	59	61	67	71
Ep(i) (<= Max spacings Em(i))	510	540	582	690	810	852	972	1098	1176
Sum(i) = $\sum_i 2p_k$	390	472	558	652	758	876	998	1132	1274
Diff = Ep(i)-Sum(i)	120	68	24	38	52	-24	-26	-34	-98
Diff/Sum(i)	30,77%	14,41%	4,30%	5,83%	6,86%	-2,74%	-2,61%	-3,00%	-7,69%

Graphs 19 and 20





In fact, we see large getaways compared to the expected ideal values in one way or another, but also very close values. We present this continuation of table 42 to show that large deviations with priori expected values may exist. Here, when the value is greater to the awaited $\sum_i 2p_k$, the relative difference is minimal (this may be actually more than what is displayed), especially for $p_i = 37$. Conversely, this relative difference may dwindle when the value is lower (for example, $p_i = 61$). But in fact no matter the exact value at a given stage as we will soon see, only matter the general trend.

6.4.6. Algorithmic background.

Research methods of the maximum spacing.

We used two methods.

The first is a systematic method by recording all of the spacings of amplitude Δ(j) throughout the cycle 1. As knowledge of intermediate maximums is got, one can make larger jumps in the search for the pair of numbers in Eras(i) in order to limit the number of verifications. It is possible to operate this way up to $p_i = 31$ (on Pari GP several weeks of calculations are however necessary).
This method ensures that the said maximum is actually the good one.

The second is a random method allied with a "Newton lift".
It is modelled in table 48 below (for the case $p_i = 19$). By the arrows ↑↓, we mean that the set of numbers below some column can be shifted by a same pace upwards or downwards. Of course, doing this, the results on the left side will be changed. The method is then to look for increasingly large values of the spacings by shifting values. These offsets are made systematically on a given column: for example in column $p_i = 11$ by shifting 1, then 2, then 3,... up to 10. Shift of 11 (and then more shifting) however would serve no purpose since giving an analogous feature to the original (then 12 to 21, etc.). The solution of larger spacing is retained then another column is chosen at random and the process is repeated. When the process reaches saturation, i.e. if the obtained maximum increases no more after many tests, the result is saved and a reset is made leading to a new maximum and the greatest of this and the previous is selected, etc. The method, employed here from $p_i = 37$ on, has the disadvantage that it does not ensure that the maximum found after many tests is actually the largest existing.

Note 1 :
However, we have a relatively good confidence in the results presented in table 47. Indeed, for $p_i = 31$, for example, the first method requires several weeks to be exhaustive, of which several days to reach the first maximum value (on the Pari GP online tool), when the second method gives the right configuration of the maximum often (as random and therefore subject to large variation) in less than a minute (on standard Excel spreadsheet).

Note 2 :
The interest and the effectiveness of the second method also reside in the fact that it is close to the real phenomenon of production of the maximum spacing, as discussed below, drawing the reason for the limitation of the maximum reached.

Table 48

Arbitrary scale	Min and max for spacing evaluation (= 30 here)	Detection of pairs (when result = 2)	Divisors identification	3	5	7	11	13	17	19
				↑↓	↑↓	↑↓	↑↓	↑↓	↑↓	↑↓
Etc.		0	2	1		1				
-26		1	0							
-24	-24	2	0							
-22		0	2	1			1			
-20		0	1		1					
-18		1	0							
-16		0	1	1						
-14		0	1			1				
-12		0	1							1
-10		0	2	1	1					
-8		0	1					1		
-6		1	0							
-4		0	1	1						
-2		0	1						1	
0		0	3		1	1	1			
2		0	1	1						
4		1	0							
6	6	2	0							
Etc.		0	1	1						

The search can also be done in a systematic way with this second method. If undertaken in this way all of the spacings are obtained with the following occurrences :

Table 49

Step i	1	2	3	4	5	6	7
p_i (guide divisor)	3	5	7	11	13	17	19
Cycle 1 size	6	30	210	2310	30030	510510	9699690
Spacings Δ	Quantity of occurrences of spacings Δ						
6	1	1	3	21	189	2457	36855
12		4	16	112	1008	13104	196560
18			6	66	714	10122	161070
24			0	24	384	6144	104832
30			10	110	1350	21150	361890
36				24	360	6132	112656
42				28	588	12012	243684
48					160	3792	83696
54					0	360	17712
60					120	3800	94520
66					132	3146	69542
72						768	33792
78						858	34060
84						168	8848
90						360	18540
96						352	14016
102						0	272
108						360	17172
114							0
120							2840

Step i	1	2	3	4	5	6	7
p_i (guide divisor)	3	5	7	11	13	17	19
Cycle 1 size	6	30	210	2310	30030	510510	9699690
Spacings Δ	Quantity of occurrences of spacings Δ						
126							1008
132							572
138							1978
144							0
150							500
Number of incidences	1	5	35	385	5005	85085	1616615
Ratio to the previous		5	7	11	13	17	19

The number of occurrences for $p_i = 3$ here is 1, because it is impossible to change positions in the first column.

If we then compare the cardinal of the spacing Δ in cycle 1 and cardinal of the occurrences of the spacing Δ by the last systematic method used here, we find a ratio with regular increment 1 when the spacing is incremented (of 6), namely the cardinal is identical for spacing 6, then doubled for spacing 12, then tripled for spacing 18, etc.

We have not tried to find here the profound nature of this result. But it promotes (a little) the research of large spacings with the random method. While in principle we get 20/22275 spacings of amplitude 108 (0.090%) for $p_i = 11$, we have 360/85085 (0,423%) chances of randomly finding (which is not surprising since bigger than others).

Note:
The same rule for ratios occurs for any other values of 2n.

6.4.7. Classes.

Of course, a vital result would be to have the number of incidences of each Δ spacing. Systematic method, although basic, finds its limit in computation time. Another way to approach the subject of this count is considering enumeration results by classes, namely $2.3\ldots p_i$, and therefore to proceed modulo 6, then modulo 30, then modulo 210, etc.

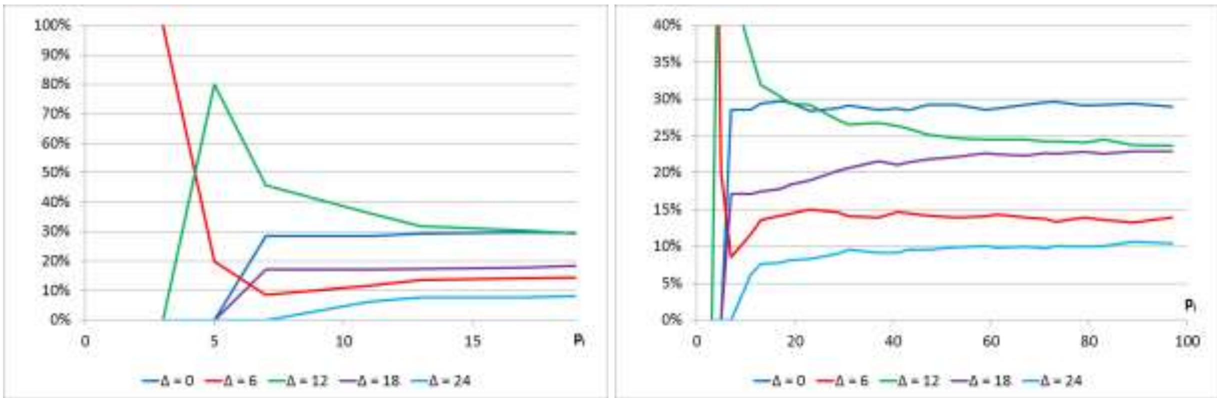
Modulo 6, count is trivial. There is a single class to 0 modulo 6.
Modulo 30, there are 5 classes with underneath tables of results :

Table 50 / Table 51

Δ mod 30 \ guide p_i	3	5	7	11	13	17	19	23	29	31	37	41
0	0%	0%	28,6%	28,6%	29,4%	29,7%	29,6%	28,4%	28,9%	29,1%	28,5%	28,8%
6	100%	20%	8,6%	11,7%	13,6%	14,2%	14,5%	15,0%	14,6%	14,1%	14,0%	14,0%
12	0%	80%	45,7%	36,4%	31,9%	30,4%	29,4%	29,2%	27,1%	26,6%	26,8%	25,5%
18	0%	0%	17,1%	17,1%	17,5%	17,8%	18,4%	19,0%	20,4%	20,7%	21,6%	22,0%
24	0%	0%	0,0%	6,2%	7,7%	7,8%	8,1%	8,3%	9,1%	9,6%	9,1%	9,7%

Δ mod 30 \ guide p_i	43	47	53	59	61	67	71	73	79	83	89	97
0	28,5%	29,2%	29,3%	28,6%	29,7%	29,2%	29,5%	29,7%	29,1%	29,2%	29,5%	29,0%
6	14,5%	14,2%	14,0%	14,1%	14,2%	14,0%	13,7%	13,3%	13,9%	13,6%	13,0%	13,9%
12	26,1%	25,2%	24,7%	24,5%	24,7%	24,5%	24,3%	24,3%	24,1%	24,5%	23,8%	23,7%
18	21,5%	21,9%	22,2%	22,7%	22,1%	22,3%	22,7%	22,6%	22,9%	22,6%	23,5%	22,9%
24	9,6%	9,5%	9,9%	10,1%	9,3%	10,0%	9,8%	10,1%	10,0%	10,1%	10,2%	10,4%

Graphs 21 and 22



In these tables, we find the exact percentages of modulo 30 spacings up to p_i = 19. Beyond that, it is a statistical assessment. The asymptotic proportions seem to be around :

Δ mod 30	0	6	12	18	24
Proportions	9/30	4/30	7/30	7/30	3/30

The modulo 210 study offers nothing remarkable statistically at the stage where we could carry it, the question being the asymptotic proportions are they integers' ratios of n_k/210 type ?

6.4.8. Configurations.

For the understanding of the presentation, let us take an example to clarify the notion of configuration with the table below :

Tableau 52

Configuration abscissa	Detection of pairs (when result = 2)	Divisors identification	3	5	7	11
/			↑↓	↑↓	↑↓	↑↓
/	1	0				
/	2	0				
0	0	2	1			1
1	0	1		1		
2	1	0				
3	0	1	1			
4	0	1			1	
5	1	0				
6	0	2	1	1		
7	1	0				
8	2	0				
Etc.	…	…	…	…	…	…
	Configuration value (here)		0	1	4	0

A configuration is identified by positions' abscissas. The position the 3-guide dividers are settled on either side of a pair of paired numbers (a pair of Eras(i) without small dividers up to the chosen stage). The abscissa just after the said pair is taken equal to 0, and then incremented, which then defines the other positions. They necessarily take, in the p_i column, values between 0 and p_i-1.
Here the previous example gives the following configuration :

0	1	4	0

For this configuration, which is limited here to p_i = 11, the spacing between pairs is 9*2 = 18.

6.4.9. Spacings generated by the sieve.

Lemma 10

The maximum spacing between pairs potentially generated by the second research method (random way or not) is less than or equal to $\sum_i 2p_k$.

Proof

Starting the configuration (0 0 0 ... 0) to which corresponds a spacing of 6, we do vary it to reach one of the configurations having maximum spacing. Let us suppose that we are omniscient. We know the final configuration and to achieve it, it is necessary not more than $\sum_i (p_k-1)$ offsets (roughly $\sum_i p_k$ offsets) of the initial elements, as a 0 modulo p_i offset of the column of divider guide p_i leads to an identical configuration (and unchanged spacing). A 1*2-shift (manipulating here only odd integers) has a mechanical effect, except of random noise, which means supplementary spacing of 2. On average, each of the efficient offsets pushing sometimes higher, sometimes lower boundaries by 2, we then consider the worst case to our argumentation (which produced the biggest spacing and therefore the maximum rarefaction of pairs of twins), namely the necessity to exhaust all of the modulo p_i paths where each of these induces a systematic (of 2) increase on the resulting spacing.
Hence the result.

Note :

In the previous lemma, we are not saying that the spacing between pairs cannot be greater than $\sum_i 2p_k$, but only what generates this spacing cannot act beyond $\sum_i 2p_k$.

Theorem 22

The maximum spacing between Eras(i) pairs is of the order of magnitude of $\sum_i 2p_k$.

Proof / Addenda to the proof

It is a simple repetition of the previous lemma to which we add a set of reframing remarks :

The attentive reader already knows that spacings Δ are all multiples of 6 and therefore evolve at least by leaps of 6. To reproduce the algorithm for our example, we do so by 3 shifts, each worth 2. Offsets can be either all positive or all negative, but must have the same sign to reproduce the algorithm leading to the maximum. They can be spread over one or more columns (up to 3 columns). To finish the total displacement in a given column i must be less than p_i, value, value called guide divisor of the said column.

The evolution of the previous configuration for a gain (or loss) of 6 can be, among others, one of the following solutions :

Example 1 (positive shift on a unique column):

Guide divisor	3	5	7	11
Initial config.	0	1	4	0
+	0	3	0	0
Final config.	0	4	4	0

Example 2 (positive shifts on several columns):

Initial config.	0	1	4	0
+	0	1	0	2
Final config.	0	2	4	2

Example 3 (negative shifts):

Initial config.	0	1 (= 6)	4	0 (= 11)
-	0	2	0	1
Final config.	0	4	4	10

The set of possibilities increases exponentially with p_i.

Having arbitrarily chosen p_i a maximum step, we try next to visualize the possibilities of gradual transition of a configuration which is associated with the minimum spacing 6 to a final configuration giving the maximum spacing. We then ask ourselves the following two questions:
- Is there a series of configurations leading from the smallest spacing to the largest one, configurations whose respective shifts correspond to the so-called spacings?
- If such series exist, is it possible to find one among them without exceeding a total p_k shift (ideally strictly inferior to p_k)

within each of the p_k columns from 3 to p_i ?

For p_i = 5, there are 5 possible configurations for which spacings are given in the last column below :

Guide divisor p_i	3	5	Spacings
Configuration 1	0	0	6
Configuration 2	0	1	12
Configuration 3	0	2	12
Configuration 4	0	3	12
Configuration 5	0	4	12

“Logical” passages from the 6-spacing configuration to the 12-spacing configuration are the following (one case in positive progress and its symmetrical in negative growth) :

p_i	3	5
6	0	0
+	0	3
12	0	3

p_i	3	5
6	0	0
-	0	3
12	0	2

Here the two previous questions find an affirmative answer.

For p_i = 7, the list of configurations is somewhat longer :

6	0	0	0
+	0	1	2
12	0	1	2

6	0	0	0
+	0	2	1
12	0	2	1

6	0	0	0
+	0	3	0
12	0	3	0

6	0	0	4
+	0	3	0
12	0	3	4

12	0	1	0
+	0	2	1
18	0	3	1

12	0	1	1
+	0	0	3
18	0	1	4

12	0	1	3
+	0	3	0
18	0	4	3

12	0	2	2
+	0	2	1
18	0	4	3

12	0	2	2
+	0	3	0
18	0	0	2

12	0	2	3
+	0	0	3
18	0	2	6

12	0	3	4
+	0	3	0
18	0	1	4

12	0	3	4
+	0	2	1
18	0	0	5

12	0	3	5
+	0	0	3
18	0	3	1

12	0	4	0
+	0	1	2
18	0	0	2

12	0	4	0
+	0	0	3
18	0	4	3

12	0	4	6
+	0	3	0
18	0	2	6

18	0	0	2
+	0	0	6
30	0	0	1

18	0	0	5
+	0	6	0
30	0	1	5

18	0	0	5
+	0	5	1
30	0	0	6

18	0	1	4
+	0	6	0
30	0	2	4

18	0	1	4
+	0	5	1
30	0	1	5

18	0	1	4
+	0	4	2
30	0	0	6

18	0	2	6
+	0	0	6
30	0	2	5

18	0	3	1
+	0	6	0
30	0	4	1

18	0	3	1
+	0	5	1
30	0	3	2

18	0	4	3
+	0	1	5
30	0	0	1

18	0	4	3
+	0	0	6
30	0	4	2

The transition solutions, answering to the first question, are :

Table 53

Guide divisor	3	5	7
6	0	0	4
+	0	3	0
12	0	3	4
+	0	3	0
18	0	1	4
+	0	4	2
30	0	0	6
	0	10	2

Guide divisor	3	5	7
6	0	0	4
+	0	3	0
12	0	3	4
+	0	2	1
18	0	0	5
+	0	5	1
30	0	0	6
	0	10	2

Guide divisor	3	5	7
+	0	3	0
12	0	3	4
+	0	3	0
18	0	1	4
+	0	5	1
30	0	1	5
	0	11	1

Guide divisor	3	5	7
+	0	3	0
12	0	3	4
+	0	2	1
18	0	0	5
+	0	6	0
30	0	1	5
	0	11	1

Guide divisor	3	5	7
+	0	3	0
12	0	3	4
+	0	3	0
18	0	1	4
+	0	6	0
30	0	2	4
	0	12	0

Negative "smooth" progressions configurations are the symmetric modulo p_i.
For all of these progressions, none satisfies the second condition, the thrusts within the column of guide divisor 5 being greater than the value of the guide. This may be due to the fact that there are no intermediate configurations corresponding to a spacing equal to 24, the change from 6 to 18 being barely achieved :

Guide divisor	3	5	7
6	0	0	4
+	0	3	0
12	0	3	4
+	0	2	1
18	0	0	5
	0	5	1

At next step p_i = 11, all the links mixing positive and negative configurations progressions of a 6n-spacing to a 6n+6-spacing are given below in table 54 and the first of such courses is to the right :

Table 54

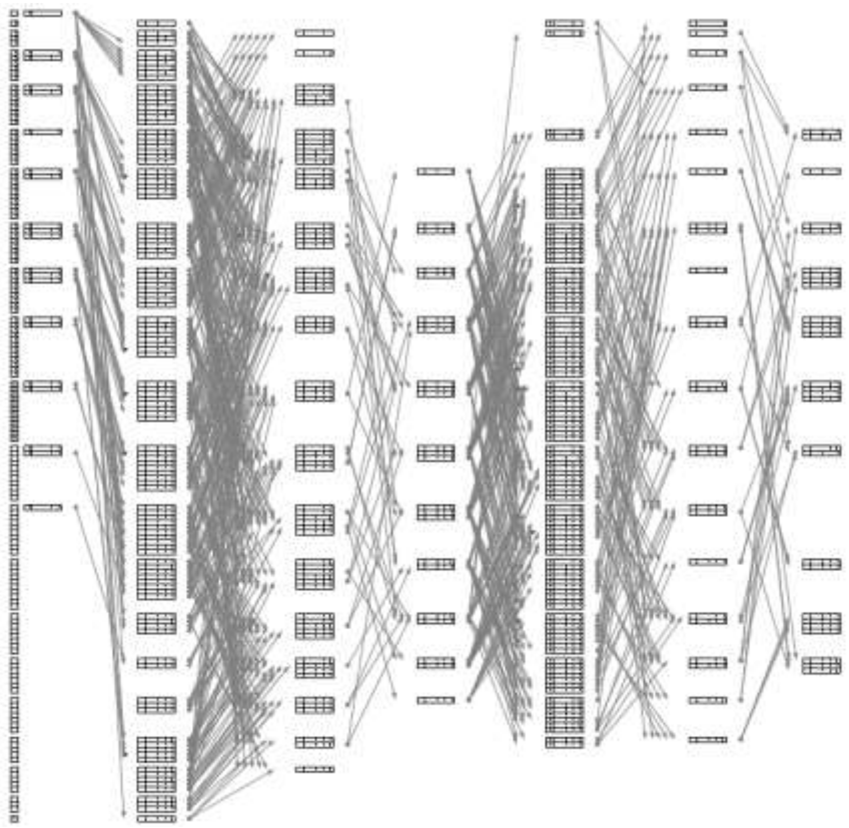


Table 55

p_i	3	5	7	11
6	0	0	0	0
+	0	1	0	2
12	0	1	0	2
-	0	0	3	0
18	0	1	4	2
-	0	2	0	1
24	0	4	4	1
+	0	0	3	0
30	0	4	0	1
+	0	1	1	1
36	0	0	1	2
-	0	1	0	2
42	0	4	1	0
	0	5	7	6

The reader can refer to appendix 8 for reading the contents of boxes.

However, the table of progressions that cross the entire table for the minimum spacing (always 6) up to the maximum spacing (here 42) continuously are less but still abundant. However, if we seek as previously the only cases where all offsets are same signs, we are reduced to 12 positive configurations (a priori if our research is indeed exhaustive). These are provided in appendix 9. There are also 12 corresponding negative configurations symmetrical modulo p_i.

Among the first, 2 sets of positive configurations are closest to ideal, namely configurations set evolving in a column of the divider guide strictly less than the value of the guide (here the guide 5 is reached again what is not completely satisfactory).

Table 56

p_i	3	5	7	11
6	0	0	0	5
+	0	1	1	1
12	0	1	1	6
+	0	0	3	0
18	0	1	4	6
+	0	2	1	0
24	0	3	5	6
+	0	1	1	1
30	0	4	6	7
+	0	1	0	2
36	0	0	6	9
+	0	0	0	3
42	0	0	6	1
	+0	+5	+6	+7

p_i	3	5	7	11
6	0	0	0	5
+	0	1	1	1
12	0	1	1	6
+	0	0	3	0
18	0	1	4	6
+	0	2	1	0
24	0	3	5	6
+	0	2	1	0
30	0	0	6	6
+	0	0	0	3
36	0	0	6	9
+	0	0	0	3
42	0	0	6	1
	+0	+5	+6	+7

The number of configurations explodes to the next rank p_i = 13 and the presence of an ideal set of configurations, answering the question becomes plausible. For the consistently positive progressions, we meet 3341 cases (and as many cases in negative progressions). Among these, however, no set of positive configurations has all thrusts in a column of the divider guide strictly less than the value of the said guide. The best choices, with 33 cases, see their 5-guide reached again (being nevertheless the only one). We give one of them below and the reader will find the remainder in appendix 10:

Table 57

p_i	3	5	7	11	13
6	0	0	0	7	3
+	0	2	1	0	0
12	0	2	1	7	3
+	0	0	0	3	0
18	0	2	1	10	3
+	0	0	0	2	1
24	0	2	1	1	4
+	0	2	1	0	0
30	0	4	2	1	4
+	0	0	0	0	3
36	0	4	2	1	7
+	0	0	2	1	0
42	0	4	4	2	7
+	0	1	1	0	1
48	0	0	5	2	8
+	0	0	0	1	5
60	0	0	5	3	0
+	0	0	1	2	0
66	0	0	6	5	0
∑	0	5	6	9	10

There are also 12 additional cases where, at the same time, the column guides 5 and 7 are reached, but without exceeding (while other guides 11 and 13 are not met).

Beyond that ($p_i > 13$), consider exhaustively all of configurations to detect the systematically positive (and negative by symmetry) progressions becomes an extravagant task.

The difficulty to find a quite satisfactory set of configurations, replicating the process near the final stage (maximum spacing), is due, it must be stressed, to the "tension" as the maximum point is reached. This may limit the full ideal achievement.

The ideal is there initially, namely for $p_i = 5$, perhaps as a simple accident. Beyond that, progress towards the ideal seems gradually. Out of scope for $p_i = 7$, it is better for $p_i = 11$, then almost reached in $p_i = 13$ by noticing that what is lacking to the ideal lies at the lower border (and not in the middle of the progression) :

+	0	2	1	0	0

If it had been in place

+	0	(1)	1	(1)	0

there it was our ideal.

To get rid of "background noise" does not seem to be a fad. Configurations that allow you to move step by step from the minimum spacing to the maximum spacing probably exist from a certain i-row.

Of course, a shift comes often, especially when it occurs on the last columns (and that p_i is large), by a non-event. Conversely, a spacing can multiply after a simple priori innocuous shift. Any change leads to random spacing evolution in a way or another (up or down). But even if the noise here is indeed stronger than the signal sent, the path progress is done at the underlying rhythm.

A shift of 1*2 (since we manipulate only odd integers) means not a shift within the boundaries of 2. It can be almost anything when the course is not followed according to a “smooth trail”. The set of the configurations is chaotic. But underlying force is one and only one and the result for the maximum spacing goes straight with it. If nothing happens after a number of shifts, then the constraint will apply with a sudden readjustment. On the contrary, if the border moves more then 2 (at least 6) and effect has been sent in advance than loosening prevails and nothing may often happen on the next stage.

Spacings of numbers near a maximum spacing (like by any other spacing) are expected to be of average amplitude (that is in $\ln^2(p_i)$ negligible in front of p_i). This maximum spacing of some $\sum_i 2p_k$ amplitude is going to increase (after step i) by negligible terms. The random hero of a given step will revert to anonymity later on. A given maximum spacing is doomed after a few rounds to become one among others and enter the rank of the second, third, etc. chap. This is normal fate since cycle 1 grows by a multiplicative factor p_i at each step, giving many new situations, and the expected scarcity of twin primes imposes increasing spacings. This is why we say that there is no inheritance notion. The mere accidental victory of the strongest cannot last and does not.

The lack of inheritance notion (on a continuum of steps) may seem a handicap because almost nothing is predictable at step i+1 from the results at step i. But in fact, it is a very positive point for our argumentation. Whatever happens at step i, for example, the maximum value of the spacings is much higher (or much lesser) than the expected value, never mind, at step i+1 almost everything is questioned again, the previous result has no lasting influence. Stage i, the work force is $2\sum_i p_k$ and produces a given result. Step i+1, the thing to consider is $2\sum_{i+1} p_k$ but very little the previous result. The latter will pass into oblivion a few steps past.

-The result is lower as the expected one : this means adverse positioning at the observed point but imposing no perennial effect.

- The result is greater to the expected one : this is coming from a merger between the spacing in question and one (or more) neighbours. The most characteristic case we found is $p_i = 37$. Substantially larger at a stage i = 11, we see however that this spacing is not sustainable as a maximum. Three steps further, this maximum enters anonymity (another maximum arose elsewhere).

6.4.10. Lower and upper bounds.

Let us give now some additional details:

Lower bound

Assuming necessity of a complete shift of 3*2 units each time in order to get maximum spacing, assuming also that each shift must take place entirely on the same column (same p_i), then the minimum to the maximum we are looking for would be $2\sum_i (p_k\text{-mod}(p_k,3))$ (for $p_i \neq 3$). Let us observe the first surveys compared to the possibility of this lower bound (for the maximum spacing):

Table 58

Step i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
p_i	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71
Em(i) = Max spacings	6	12	30	42	66	108	150	204	258	348	510	540	582	690	810	852	972	1098	1176
Min of maximum	6	12	24	42	66	96	132	174	228	288	360	438	522	612	714	828	948	1080	1218
Difference	0	0	6	0	0	12	18	30	30	60	150	102	60	78	96	24	24	18	-42

We note that the maximum spacing's minimum is not far from being again achieved at steps 16 up to 18 after the first cases reached in steps 1, 2, 4 and 5. It might even not reached at step 19, which is not however detrimental to our argumentation. Nothing forbids low values (set of configurations which cannot express completely).

Below this bound, we get however generally spacings for almost all the a priori allowed values, namely the multiples of 6. Of course, exceptions may exist as mentioned, for example for the $p_i = 7$ case, the spacing values are 6, 12, 18 and 30, the spacing 24 never occurs and for the case $p_i = 13$, the observed values are 6, 12, 18, 24, 30, 36, 42 48, 60 and 66, the spacing 54 not appearing.

Upper bound

The upper bound can be superior to $2\sum_i p_k$ as shown in the numerical results. Excess compared to the expected value is the result of the collision with the environment as mentioned previously. However, this unexpected value is easily identifiable as an exception by its isolation from the other values of standard spacings. Case $p_i = 37$ is the most typical among the values discussed here, the spacing of amplitude 510 is followed by the spacing 432, then 426, etc. Thus, it is rather the spacing 432 (instead of 510) which is to be compared with $2\sum_i p_k = 390$. Even though spacing (432) is still significantly above $2\sum_i p_k$ (390), the same remark about the possibility of collision with the environment is still at this point as we observe other holes between 420 and 408 and 390 and 378. Similar remarks can be made to a lesser extent for $p_i = 31$ (348 isolated from 330, isolated itself from 318, 318 to retain and compare to 316), $p_i = 41$ (540 isolated from 528 and several holes are recognized down to 480, 480 to retain and compare to 472), $p_i = 43$ (582 isolated from 570, isolated itself from 558 to retain and compare to 558), $p_i = 53$ (810 isolated from 768 to retain and compare to 758), etc.

To do an inventory of all values obtained when searching randomly enables to have more or less insurance on the proximity (or the actual achievement) of the maximum, the appearance of holes after systematic series of 6-distant spacings announcing some way such proximity to the maximum, or at least the approximate logical value.

Table 59

p_i	Max spacings found	Followers (etc. meaning that all admissible spacings exist under the previous value)	$\sum 2p_k$	« Min » of max = $2\sum_i (p_k - mod(p_k,3))$
3	6	/	6	6
5	12	etc.	16	12
7	30	18, etc.	30	24
11	42	etc.	52	42
13	66	60, 48, etc.	78	66
17	108	96, etc.	112	96
19	150	138, etc. except 114	150	132
23	204	etc. except 144	196	174
29	258	240, etc.	254	228
31	348	330, 318, etc.	316	288
37	510	432, 426, 420, 408, 390, 378, etc. except 354	390	360
41	540	528, 516, 510, 498, 492, 480, 474, 468, 462, 450, 438, etc.	472	438
43	582	570, 558, etc. except 534	558	522
47	690	678, 672, 660, 648, 642, 636, 630, 618, etc.	652	612
53	810	798, 768, 762, 750, 720, 714, 708, 702, 690, etc.	758	714
59	852	846, 834, 822, 816, 810, 798, 780, 768, etc.	876	828
61	972	942, 924, 912, 906, 900, 882, etc.	998	948
67	1098	1050, 1038, 1026, 1020, 1008, 996, etc. except 966	1132	1080
71	1176	1146, 1128, 1122, 1098, 1092, 1080, 1068, etc. except 1044	1274	1218

Note:

Missing numbers (like 54, 114, 144, 244, 354, 444, 534, 624, 774, 894, 1044) below the minimum for the maximum (or slightly above) are often valued 24 modulo 30. In a general way, configurations giving a 24 mod 30 spacing are rarer than those that surround them (see table 49 and the paragraph 6.4.7 page 70).

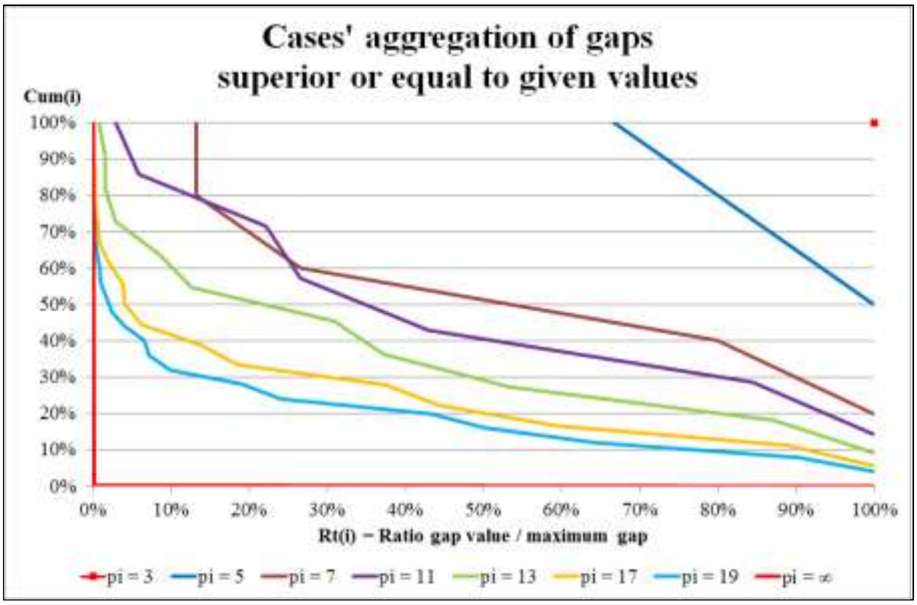
An asymptotic evaluation of the upper bound is easy as part of statistical considerations. To do this, we start from table 34 to build the following table:

Table 60

Steps i	1	2	3	4	5	6	1	2	3	4	5	6
p_i	3	5	7	11	13	17	3	5	7	11	13	17
Spacings Δ	Cum(i) Total cases with spacings >= given value (in abscissa)						Rt(i) Relative sizes of spacings to maximum spacing					
6	1/1	3/3	15/15	135/135	1485/1485	22275/22275	6/6	6/12	6/30	6/42	6/66	6/108
12		2/3	12/15	114/135	1296/1485	19818/22275		12/12	12/30	12/42	12/66	12/108
18			4/15	58/135	792/1485	13266/22275			18/30	18/42	18/66	18/108
24			2/15	36/135	554/1485	9892/22275			24/30	24/42	24/66	24/108
30			2/15	30/135	458/1485	8356/22275			30/30	30/42	30/66	30/108
36				8/135	188/1485	4126/22275				36/42	36/66	36/108
42				4/135	128/1485	3104/22275				42/42	42/66	42/108
48					44/1485	1388/22275					48/66	48/108
54					24/1485	914/22275					54/66	54/108
60					24/1485	874/22275					60/66	60/108
66					12/1485	494/22275					66/66	66/108
72						208/22275						72/108
78						144/22275						78/108
84						78/22275						84/108
90						66/22275						90/108
96						42/22275						96/108
102						20/22275						102/108
108						20/22275						108/108

This table reads more easily using the following graph :

Graph 23



We have represented (without proof nevertheless) the asymptotic trend of the percentage of spacings having significant value compared to the maximum spacing. This percentage (a priori) drops to zero by observing the trend of the first steps i (resulting in the orange curve). In other words, it is less and less likely that the maximum spacing be significantly greater than $2\sum_i p_k$ when p_i diverges. It should be noted that even if this was not the case, the result developed in paragraph 6.5.3 would not be called into question.

The reader will refer to appendix 11 for other developments related to the 2n = 2 gap.

6.4.11. Futher horizons for spacings. Entities viewed with a telescope.

The aim here is to expose the similarity of the spacings between pairs of numbers on one hand and isolated numbers on the other hand and to show the continuous path that can be followed from one to the other.

We first studied the evolution of the quantities of spacings of amplitude Δ between sieved numbers. We got table 5.
We then looked at the evolution of the amounts #S(j,i) of spacings Δ(j) between pairs of numbers. We got table 16.

These latter quantities arise from the application of the Eratosthenes sieve and are determined simply by using the algorithm given in Appendix 14 (Direct evaluation method) where fac, expo, qtpr are adjustable parameters. The first two parameters fac and expo define the type of pairs studied using $2n = fac.2^{expo}$, fac being odd and qtpr being the current step, that is qtpr = 2, p = 3, qtpr = 3, p = 5, qtpr = 4, p = 7, qtpr = 5, p = 11, etc.

We find the quantities of tables 34 and 5 at heads and ends in the following two tables in which we adjust the value of the "fac" parameter in two different ways.

Table 61

fac / Δ	1	3	15	105	1155	1	11	77	385	1155
2	0	21	84	105	135	0	0	0	0	135
4	0	42	63	105	135	0	0	0	0	135
6	21	104	86	130	142	21	36	90	135	142
8	0	28	28	34	28	0	0	0	0	28
10	0	20	54	40	30	0	0	0	0	30
12	56	0	26	12	8	56	54	13	71	8
14	0	22	10	6	2	0	0	0	0	2
16	0	4	4			0	0	0	0	
18	22	8	4			22	22	45	28	
20	0	4	0			0	0	0	0	
22	0	2	1			0	0	0	0	
24	6	4				6	19	26	6	
26	0	0				0	0	0		

fac / Δ	1	3	15	105	1155	1	11	77	385	1155
28	0	8				0	0	0		
30	22	2				22	17	6		
32	0	1				0	0			
34	0					0	0			
36	4					4	0			
38	0					0	0			
40	0					0	0			
42	4					4	2			

What's going on here?

For the first table, we determine the quantities of spacings of amplitude Δ for pairs that are at a distance of 2.1 = 2 (the almost twins) and then for the pairs at distance 2.3 = 6 (the almost sexy), then for the pairs at a distance 2.3.5 = 30, and then for the pairs at distance 2.3.5.7 = 210, then for pairs at a distance 2.3.5.7.11 = 2310.
At this last step, as the cycles are of size 2.3.5.7.11, there is trivially, for a number in position x, another one in position x-2.3.5.7.11 and therefore a pair (x, x+2.3.5.7.11) finds as many counterparts as desired (y, y+2.3.5.7.11). The table therefore reproduces, not the counting of constrained pairs, but rather that of isolated integers, hence the return to the populations of table 5.

For the second table, the result is the same, starting with the biggest multiplier factors, namely 11, then 11.7, then 11.7.5, then 11.7.5.3. The interest in this case is to see that Δ's that are non-dividers of 6 are only reached when factor 3 occurs at the last step (in the fac parameter).

The title of the paragraph comes from the fact that when the algorithm is implemented, the observed paired pairs are at exponentially growing distances.

Taking in account the last step, which meets a range of values Δ(j) equal to some $2p_i$ (see paragraph 3.2.2), going backwards, additional amplitudes should be $2p_k$, k = 1 to i, thus a total of $\sum 2p_k$.

Therefore another way to see the approximate amplitude $\sum 2p_k$ of the largest spacing Δ is that it results thanks to some peculiar telescope from the maximum spacing observed at each of the previous steps.

Appendix 13 gives the tables for i = 1 up to 7. Some populations are equal (or in a 2-ratio) systematically between elements of certain columns and lines (colour fonts in the previous table) from one table to another. However, these identifications do not lead to the possibility of a comprehensive study.

Iterative formulas, as those proposed previously for the first and last columns of these tables, are also at work here giving the populations of the intermediate columns. We give a few more examples, in addition to the study below, in appendix 13 already mentioned, some of which are sometimes weird.

In the previous right-hand tables, the last column concerns the pseudo-primes. Let us move on to the penultimate column to the left of each of them. We then get :

Table 62

	i	1	2	3	4	5	6	7	8	
	p_i	3	5	7	11	13	17	19	23	
	fac	1	5	35	385	5005	85085	1616615	37182145	…
j	Δ	#SPD3(j,i)								
1	6	(1)	3	15	135	1485	22275	378675	7952175	…
2	12		(1)	(7)	71	845	13315	235315	5084975	…
3	18			(2)	(28)	(394)	6812	128810	2918020	…
4	24				(6)	(132)	(2766)	59160	1451310	…
5	30				(0)	(24)	(816)	(22488)	641424	…
6	36						(72)	(3384)	(124992)	…
7	42						(24)	(1392)	(58536)	…
8	48							(192)	(12816)	…
9	54							(24)	(2952)	…
10	60								(480)	…

Here the factor fac is simply divided by 3 compared to its populations' evaluation for pseudo-primes. At this stage, recursive formulas remain "classic":

Table 63

j	Formulas
1	#SPD3(1,1) = 1 #SPD3(1,i) = (p_i-2).#SPD3(1,i-1)
2	x1(4) = 8 x1(i) = (p_{i-1}-3).x1(i-1) #SPD3(2,3) = 7 #SPD3(2,i) = (p_i-2).#SPD3(2,i-1)+x1(i)
3	x1(5) = 6 x1(i) = (p_{i-2}-4).x1(i-1) x2(4) = 10 x2(i) = (p_{i-1}-3).x2(i-1)+x1(i) #SPD3(3,3) = 2 #SPD3(3,i) = (p_i-2).#SPD3(3,i-1)+x2(i)
4	x1(6) = 126 x1(i) = (p_{i-2}-4).x1(i-1) x2(5) = 66 x2(i) = (p_{i-1}-3).x2(i-1)+x1(i) #SPD3(4,4) = 6 #SPD3(4,i) = (p_i-2).#SPD3(4,i-1)+x2(i)
5	x1(7) = 288 x1(i) = (p_{i-3}-5).x1(i-1) x2(6) = 216 x2(i) = (p_{i-2}-4).x2(i-1)+x1(i) x3(5) = 24 x3(i) = (p_{i-1}-3).x3(i-1)+x2(i) #SPD3(5,4) = 0 #SPD3(5,i) = (p_i-2).#SPD3(5,i-1)+x3(i)
6	?
…	...

However, going a stage ahead, an interesting evolution manifests itself. The factor fac is now divided by 3*5 compared to the population's evaluation for pseudo-primes.
The populations' table is as follows :

Table 64

	i	2	3	4	5	6	7	8	
	p_i	5	7	11	13	17	19	23	
	fac	1	7	77	1001	17017	323323	7436429	…
j	Δ	#SPD15(j,i)							
1	6	(1)	10	90	495	14850	252450	2650725	…
2	12	(2)	(1)	13	990	2945	54545	5301450	…
3	18		(5)	(45)	350	7425	126225	2434250	…
4	24		(2)	(26)	175	5890	109090	1217125	…
5	30			(6)	(132)	(2766)	59160	1451310	…
6	36				(6)	(408)	(11244)	160356	…
7	42				(12)	(24)	(1152)	(340788)	…
8	48					(204)	(5622)	(66276)	…
9	54					(48)	(2256)	(24612)	…
10	60						(312)	(31380)	…
11	66						0	(3312)	
12	72						0	(3504)	
13	78						(24)	((384)	
14	84							(240)	
15	90							(48)	

Recursive formulas are no longer with unique initial values for the entire line. Distinctions, modulo the columns' number (thus i), are to be taken into account.

Table 65

j	Formulas Columns i = 2 mod 3	Formulas Columns i = or(0,1) mod 3
1	#SPD15(1,2) = 1 #SPD15(1,i) = (p_i-2).#SPD15(1,i-1)	#SPD15(1,2) = 2 #SPD15(1,i) = (p_i-2).#SPD15(1,i-1)
2	x1(4) = 0 x1(i) = 0 #SPD15(2,3) = 10 #SPD15(2,i) = (p_i-2).#SPD15(2,i-1)+x1(i)	x1(4) = 4 x1(i) = 0 x1(i) = (p_{i-1}-3).x1(i-1) #SPD15(2,3) = 1 #SPD15(2,i) = (p_i-2).#SPD15(2,i-1)+x1(i)
3	x1(4) = 0 x1(i) = 0 #SPD15(2,3) = 5 #SPD15(2,i) = (p_i-2).#SPD15(2,i-1)+x1(i)	x1(4) = 8 x1(i) = (p_{i-1}-3).x1(i-1) #SPD15(2,3) = 2 #SPD15(2,i) = (p_i-2).#SPD15(2,i-1)+x1(i)
4	x1(4) = 4 x1(i) = (p_{i-1}-3).x1(i-1) #SPD15(2,3) = 1 #SPD15(2,i) = (p_i-2).#SPD15(2,i-1)+x1(i)	x1(4) = 8 x1(i) = (p_{i-1}-3).x1(i-1) #SPD15(2,3) = 2 #SPD15(2,i) = (p_i-2).#SPD15(2,i-1)+x1(i)
5	x1(6) = 126 x1(i) = (p_{i-2}-4).x1(i-1) x2(5) = 66 x2(i) = (p_{i-1}-3).x2(i-1)+x1(i) #SPD15(5,4) = 6 #SPD15(5,i) = (p_i-2).#SPD15(5,i-1)+x2(i)	Same formula
6	x1(7) = 72 x1(i) = (p_{i-3}-5).x1(i-1) x2(6) = 54 x2(i) = (p_{i-2}-4).x2(i-1)+x1(i) x3(5) = 6 x3(i) = (p_{i-1}-3).x3(i-1)+x2(i) #SPD15(5,4) = 0 #SPD15(5,i) = (p_i-2).#SPD3(5,i-1)+x3(i)	x1(7) = 144 x1(i) = (p_{i-3}-5).x1(i-1) x2(6) = 108 x2(i) = (p_{i-2}-4).x2(i-1)+x1(i) x3(5) = 12 x3(i) = (p_{i-1}-3).x3(i-1)+x2(i) #SPD15(5,4) = 0 #SPD15(5,i) = (p_i-2).#SPD3(5,i-1)+x3(i)
…	...	...

The previous table is still very simple to put together. It is likely that as the parameter fac evolves more (modulo) cases will occur. It should be noted, however, the economy on need of new initial values (e.g. ratios of 2 or reuse of values in different lines).

6.5. Landscaping of spacings between relative integers.

Let us focus now on to a comprehensive study of all gaps.

Theorem 23

At the given step i, the populations are identical for any gap 2n modulo p_i#.

Proof

This is trivial, the cycles generated by the Eratosthenes sieve being of period p_i.

It is therefore sufficient to consider, at stage i, the even gaps 2n between 0 and p_i#-2 to be exhaustive. We give the example of all the populations at step i = 2 below :

Table 66

	Δ / 2n	2	4	6	8	10	12	14	16	18
		#R(2n,Δ)								
Pseudo isolated	0	3	3	2	0	0	0	0	0	0
Pseudo twins	2	0	0	1	0	0	2	0	0	0
Pseudo cousins	4	0	0	2	0	0	0	0	0	1
Pseudo sexys	6	1	2	2	1	0	0	0	0	0
etc.	8	0	0	1	0	0	2	0	0	0
	10	0	0	3	0	0	1	0	0	0
	12	2	1	2	0	1	0	0	0	0
	14	0	0	2	0	0	0	0	0	1
	16	0	0	2	0	0	0	0	0	1
	18	2	1	2	0	1	0	0	0	0
	20	0	0	3	0	0	1	0	0	0
	22	0	0	1	0	0	2	0	0	0
	24	1	2	2	1	0	0	0	0	0
	26	0	0	2	0	0	0	0	0	1
	28	0	0	1	0	0	2	0	0	0

The table's exploitation is improved by sorting according to the increasing modulo $p_i\#$ values of the square of the 2n-gap :

Table 67

2n mod 30	(2n)² mod 30	Δ	2	4	6	8	10	12	14	16	18
		$\sum_\Delta$ #R(2n,Δ)	#R(2n,Δ)								
0	0	8	3	3	2	0	0	0	0	0	0
2	4	3	0	0	1	0	0	2	0	0	0
8	4	3	0	0	1	0	0	2	0	0	0
22	4	3	0	0	1	0	0	2	0	0	0
28	4	3	0	0	1	0	0	2	0	0	0
6	6	6	1	2	2	1	0	0	0	0	0
24	6	6	1	2	2	1	0	0	0	0	0
10	10	4	0	0	3	0	0	1	0	0	0
20	10	4	0	0	3	0	0	1	0	0	0
4	16	3	0	0	2	0	0	0	0	0	1
14	16	3	0	0	2	0	0	0	0	0	1
16	16	3	0	0	2	0	0	0	0	0	1
26	16	3	0	0	2	0	0	0	0	0	1
12	24	6	2	1	2	0	1	0	0	0	0
18	24	6	2	1	2	0	1	0	0	0	0

Conjecture 5

At the given step i and for 4n² modulo $p_i\#$ set in advance, the populations are the same. Conversely, identical populations lead to constant 4n² modulo $p_i\#$.

Let us rewrite the table in an ultimate form :

Table 68

Families 2n mod 30	Multiplicands	(2n)² mod 30	Δ	2	4	6	8	10	12	14	16	18
			$\sum_\Delta$ #R(2n,Δ)	#R(2n,Δ)								
(0)	1	0	8	3	3	2	0	0	0	0	0	0
(6,24)	2	6	6	1	2	2	1	0	0	0	0	0
(12,18)	2	24	6	2	1	2	0	1	0	0	0	0
(10,20)	2	10	4	0	0	3	0	0	1	0	0	0
(2,8,22,28)	4	4	3	0	0	1	0	0	2	0	0	0
(4,16,14,26)	4	16	3	0	0	2	0	0	0	0	0	1

We call « multiplicand » the number of solutions 2n with the same distribution of populations #R(2n,Δ). This word is chosen so because, as we will see below, its values can be anticipated (thus intervening in some way first in the multiplication). Several families can have multiplicands of equal value.

Conjecture 6

A multiplicand is a power of 2.

This last result is demonstrated by admitting that the expression $(2n)^2$ mod $p_i\#$ is actually at work here.
So we are going to study the latter and establish that result in that context. What we call families below is also understood in this context.

Theorem 24

At the given step i, the number of families nbf(m,i) of 2^m multiplicands is given by :

$$\begin{gathered} nbf(0,i) = 1,\ i \geq 0 \\ \text{et} \\ nbf(m,i) = nbf(m,i-1)+((p_i-1)/2).nbf(m-1,i-1) \end{gathered} \qquad (99)$$

Numerical application

Table 69

		i	0	1	2	3	4	5	6	7	8	9	10	11
		p_i	2	3	5	7	11	13	17	19	23	29	31	37
m	2^m	$(p_i-1)/2$		1	2	3	5	6	8	9	11	14	15	18
			nbf(m,i)											
0	1		1	1	1	1	1	1	1	1	1	1	1	1
1	2			1	3	6	11	17	25	34	45	59	74	92
2	4				2	11	41	107	243	468	842	1472	2357	3689
3	8					6	61	307	1163	3350	8498	20286	42366	84792
4	16						30	396	2852	13319	50169	169141	473431	1236019
5	32							180	3348	29016	175525	877891	3415006	11936764
6	64								1440	31572	350748	2808098	15976463	77446571
7	128									12960	360252	5270724	47392194	334968528
8	256										142560	5186088	84246948	937306440
9	512											1995840	79787160	1596232224
10	1024												29937600	1466106480
11	2048													538876800

Proof

Let us first illustrate the subject by going back to Table 68 and analysing the groupings of families 2n modulo 30 for which $(2n)^2$ modulo 30 leads to a given value that is indicated (in italics) under each column corresponding to a family below :

Families 1 : Divisors 3 and 5 :

Table 70

0
0

Families 2 : Divisors 3 or 5 :

Table 71

6	12	10
24	18	20
6	*24*	*10*

Families 2 : Divisors not 3, nor 5 :

Table 72

2	4
8	16
22	14
28	26
4	*16*

When we look at the next step modulo 210, we find :

Family 1 : Divisors 3, 5 and 7 :

Table 73

0
0

Families 2 : Divisors (3 and 5) or (3 and 7) or (5 and 7) :

Table 74

30	60	120	42	84	70
180	150	90	168	126	140
60	*30*	*120*	*84*	*126*	*70*

Families 3 : Divisors 3 or 5 or 7 :

Table 75

6	12	24	48	96	192	10	20	40	14	28
36	72	144	78	156	102	80	160	110	56	112
174	138	66	132	54	108	130	50	100	154	98
204	198	186	162	114	18	200	190	170	196	182
36	*144*	*156*	*204*	*186*	*114*	*100*	*190*	*130*	*196*	*154*

Families 4 : Divisors not 3, nor 5, nor 7 :

Table 76

2	4	8	16	32	64
58	116	22	44	88	176
68	136	62	124	38	76
82	164	118	26	52	104
128	46	92	184	158	106
142	74	148	86	172	134
152	94	188	166	122	34
208	206	202	194	178	146
4	*16*	*64*	*46*	*184*	*106*

These examples show in the first place that if a family $2n_k$ has an invariant $(2n_k)^2 \bmod p_i\# = c \bmod p_i\#,(2n_k)^2 \bmod p_i\# = c \bmod p_i\#$, then the family $4n_k$ has the invariant $(4n_k)^2 \bmod p_i\# = 4c \bmod p_i\#$, which is trivial. The number of families $2n_k$ of a given dividers characteristic is therefore equal to the period t of $2^t.n_{ki} = 2n_{kj} \bmod p_i\#$, $2n_{ki}$ and $2n_{kj}$ being one or the other of their representatives. For example, in the last table, the integers 2, 4, 8, 16, 32, 64 do not meet in two columns at once, but 128 ends up in the first column with integer 2 completing the cycle and the period t is equal to 6 for all table elements (such as 58, 116, 22, 44, 88, 176, 142, etc.).

Let us see how to move from elements at step i to those at step i+1. The first table at each new step is 0 since the only even number between 0 to $p_i\#$-2 divisible by all prime numbers between 2 and p_i. The n+1-table at the i+1 step is deduced, in part, from the n^{th} table at the i-step. Let us take, for example, the following two tables in correspondence:

6	12	10
24	18	20
6	*24*	*10*

6 = 6+0.30	12 = 12+0.30	24	48	96	192	10 = 10+0.30	20	40	14	28
36 = 6+1.30	72 = 12+2.30	144	78	156	102	80 = 20+2.30	160	110	56	112
174 = 24+5.30	138 = 18+4.30	66	132	54	108	130 = 10+4.30	50	100	154	98
204= 24+6.30	198 = 18+6.30	186	162	114	18	200 = 20+6.30	190	170	196	182
36	*144*	*156*	*204*	*186*	*114*	*100*	*190*	*130*	*196*	*154*

Consider 2n and $2n+k.p_i\# \bmod p_{i+1}\#$ where k varies from 0 to p_{i+1}-1. If 2n is not divisible by some prime number $p_k < p_{i+1}$ then there is effectively some k, according to the Chinese theorem, such that $2n+k.p_i\# \bmod p_{i+1}\#$ is not divisible by the same p_k. This proves the existence.

An existing element creates two new elements systematically in a column because if 2n is present at the i-step then

$2n+k.p_i\#$ is generated at the same time as $p_{i+1}\# - (2n+k.p_i\#)$ in the same family. Indeed, they both admit the same prime dividers lower or equal to p_i and one of them is necessarily larger than $p_i\#$ and therefore absent in the same family at the previous rank. This proves the doubling of lines.

Tables' increasing is active evenly in all parts of themselves, i.e. systematically by multiplication by 2 from one column to another. By moving from step i to step i+1, the total number of items increases by a p_{i+1} factor, while the number of lines doubles. Thus, the number of columns of the parts of tables in correspondence necessarily increases by a factor close to $p_{i+1}/2$, knowing however that new elements appear with divider p_{i+1}. These are exactly at the number of $p_{i+1}\#/p_{i+1}$, or also exactly $(p_{i+1}-1).p_i\#$ elements without the said p_{i+1} divider. The number of columns in each part of tables is hence multiplied by $(p_{i+1}-1)/2$.

Then let us focus on the new elements that appear, the divider of which is p_{i+1}. They correspond to the multiplication by p_{i+1} from the n+1st table of the previous step since this factor is introduced at stage i+1:

2	4
8	16
22	14
28	26
4	*16*

$p_i = 7$ $\rightarrow$

14	28
56	112
154	98
196	182
196	*154*

The two generation processes described above either leave the size of a family unchanged or double its size. Starting from the unit, the size of the families (the multiplicand) is therefore necessarily a power of 2.
This completes the proof.

Note 1

Trivially, the sum of the products of multiplicands by the number of families is equal to the sum of the even numbers in a cycle, i.e. $p_i\#/2$ in step I :

$$p_i\#/2 = \sum 2^m.nbf(m,i) \qquad (100)$$

Note 2

A witty property of the previous triangular table is worth noting: The values on the lower edge grow multiplicatively at the same pace that the values of the m-1 line grow additively.

Line m = 1	LS(i)	1	3	6	11	17	25	34	45	59	74	92	…
Lower edge	LI(i)	1	2	6	30	180	1440	12960	142560	1995840	29937600	538876800	…
Difference	LS(i)-LS(i-1)	1	2	3	5	6	8	9	11	14	15	18	$(p_i-1)/2$
Quotient	LI(i)/LI(i-1)	1	2	3	5	6	8	9	11	14	15	18	$(p_i-1)/2$

Note 3

The study is conducted here on 2n mod $p_i\#$ and the square $(2n)^2$ mod $p_i\#$. The same exercise with n mod $p_i\#$ and the square n^2 mod $p_i\#$ (n = 0 to p_i-1) would give a table where nbf(m,i) would simply be replaced by 2.nbf(m,i), which is to keep the same formula $nbf(m,i) = nbf(m,i)+((p_i-1)/2).nbf(m-1,i-1)$ but adjusting the initial values $nbf(0,i) = 2$, $i \geq 0$. Similarly, with n mod $p_i\#$ and n^4 mod $p_i\#$ (n = 0 to p_i-1), the formula is still unchanged, but requires the initial values $nbf(0,i) = 2$, $i \geq 1$, $nbf(1,1) = 2$, $nbf(1,2) = 2$ $nbf(2,2) = 2$ and $nbf(3,2) = 2$. It is likely that the reuse of the same formula is appropriate for the transition to power $n^{2^\wedge r}$ with appropriate initial values. More general problems will eventually lead to adjustments to the recursive formula.

Having given a general view of the situation, let us now split our analysis.

6.5.1. Periodicity of the entities.

4.4.11.1 Periodicity focusing on even components.

At paragraph 6.4.11, we have changed the "fac" parameter. We will proceed now on the "expo" parameter, i.e. we consider pairs whose gap 2n gradually doubles : 2n = 2, 4, 8, 16, etc. (fac = 1, expo = 1, 2, 3,4, etc.).

The populations' tables #S (j,i) of spacings of amplitude Δ as follows:

Step 1, qtpr = 2, $p_i = 3$.

Δ \ 2n	2	4	…
6	1	1	…

Step 2, qtpr = 3, $p_i = 5$.

2n \ Δ
6
12
18

2	4	8	…
1	2	1	…
2	0	2	…
0	1	0	…

Step 3, qtpr = 4, $p_i = 7$.

2n \ Δ
6
12
18
24
30

2	4	8	16	32	64	128	…
3	6	4	6	3	8	3	…
8	2	6	2	6	0	8	…
2	4	2	3	4	3	2	…
0	2	2	4	2	2	0	…
2	1	1	0	0	2	2	…

As there is no divider of 3 in 2n, the set of Δ's contains multiples of 6.

We observe that, when the "expo" parameter is incremented, at some stage, the same populations show up.
The evolution of periodicity extends as follows for the parameter examined:

Theorem 25

The periodicity of the population of pairs of gaps 2n, power of 2, is half-value of the order of the monogenic group of generator 2 modulo the primorial $p_i\#$ at step i.

$$\#ord2_i = \min(r/2) \setminus 2^r = 1 \bmod p_i\# , i > 0, r > 0 \qquad (101)$$

Proof

This is an immediate and trivial consequence of the periodicity of $p_i\#$-sized cycles produced the Eratosthenes algorithm. It gives an order equal to that of 2, modulo $p_i\#$, for the family 2n. As it is the squared values $(2n)^2 \bmod p_i\#$ that must be taken into account for families, this order is therefore divided by 2.

Note :

The $p_i\#$ factor increases exponentially with i. It would be interesting to be able to evaluate the order $\#ord2_i$ from the modulo p_i study instead. Let us note the order of 2 modulo p_i as follows :

$$\#ordel2_i = \min(r) \setminus 2^r = 1 \bmod p_i , i > 0, r > 0 \qquad (102)$$

The order of a subgroup is an integer divider of a group. The order $2.\#ord2_i$ is therefore a divider of the product of orders $\#ordel2_i$, the multiplicative factor $\#fm2_i$ (see table below) being a divider of this order, itself a divider of (p_i-1). As only the even numbers are involved here, it is $(p_i-1)/2$ that is to be taken into account. The evolution of periodicity shows as follows :

Table 77

Step	p_i	Periodicity $\#ord2_i$	$\#fm2_i$ = multiplicative factor	$(p_i-1)/2$	Factors accumulation	Verification
1	3	1	1	1	1	fully done
2	5	2	2	2	2	fully done
3	7	6	3	3	3	fully done
4	11	30	5	5	2.3.5	fully done
5	13	30	1	2.3	2.3.5	fully done
6	17	60	2	2^3	$2^3.3.5$	fully done
7	19	180	3	3^2	$2^3.3^2.5$	fully done
8	23	1980	11	11	$2^3.3^2.5.11$	fully done
9	29	13860	7	2.7	$2^3.3^2.5.7.11$	fully done
10	31	13860	1	3.5	$2^3.3^2.5.7.11$	by incomplete way
11	37	13860	1	2.3^2	$2^3.3^2.5.7.11$	by incomplete way
12	41	13860	1	$2^2.5$	$2^3.3^2.5.7.11$	by incomplete way
13	43	13860	1	3.7	$2^3.3^2.5.7.11$	by incomplete way
14	47	318780	23	23	$2^3.3^2.5.7.11.23$	by incomplete way

Step	p_i	Periodicity $\#ord2_i$	$\#fm2_i$ = multiplicative factor	$(p_i-1)/2$	Factors accumulation	Verification
15	53	4144140	13	2.13	$2^3.3^2.5.7.11.13.23$	by incomplete way
16	59	120180060	29	29	$2^3.3^2.5.7.11.13.23.29$	by incomplete way
17	61	120180060	1	2.3.5	$2^3.3^2.5.7.11.13.23.29$	by incomplete way
18	67	120180060	1	3.11	$2^3.3^2.5.7.11.13.23.29$	by incomplete way
19	71	120180060	1	5.7	$2^3.3^2.5.7.11.13.23.29$	by incomplete way
…						

Due to the exponential growth of the calculations, a full verification of the results (on Pari GP) could only be carried out until stage 9 with populations as follows (verifying also that the periodicity is not 2.1980 = 3960):

2n / Δ	2^1	$2^{1+1.1980}$	$2^{1+7.1980}$
6	17506125	17506125	17506125
12	46683000	48550320	46683000
18	27184430	26844090	27184430
24	14178528	13478400	14178528
30	39735054	39088254	39735054
36	10497320	10534680	10497320
42	22680468	21998100	22680468
48	8256720	8178960	8256720
54	2479200	2422518	2479200
60	7815766	7686076	7815766
66	5067262	5228158	5067262
72	3197558	3388718	3197558
78	3028200	2957192	3028200
84	1026404	1149446	1026404
90	1711068	1847620	1711068
96	948278	1010116	948278
102	264346	298490	264346
108	1194016	1239080	1194016
114	54546	61360	54546
120	387506	392990	387506
126	205068	236824	205068
132	150588	145766	150588
138	278558	282968	278558
144	1180	1802	1180
150	88548	85216	88548
156	29724	32814	29724
162	15172	16162	15172
168	24418	25978	24418
174	2054	1974	2054
180	10862	11334	10862
186	2090	2620	2090
192	2764	2428	2764
198	748	942	748
204	548	426	548
210	442	498	442
216	38	38	38
222	84	126	84
228	22	50	22
234	12	24	12
240	8	30	8
246	0	0	0
252	0	4	0
258	2	4	2
264		0	
270		4	

The time required to calculate each column is in the order of one day. Beyond that the step, we have adopted another verification strategy, namely in the algorithm given in Appendix 14, we continue to increment qtpr, but the program sequences

if(Mod(ac, 3) <> 0,
if(Mod(a, 3) <> 0,
if(Mod(ac, 5) <> 0,
if(Mod(a, 5) <> 0,
…,
…,
if(Mod(ac, p_i) <> 0,
if(Mod(a, p_i) <> 0,

is limited to

if(Mod(ac, 3) <> 0,
if(Mod(a, 3) <> 0,
if(Mod(ac, 5) <> 0,
if(Mod(a, 5) <> 0,
f(Mod(ac, 7) <> 0,
if(Mod(a, 7) <> 0,
if(Mod(ac, 11) <> 0,
if(Mod(a, 11) <> 0,

In this case, for steps 5 to 9, we find the periodicities already mentioned and we hypothesize that the behaviour is the same afterwards. The search is then extremely fast and could be extended well beyond the values given in Table 38. A second limit then occurs however, which are the sizes of the pairs considered (x, $x+2^{expo}$) as the expo setting increases (on our version of Pari GP we are limited to expo = 120 180 060 by the memory stack).

Conjecture 7

The multiplicative factor is equal to the product of new factors in $(p_i-1)/2$ compared to all factors previously contained in the $(p_j-1)/2$, j = 1 to i, at their maximum powers.

Note: By new factor, we mean if p_k^{n1} is present in $(p_i-1)/2$ and if p_k^{n2} appears in one of the terms $(p_j-1)/2$, j = 1 to i, then the multiplicative factor is equal to $\prod p_k^{si(n1-n2 \geq 1,\ 1,\ 0)}$, where the product deals with all the prime factors of $(p_i-1)/2$. In particular, if pd_i is a prime number, then $\#fm2_i = (p_i-1)/2$.

Examples: Factors 2 and 3 already present in stages 2 and 3 will be ignored in step 5. The integer pd_i has factor 2^3 at step 7, with exponent larger than its power in the column cumulating maximum exponents at step 6 (difference for exponent equal to 3-1 = 2) and we have a multiplicative factor $\#fm2_i = 2^1$ (and not 2^2).

4.4.11.2 Periodicity focusing on odd components.

We were interested in the evolution of the populations #S(j,i) when 2n is replaced by $2n.2^{expo}$.
What happens with the change from 2 to $2q^{expo}$, with odd q?

Conjecture 8
Case q prime number.

The multiplicative factor $\#fm2_i$ at step i is a divider of $(p_i-1)/2$. The populations' tables form classes function of modulo(q,p_j), j = 1 to i. The populations #S(j,i) have amplitude Δ multiple of 6.

Example : Step 3.

q	qd = (q-1)/2	Mod(qd,3)	Mod(qd,5)	Mod(qd,7)	Class	Periodicity
73	36	0	1	1	1	6
193	96	0	1	5	1	6
157	78	0	3	1	1	6
67	33	0	3	5	1	6
353	176	2	1	1	1	6
53	26	2	1	5	1	6
17	8	2	3	1	1	6
137	68	2	3	5	1	6
103	51	0	1	2	2	6
163	81	0	1	4	2	6
397	198	0	3	2	2	6
37	18	0	3	4	2	6
173	86	2	1	2	2	6
23	11	2	1	4	2	6
47	23	2	3	2	2	6
107	53	2	3	4	2	6

q	qd = (q-1)/2	Mod(qd,3)	Mod(qd,5)	Mod(qd,7)	Class	Periodicity
61	30	0	0	2	3	3
331	165	0	0	4	3	3
19	9	0	4	2	3	3
79	39	0	4	4	3	3
131	65	2	0	2	3	3
191	95	2	0	4	3	3
89	44	2	4	2	3	3
149	74	2	4	4	3	3
31	15	0	0	1	4	3
151	75	0	0	5	4	3
199	99	0	4	1	4	3
109	54	0	4	5	4	3
101	50	2	0	1	4	3
11	5	2	0	5	4	3
59	29	2	4	1	4	3
179	89	2	4	5	4	3
5	2	2	2	2	5	3
7	3	0	3	3	6	2
43	21	0	1	0	7	2
13	6	0	1	6	7	2
127	63	0	3	0	7	2
97	48	0	3	6	7	2
113	56	2	1	0	7	2
83	41	2	1	6	7	2
197	98	2	3	0	7	2
167	83	2	3	6	7	2
211	105	0	0	0	8	1
181	90	0	0	6	8	1
379	189	0	4	0	8	1
139	69	0	4	6	8	1
71	35	2	0	0	8	1
41	20	2	0	6	8	1
29	14	2	4	0	8	1
419	209	2	4	6	8	1

Population tables #S(j,i) are as follows :

Class 1 : q = 17, 53, 67, 73,... Periodicity 6.

Condition : mod(qd,3) = or(0,2) and mod(qd,5) = or(1,3) and mod(qd,7) = or(1,5)

8	3	6	4	6	3
0	6	2	6	2	8
3	4	3	2	4	2
2	2	4	2	2	0
2	0	0	1	1	2

Class 2 : q = 23, 37, 47,... Periodicity 6.

Condition : mod(qd,3) = or(0,2) and mod(qd,5) = or(1,3) and mod(qd,7) = or(2,4)

6	4	6	3	8	3
2	6	2	6	0	8
4	2	3	4	3	2
2	2	4	2	2	0
1	1	0	0	2	2

Class 3 : q = 19, 61, 79, 89,... Periodicity 3.

Condition : mod(qd,3) = or(0,2) and mod(qd,5) = or(0,4) and mod(qd,7) = or(2,4)

3	4	3
6	6	8
4	2	2
2	2	0
0	1	2

Class 4 : q = 11, 31, 59,... Periodicity 3.

Condition : mod(qd,3) = or(0,2) and mod(qd,5) = or(0,4) and mod(qd,7) = or(1,5)

4	3	3
6	6	8
2	4	2
2	2	0
1	0	2

Class 5 : q = 5. Periodicity 3. (unique as prime number)

Condition : mod(qd,3) = or(0,2) and mod(qd,5) = 2 and mod(qd,7) = or(2,4)

9	12	9
7	3	8
4	3	2
0	2	1

Class 6 : q = 7. Periodicity 2. (unique as prime number)

Condition : mod(qd,3) = or(0,2) and mod(qd,5) = or(1,3) and mod(qd,7) = 3

10	5
1	10
5	2
2	1

Class 7 : q = 13, 43, 83, 97,... Periodicity 2.

Condition : mod(qd,3) = or(0,2) and mod(qd,5) = or(1,3) and mod(qd,7) = or(0,6)

6	3
2	8
3	2
4	0
0	2

Class 8: q = 1, 29, 41, 71,... Periodicity 1.

Condition : mod(qd,3) = or(0,2) and mod(qd,5) = or(0,4) and mod(qd,7) = or(0,6)
Condition : q = 1

3
8
2
0
2

When q is not a prime number, the process is the same but new families are possible and have to be taken into account.

Class 9 : q = 25, 95, 115, 185,... Periodicity 3.

Condition : mod(qd,3) = or(0,2) and mod(qd,5) = 2 and mod(qd,7) = or(1,5)

12	9	9
3	7	8
3	4	2
2	0	1

Class 10: q = 55,... Periodicity 1.

Condition : mod(qd,3) = or(0,2) and mod(qd,5) = 2 and mod(qd,7) = or(0,6)

9
8
2
1

Class 11: q = 49, 91, 119, 161,... Periodicity 1.

Condition : mod(qd,3) = or(0,2) and mod(qd,5) = or(0,4) and mod(qd,7) = 3

5
10
2
1

Class 12: q = 35, 175,... Periodicity 1.

Condition : mod(qd,3) = or(0,2) and mod(qd,5) = 2 and mod(qd,7) = 3

15
7
2

We have only included here cases where q is not divisible by 3. Other classifications are then added on the same modulo pattern. The populations #S (j,i) have then Δ's multiple of 2.

What happens passing from 2p to $2p.q^{expo}$, p and q whatever integers? The same thing considering then the different families modulo qd.

The number of cases increases exponentially with step i which quickly makes any comprehensive study extremely long and tedious.

6.5.2. Sums of products.

Let us go back to the $2n = 2^m$ case even though if what follows applies in a more general way.
Let us have i a given depletion step. If k = 0 or k = 1, then we have seen that the expression $(\Delta(j))^k$.#S(j,i) is constant whatever choice of m, the different solutions for #S(j,i) forming a set of values that return periodically.
The question here is whether the elements of #S(j,i) can be obtained in a unique way from the value of $(\Delta(j))^2$.#S(j,i), if not by adding $(\Delta(j))^3$.#S(j,i), and so on, in other words, if certain distributions #S(j,i) would not be some kind of Carmichael series where the initial data $(\Delta(j))^k$.#S(j,i), k = 2, k = 3, k = 4, ... would not allow to distinguish them by a backward evaluation.

Here we give only a few examples to set ideas down on this subject.

Step 2 : $p_i = 5$

k \ m	0	1
0	3	3
1	30	30
2	324	396

That is, a distinction that appears as early as k = 2.

Step 3 : $p_i = 7$

k \ m	0	1	2	3	4	5
0	15	15	15	15	15	15
1	210	210	210	210	210	210
2	3708	3852	3708	3780	3420	4212
3	80136	82728	77544	77544	61992	100872

That is a partial distinction as early as k = 2 and total one by adding k = 3.
Equal values can be found in :

k	m
2	(0,2)
3	(2,3)

Step 4 : $p_i = 11$

k \ m	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	135	135	135	135	135	135	135	135	135	135	135	135	135	135	135
1	2310	2310	2310	2310	2310	2310	2310	2310	2310	2310	2310	2310	2310	2310	2310
2	51444	52164	51444	55260	50868	56700	52020	51804	53460	53244	49716	57348	51732	56052	53172
3	1389528	1371384	1381752	1633176	1368792	1685016	1441368	1345464	1511352	1482840	1296216	1755000	1423224	1695384	1511352

k \ m	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
0	135	135	135	135	135	135	135	135	135	135	135	135	135	135	135
1	2310	2310	2310	2310	2310	2310	2310	2310	2310	2310	2310	2310	2310	2310	2310
2	52668	51588	56988	53820	53460	51300	52524	49284	60012	55044	52020	51732	51948	50724	56556
3	1446552	1503576	1726488	1615032	1508760	1358424	1415448	1249560	1923480	1729080	1366200	1397304	1366200	1348056	1672056

Hence again a partial distinction as early as k = 2 and total one by adding k = 3.
Equal values can be found in :

k	m
2	(0,2), (6,25), (8,19), (12,26)
3	(8,14), (25,27)

Step 5 : $p_i = 13$

With a periodicity of 30 here, the equality of values can be found in :

k	m
2	(8,14)
3	/

Step 6 : $p_i = 17$

With a periodicity of 30 here, the equality of values can be found in :

k	m
2	(41,47)
3	/

Step 7 : $p_i = 19$

Although of periodicity 180, no equality is found :

k	m
2	/
3	/

Thus equality, although surprising, is not uncommon. However, more possibilities do not lead to more redundancies (see step 7). A priori, it must be very rare to have to lay down more than 2 sets of additional products' sums (i.e. more than those for k = 2 and k = 3) to get all the different solutions within an eligible set (when m varies).

6.5.3. Divergence of solutions.

Theorem 26

There are infinitely many twin prime numbers.

Proof

The sum $2\sum_i p_k$ can be estimated elementarily with the PNT.
We have $\sum_i p_k \approx \sum k.\ln(p_k) < \ln(p_i).\sum k = \ln(p_i).i.(i+1)/2 \approx (1/2).\ln(p_i).i^2$. As the logarithm varies very slowly asymptotically, we have actually $\sum 2p_k \rightarrow \ln(p_i).i^2 \approx p_i^2/\ln(p_i)$ when i diverges.
The maximum spacing between pairs of numbers in Eras(i) is that of full cycle 1, i.e. pairs between p_i+2 and $p_i+2+2.3\ldots p_i$ and thus of course also pairs between p_i+2 and p_{i+1}^2, a space with magnitude size p_i^2 asymptotically, where only prime numbers can exist. Thus, even if all the spacings between integers happen to be within this range to their maximum (which is far from being the case here), there would be at least the integer part of $\ln(p_i)$ twin prime numbers actually present in the said interval.
So, when i diverges neglecting p_i in front of p_i^2 (what is legitimate asymptotically), cardinal of the twin prime numbers below p_i^2 diverges (in $\ln(p_i)$ at least).

Nota 1

We might consider that a maximum spacing can hit another under the abscissa p_i^2, giving a spacing of double size. This would still give room for $\ln(p_i)/2$ twin primes at the condition that the same type of unusual encounter realize repeatedly under p_i^2. This would still not change the result of the divergence as, in addition, this type of accidents should then repeat continuously so to apply asymptotically (which is quite more unlikely than the existence of an infinite number of twin prime numbers).

Nota 2

The information on the order of magnitude of the cardinal of twin prime numbers is of course very pessimistic here. Only 2 to 3 twin primes would show up at the increase of a decade of i. As we have seen earlier, this divergence is much faster in the real world.

6.6. Comparison of families.

The generalization of the case of twin numbers makes it possible to find interesting additional properties.
Let us go back to Table 34. We have two relationships for 2n = 2:

$$\sum_{j=jmin}^{jmax} \#S(j,i) = \prod_{k=1}^{i} (p_k-2) \qquad (103)$$

$$\sum_{j=jmin}^{jmax} \Delta(j).\#S(j,i) = \prod_{k=1}^{i} p_k \qquad (104)$$

These two become in the general case:

$$\sum_{j=jmin}^{jmax} \#S(j,i) = \prod_{\substack{p_k \backslash n \\ p_k > 2}} (p_k-1)/(p_k-2) \prod_{k=1}^{i} (p_k-2) \qquad (105)$$

$$\sum_{j=jmin}^{jmax} \Delta(j).\#S(j,i) = \prod_{k=0}^{i} p_k \qquad (106)$$

When the dividers of two numbers are the same, the members are on the left are identical. So how many solutions to such equations?
We can look for solutions in two different ways, either systematically or as solutions of the cases $2n = r.2^m$, m - 1, 2, 3,

etc. In the first case, a large number of solutions are found, far more than in the second way of proceeding where the following conjecture, with i fixed, is observed:
The number of distinct solutions is

$$nbs = 2 \prod_{k=1}^{i-2} p_k \qquad (107)$$

and the quantities #S(j,i) show up with period nbs, that is identical for all $2n = r.2^{m+nbs.x}$, where x is any natural integer, r and m are given and nbs deducted by the previous formula.

Let us take the case r =1 and therefore $2n = 2^m$.
For i = 4, $p_0.p_1.p_2.p_3.p_4 = 2.3.5.7.11 = 2310$, $(p_1\text{-}2).(p_2\text{-}2).(p_3\text{-}2).(p_4\text{-}2) = 1.3.5.9 = 135$, $p_0.p_1.p_2 = 2.3.5 = 30$.
The table of 30 distinct results is here (column i = 4 of Table 34 and quantities in tables in correspondence):

2n Spacings Δ(j)	2^{1}	2^{2}	2^{3}	2^{4}	2^{5}	2^{6}	2^{7}	2^{8}	2^{9}	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}	2^{15}	2^{16}	2^{17}	2^{18}	2^{19}	2^{20}	2^{21}	2^{22}	2^{23}	2^{24}	2^{25}	2^{26}	2^{27}	2^{28}	2^{29}	2^{30}	2^{31}	...
6	21	42	28	48	21	56	21	42	32	42	21	56	21	48	28	42	21	56	24	42	28	42	21	64	21	42	28	42	24	56	21	...
12	56	16	42	14	48	6	56	16	42	20	42	6	56	14	48	18	42	6	56	18	42	18	42	0	64	16	42	18	42	6	56	...
18	22	32	22	21	32	21	24	28	18	21	36	24	22	28	18	21	40	21	18	28	22	24	36	21	18	28	24	21	32	21	22	...
24	6	20	16	30	14	24	6	26	16	32	20	18	6	20	14	36	18	22	6	26	18	30	18	18	0	28	16	32	18	22	6	...
30	22	15	24	16	10	18	18	16	19	16	10	23	24	15	19	14	8	24	24	13	19	16	12	20	22	13	19	20	11	22	22	...
36	4	10	0	2	4	6	4	6	0	0	2	4	2	6	2	2	0	2	2	6	2	2	2	4	4	6	0	0	2	4	4	...
42	4		2	0	6	0	6	1	8	0	2	0	2	0	4	0	4	0	4	0	4	1	4	2	4	2	6	0	6	0	4	...
48			0	2		4				4	2	2	2	2	2	0	0	2	0	0		2		6	0			2		4		...
54			1	2								2		2		2	0	2	0	2					0							...
60																	2		0						2							...
66																			1													...

The quantities for $2n = 2^{31}$ are the same for $2n = 2^1$, those of $2n = 2^{32}$ are the same for $2n = 2^2$, etc.
In addition, each column is indeed distinct here.

Two other examples are:

Case $2n = 3.2^m$.
The sum per column is equal to 270.

2n/3 Spacings Δ(j)	2^{1}	2^{2}	2^{3}	2^{4}	2^{5}	2^{6}	2^{7}	2^{8}	2^{9}	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}	2^{15}	2^{16}	2^{17}	2^{18}	2^{19}	2^{20}	2^{21}	2^{22}	2^{23}	2^{24}	2^{25}	2^{26}	2^{27}	2^{28}	2^{29}	2^{30}	2^{31}	...
2	21	56	24	42	28	42	21	64	21	42	28	42	24	56	21	42	28	48	21	56	21	42	32	42	21	56	21	48	28	42	21	...
4	42	21	56	24	42	28	42	21	64	21	42	28	42	24	56	21	42	28	48	21	56	21	42	32	42	21	56	21	48	28	42	...
6	104	42	60	56	64	48	92	42	60	56	72	42	88	42	64	64	64	42	88	42	72	56	60	42	92	48	64	56	60	42	104	...
8	28	22	21	26	24	30	28	18	21	28	21	32	28	22	24	20	21	30	28	24	21	26	21	32	32	18	21	20	21	36	28	...
10	20	60	32	39	43	48	28	53	22	43	39	60	27	51	30	39	48	54	20	53	22	48	43	48	26	51	32	43	39	54	20	...
12	0	30	16	48	12	32	8	28	20	48	8	26	8	36	18	52	8	24	10	36	18	54	8	28	8	40	18	48	14	28	0	...
14	22	5	40	8	38	11	30	2	40	8	34	11	30	4	38	8	40	9	30	4	34	5	40	11	24	4	40	4	40	12	22	...
16	4	4	4	5	4	8	2	8	4	4	4	7	4	8	4	2	2	7	6	8	4	2	8	9	4	4	2	4	4	7	4	...
18	8	20	4	14	4	16	4	20	2	10	10	12	4	20	2	10	8	20	2	18	10	6	8	22	2	18	4	14	4	14	8	...
20	4	2	4	0	3	3	0	2	6	0	2	8	0	1	5	2	0	4	2	0	2	2	0	4	4	0	4	4	2	2	4	...
22	2	4	1	8	0	2	0	8	2	10	0	0	0	2	0	10	1	0	0	4	0	4	0		0	4	0	8	0	2	2	...
24	4	0	4		4	2	2	0	4		10	0	4	0	2		4	2	4	0	6	0	8		2	0	2		10	0	4	...
26	0	4	4		0		0	4	4			2	0	2	2		4	0	0	2	4	0			0	4	4			1	0	...
28	8				4		7						7	0	2			2	7	0		0			9	2	0			2	8	...
30	2						6						2	0	0				4	2		2			4	0	2				2	...
32	1												0	0	2							0									1	...
34													2	0								0										...
36														2								2										...

Here, there are not only multiple deviations of 6, but also intermediate even integers.
Each column is still district following the conjecture.

Case $2n = 15.2^m$
The sum per column is equal to 360.

2n/15 Spacings Δ(j)	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}	2^{15}	2^{16}	2^{17}	2^{18}	2^{19}	2^{20}	2^{21}	2^{22}	2^{23}	2^{24}	2^{25}	2^{26}	2^{27}	2^{28}	2^{29}	2^{30}	2^{31}	…
2	84	63	63	96	63	63	84	63	72	84	63	63	84	72	63	84	63	63	96	63	63	84	63	72	84	63	63	84	72	63	84	...
4	63	84	63	63	96	63	63	84	63	72	84	63	63	84	72	63	84	63	63	96	63	63	84	63	72	84	63	63	84	72	63	...
6	86	96	128	78	78	128	96	86	120	78	86	144	86	78	120	86	96	128	78	78	128	96	86	120	78	86	144	86	78	120	86	...
8	28	39	30	20	43	32	20	43	22	27	48	22	26	39	32	28	39	30	20	43	32	20	43	22	27	48	22	26	39	32	28	...
10	54	24	32	54	24	30	48	36	32	48	34	22	60	38	22	54	24	32	54	24	30	48	36	32	48	34	22	60	38	22	54	...
12	26	34	32	20	30	28	28	28	32	30	26	20	18	26	30	26	34	32	20	30	28	28	28	32	30	26	20	18	26	30	26	...
14	10	10	6	13	18	6	11	16	9	13	10	10	15	15	10	10	10	6	13	18	6	11	16	9	13	10	10	15	15	10	10	...
16	4	4	2	10	4	4	4	2	4	4	5	6	2	4	7	4	4	2	10	4	4	4	2	4	4	5	6	2	4	7	4	...
18	4	6	0	4	4	2	4	2	2	2	2	6	6	2	0	4	6	0	4	4	2	4	2	2	2	2	6	6	2	0	4	...
20	0		0	0		0	0		0	0	2	0		0	0	0		0	0		0	0		0	0	2	0		0	0	0	...
22	1		2	2		4	0		4	0		4		0	4	1		2	2		4	0		4	0		4		0	4	1	...
24			0				2			2				2				0				2			2				2			...
26			0															0														...
28			0															0														...
30			2															2														...

Again, each column is distinct.
We also note, by comparing the three examples, that the maximum spacings Δ(j) are reduced at least approximately in the inverse ratio to the characteristic ratio of related prime numbers:

$$\prod_{\substack{p_k \backslash n \\ p_k > 2}} (p_k-2)/(p_k-1)$$

Note: In the case of systematic research, the extent of the spacings is much larger, with the largest of the spacing values being given by (i.e. asymptotically $\prod p_k$) :

$$6 + \prod_{k=0}^{i} p_k - 6 \prod_{\substack{p_k \backslash n \\ p_k > 2}} (p_k-1)/(p_k-2) \prod_{k=1}^{i} (p_k-2) \qquad (108)$$

The quantities that appear are therefore very specific values and limited to a small domain.

7. Theorem of density of prime numbers.

Here we outline a process that can be applied to many Diophantine equations with asymptotic branches. It leads systematically for all the mathematical literature’s standards to their known Euler products (also called singular series). It enables also to find many more of these products as we have proposed in other articles.

7.1. Equivalent of a prime number variable.

We want to restore somehow the Euler product of Hardy-Littlewood formula.
To do this, we seek to solve the problem by creating local equivalents (i.e. modulo p_i) of global variables p and q (hence of the set of primes P) in the equation p-q = 2n. These equivalents then enable the Euler product evaluation.

Theorem 27

The Chebotariov density theorem extends the Dirichlet theorem on the infinite number of prime numbers in arithmetic progression by trivial application to a cyclotomic extension of Q. Thus, if c, a ≥ 1 are two relative prime integers, the natural density of the set of prime numbers p = c mod a is 1/φ(a), a some constant.

Corollary on the variables of prime numbers

Let us have p a prime number.
We project the prime numbers set P on the classes of congruencies modulo p.

$$\begin{array}{ccc} P & \xrightarrow{\text{modulo}} & \{0, 1, 2, \dots, p-1\} \\ p_i & & p_i \text{ mod } p \end{array} \qquad (109)$$

This application projects a unique number to 0. That is p. The other classes are images in same density of all the other prime numbers. By assigning a probability density to the quantities of numbers projected on each of the congruencies 0, 1, 2, ..., p-1 and arbitrarily adding all densities up to p (i.e. an average density of 1 for each class), we obtain the following correspondence :

Congruencies	0	1	2	…	p-1
Normalized probability densities ($\sum = 1$)	$\to 0$	$\to p/(p-1)$	$\to p/(p-1)$		$\to p/(p-1)$

7.2. Reconstruction of De Polignac formula.

We start with a formula such as $\#(p\text{-}q = 2n) = c_n.x/\ln^2(x)$, for n an even integer. Here, c_n is an infinite product (so called also Euler product).

To evaluate the solutions of a Diophantine equation $q_1\text{-}q_2 = n$, n a given integer (even or odd at this stage), q_1 and q_2 variables representative of prime numbers, we transform the initial global problem in a series of local problems $q_1\text{-}q_2 = n$ mod p, the generation of the infinite product being related to equality $\#(q_1\text{-}q_2 = n \bmod \Pi\, p_i) = \Pi\, \#(q_1\text{-}q_2 = n \bmod p_i)$ issued from Chinese theorem.

Heuristically, the independent variables of a Diophantine equation with asymptotic branches induce class instances in crossed charts based on $\{0^n, 1^n, 2^n, \ldots, (p\text{-}1)^n\}$ for variables x^n of natural integers and based on $\{1^n, 2^n, \ldots, (p\text{-}1)^n\}$ for variables of prime numbers. Let us note that the "mechanics" of these crossed tables allows changing the problem of enumeration essentially into a product of matrices problem that we will not develop here. The interested reader can refer to our articles on asymptotic enumerations in hyperplanes on free access [7].

Here $q_1\text{-}q_2 = n$, n being a given integer (even or odd at this stage), q_1 and q_2 the representative of the prime numbers variables, we look at the classes of congruence modulo p such as $cq_1\text{-}cq_2 = n$. For each variable, representative classes are locally :

		cq_2 mod p			
	cq_1-cq_2 mod p	1	2	…	p-1
cq_1 mod p	1	0	p-1		2
	2	1	0		3
	…	…	…		…
	p-2	p-3	p-4		p-1
	p-1	p-2	p-3		0

Thus we have for the classes collected inside the table :

$$\#\{n = 0 \bmod p\} = p\text{-}1 \quad \text{(principal diagonal)}$$
$$\#\{n \neq 0 \bmod p\} = p\text{-}2 \quad \text{(other diagonals)}$$

This gives the density, to a given factor, of the numbers n at the sequence p (including for p = 2).
The overall proportion is then rendered by the product of these values for p = 2 to ∞.

To obtain the Euler factor, one simply adjusts the average of the frequencies to 1. In the classes [0, 1, 2, p-1], one has the target 0 with cardinal #(0) and p-2 other targets with equal cardinal $\#\{c\neq 0\}$. The adjustment factor f is then given using $f.(1.\#(0)+(p\text{-}1).\#\{c \neq 0\}) = p$ the number of elements, that is $f.((p\text{-}1)+(p\text{-}1).(p\text{-}2)) = p$, so that $f = p/(p\text{-}1)^2$.
Hence :

$$\#_{adjusted}(n = 0 \bmod p) = f.(p\text{-}1) = p/(p\text{-}1) = 1+1/(p\text{-}1)$$
$$\#_{adjusted}(n \neq 0 \bmod p) = f.(p\text{-}2) = p.(p\text{-}2)/(p\text{-}1)^2 = 1\text{-}1/(p\text{-}1)^2$$

The cardinals of the twin and distant relative prime numbers are then :

$$\pi(p\text{-}q = 2n) = \prod_{p \nmid n} \left(1 - \frac{1}{(p\text{-}1)^2}\right) \prod_{p \backslash n} \left(1 + \frac{1}{(p\text{-}1)}\right) \frac{x}{\ln^2(x)} \qquad (110)$$

This process is reproducible to many Diophantine equations with asymptotic branches (infinite number of solutions), as for example Iwaniec/Friedlander equation generalized to $x^2+x^4 = p+c$, c a given constant, but also an more complicated equation as for example $p = x^3+x^2y+xy^2+y^3+5t^2+9u^4+c$, giving their Euler products, parametrized in c, which seems impossible to achieve by any other means (indispensable complement in reference [7]).

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Numeric example

This example uses as reference axis p_i and rounding to an integer for withdrawals. It shows that the coefficient c is close to 1. The value of c is less than 1 indicates a reduction in the number of solutions (when c is taken as equal to 1).

M = 1260799, 10000th twin prime number (with M-2).
i = 1408, p_i = 11731.
c = 0,994575.
Initial number of odd integers: 630398. Total number of withdrawals : 620398.

List of withdrawals:

p_i	nb removals	p_i	nb removals	p_i	nb removals	p_i	nb removals
2	-417984	157	-242	367	-78	683 à 691	-34
3	-83596	163	-233	373	-77	701 à 709	-33
5	-35827	167	-222	379	-76	719 à 733	-32
7	-16285	173	-212	383	-74	739 à 751	-31
11	-11274	179	-208	389	-72	757 à 769	-30
13	-7295	181	-195	397	-71	773 à 787	-29
17	-5759	191	-191	401	-69	797 à 811	-28
19	-4256	193	-185	409 à 419	-67	821 à 829	-27
23	-3082	197	-181	421	-65	839 à 859	-26
29	-2684	199	-169	431	-64	863à 883	-25
31	-2104	211	-158	433	-63	887 à 911	-24
37	-1796	223	-154	439	-62	919 à947	-23
41	-1629	227	-151	443	-61	953 à 977	-22
43	-1421	229	-148	449	-60	983 à 1019	-21
47	-1206	233	-143	457	-59	1021 à 1061	-20
53	-1043	239	-140	461 à 463	-58	1063 à 1097	-19
59	-974	241	-133	467	-56	1103 à 1151	-18
61	-858	251	-129	479	-55	1153 à 1193	-17
67	-785	257	-125	487	-54	1201 à 1249	-16
71	-742	263	-122	491	-53	1259 à 1321	-15
73	-667	269	-120	499 à 503	-52	1327 à 1399	-14
79	-619	271	-116	509 à 521	-50	1409 à 1487	-13
83	-563	277	-114	523	-48	1489 à 1579	-12
89	-505	281	-112	541	-47	1583 à 1697	-11
97	-475	283	-108	547	-46	1699 à 1823	-10
101	-456	293	-102	557 à 569	-45	1831 à 1993	-9
103	-431	307	-100	571	-44	1997 à 2161	-8
107	-415	311	-99	577	-43	2179 à 2417	-7
109	-393	313	-97	587 à 599	-42	2423 à 2741	-6
113	-343	317	-92	601	-41	2749 à 3181	-5
127	-328	331	-90	607 à 617	-40	3187 à 3797	-4
131	-308	337	-87	619	-39	3803 à 4799	-3
137	-300	347	-86	631 à 643	-38	4801 à 6661	-2
139	-275	349	-84	647 à 653	-37	6673 à 11731	-1
149	-268	353	-82	659 à 661	-36		
151	-254	359	-80	673 à 677	-35		

Appendix 2

Numeric example

This example uses as reference axis p_i^2 (squared) and rounding to an integer for withdrawals. It shows that the coefficient c is close to 1. The value of c, less than 1, indicates a mark-up of the number of solutions (when c is taken equal to 1).

M = 1260799, 10000th twin prime number (with M-2).
i = 183, p_i = 1093.
c = 1,00406412.
Initial number of odd integers: 630398. Total number of withdrawals : 620398.

List of withdrawals:

p_i	nb removals	p_i	nb removals	p_i	nb removals	p_i	nb removals
2	-421972	149	-266	347	-78	563	-34
3	-84393	151	-252	349	-77	569	-33
5	-36168	157	-239	353	-75	571	-33
7	-16439	163	-230	359	-73	577	-32
11	-11380	167	-220	367	-71	587	-31
13	-7363	173	-209	373	-69	593 à 599	-30
17	-5812	179	-204	379	-68	601 à 607	-29
19	-4296	181	-191	383	-66	613 à 619	-28
23	-3110	191	-187	389	-64	631 à 641	-26
29	-2708	193	-181	397	-63	643 à 647	-25
31	-2122	197	-177	401	-61	653 à 659	-24
37	-1811	199	-165	409	-59	661 à 673	-23
41	-1642	211	-154	419	-58	677 à 683	-22
43	-1432	223	-149	421	-56	691	-21
47	-1216	227	-147	431	-55	701 à 709	-20
53	-1050	229	-143	433	-54	719 à 727	-19
59	-981	233	-138	439	-53	733 à 739	-18
61	-863	239	-135	443	-52	743 à 751	-17
67	-790	241	-129	449	-51	757 à 769	-16
71	-746	251	-124	457	-50	773	-15
73	-671	257	-120	461	-49	787 à 797	-14
79	-622	263	-116	463	-48	809 à 821	-13
83	-565	269	-114	467	-47	823 à 829	-12
89	-507	271	-111	479	-45	839 à 857	-11
97	-476	277	-108	487	-44	859 à 877	-10
101	-457	281	-106	491	-43	881 à 887	-9
103	-431	283	-102	499	-42	907 à 911	-8
107	-415	293	-96	503	-41	919 à 941	-7
109	-393	307	-93	509	-40	947 à 953	-6
113	-343	311	-92	521	-39	967 à 983	-5
127	-327	313	-90	523	-38	991 à 1009	-4
131	-307	317	-86	541	-36	1013 à 1033	-3
137	-298	331	-83	547 à	-35	1039 à 1063	-2
139	-274	337	-80	557	-35	1069 à 1093	-1

APPENDIX 3

Research of the centres M_1 and M_2 of the maximal spacings in cycle 1.

Code

https ://pari.math.u-bordeaux.fr/

```
{infini = 49; pd = 1;
for(c = 1, infini, q = primes(c)[c]; pd = pd*q; p1 = primes(c+1)[c+1] ; p2 = primes(c+2)[c+2] ;
for(k1 = 1, p1*p2, M1 = pd*k1;
if(Mod(M1-1, p1) == 0, if(Mod(M1+1, p2) == 0, print("i="c+1", pi="p2", M1="M1", k1="k1)))))}

{nb = 49; pd = 1;
for(c = 1, nb, q = primes(c)[c]; pd = pd*q; p1 = primes(c+1)[c+1] ; p2 = primes(c+2)[c+2] ;
for(k2 = 1, p1*p2, M2 = pd*k2;
if(Mod(M2+1, p1) == 0, if(Mod(M2-1, p2) == 0, print("i="c+1", pi="p2", M2="M2", k2="k2)))))}
```

Note 1 :

The code makes no distinction between p_j and its multiples. For c = 1 and c = 2, it gives a result for M_1, which is not to be taken literally. One has to take M_1+2.3.5 and M_1+2.3.5.7 respectively.

Note 2 :

$M_1+M_2 = 2.3.5\ldots p_i$ and $k_1(i)+k_2(i) = p_{i-1}.p_i$.

List of values

i=2, pi=5, M1=4, k=2
i=3, pi=7, M1=6, k=1
i=4, pi=11, M1=120, k=4
i=5, pi=13, M1=9450, k=45
i=6, pi=17, M1=217140, k=94
i=7, pi=19, M1=9639630, k=321
i=8, pi=23, M1=193483290, k=379
i=9, pi=29, M1=417086670, k=43
i=10, pi=31, M1=125601285810, k=563
i=11, pi=37, M1=2723740849830, k=421
i=12, pi=41, M1=79622514581610, k=397
i=13, pi=43, M1=6136950437487870, k=827
i=14, pi=47, M1=223928193956026560, k=736
i=15, pi=53, M1=9171015693500691030, k=701
i=16, pi=59, M1=522656315200217698500, k=850
i=17, pi=61, M1=102036655192082030049630, k=3131
i=18, pi=67, M1=6235511815550111588504010, k=3243
i=19, pi=71, M1=334506463637028681244286040, k=2852
i=20, pi=73, M1=28478557301114887810505822160, k=3624
i=21, pi=79, M1=2843824411155784604050916242830, k=5097
i=22, pi=83, M1=113432160468908532259480385863950, k=2785
i=23, pi=89, M1=5778890002143848542586755859217480, k=1796
i=24, pi=97, M1=184675112599134251212414008442014285 0, k=6915
i=25, pi=101, M1=72708581460921039807419994522555070290, k=3059
i=26, pi=103, M1=1928607601840426162305969946243013952555 0, k=8365
i=27, pi=107, M1=127375713304098056412334634333637527599290 0, k=5470
i=28, pi=109, M1=249658028112582700049702183737147717646149890, k=10409
i=29, pi=113, M1=27763056840142703665840289166348895092331376460, k=10818
i=30, pi=127, M1=2574121440717901122712497241546767984629324510460, k=9202
i=31, pi=131, M1=521439461348328858073243322985304256489059236587040, k=16496
i=32, pi=137, M1=25291204717798127991294979553864084360869077060699 0, k=63
i=33, pi=139, M1=609356250278279763231788501267803683477937927987362381 0, k=11587
i=34, pi=149, M1=14539249597776378098009807524942143191198031172775880558 00, k=20180
i=35, pi=151, M1=185841797895169170082768839224027937644547072159440436851130, k=18557
i=36, pi=157, M1=28828963020146876463537479231418049444339320239475022727521200, k=19320
i=37, pi=163, M1=53400729793063989026946985986271104320112383716878500949782670, k=237
i=38, pi=167, M1=113837287385782898397165898489028712874401262344909947345379321660, k=3218
i=39, pi=173, M1=10898027695754548635554660766785293839408575789495465344527970050900, k=1890
i=40, pi=179, M1=1046531256855867331925489147441352897650668311142832370362069122312236 0, k=10868
i=41, pi=181, M1=236924161166333927003447281128044655640144480745831028803188906970165562 0, k=14222
i=42, pi=191, M1=41714628337047975641145345156120010343639897642136409568631551252243752101 0, k=13989
i=43, pi=193, M1=1859979505963720510623212973307129554920606096138858879377902292332164710476 90, k=34461
i=44, pi=197, M1=15463397128887900127493413332535115564979357724493864627891500732804189940850 0, k=150
i=45, pi=199, M1=11245353513658374282316270017339516881626548441837326210153768728912794225544700 40, k=5652
i=46, pi=211, M1=139775386898730614296693311221521353093754097248618356911036498033309542658879093259 0, k=35661
i=47, pi=223, M1=3272379293367879460050168993069610113825682173236135763504330844506335329882538577827 40, k=41954
i=48, pi=227, M1=7432358515391170111013883847643127727562951629176136010356468412522807798534475449029560 0, k=45160
i=49, pi=229, M1=5059963172703425132303431555239735128768826914715159286974528895210315566740930757864665910, k=13787
i=50, pi=233, M1=128974082846109606552651042480693835301100584238355853116814091351798456738511722907000483791 0, k=15481

i=100, pi=547,
M1=80412016839434108281095506348288547775055372634944231497489040955191413286856337240809303257918560584677530551908923495673445362244076882800327681045087287866730366491280307321581179108683251508564240264588072798167318925O, k1=92325
i=150, pi=877,
M1=30759861815777997528759427699830716416669506094742302068530139209761229815155668993038847346593920284706826977723711020622682366581397852101794145065611016075132772586976511414739703279463409866182158852140300093875736482536407248052011367535802492904042534164627876795442191138848011141606041097369544579245117181553387123860884412525715809794800030499661020149540, k1=410114
i=200, pi=1229,
M1=52471110922700994798602150688716541272004139399046591933812949919422347095369762984106978127003320393549928245844746913541091005422910485981695314174403881214217138929401694464579842759856610463325321827479240290421796085223297097744066933636889569384560356763970555049402253209837923179783332960608475211196527586449302888756228881667226924071140532962391832782516018986990863852615631575635155628543704707022537258764799391798318648271273332933347472654157095704534791585990961382617049958339935177090932805184440, k1=12972
i=250, pi=1597,
M1=124331663774261465714573774927397749906275079803921917178820772051241939205194836356881897262886072913265100594165176904678205706106466970445242592120829878315033595789312866953303779388327876531548881472479886557904421024907313146513725626989984459074519559193143583123141281686770209660132797233800485410651431623263928040430949288548807251130161724537221325028211585609482008814810417864376320459257846095951125012230759826238475114493987667995795117964983080089211436148697391807548458717120745935024294389862152270524801907880127861970611261448595260891774019179367631628518624683587699350709944435750816676421106155724678491059366186364291037610910485852183731401380, k1=156866
i=300, pi=1993,
M1=47363098369411624698393817097247053349398585751234758952663935169270736562354097324368393946517432879758586170791784281639207528251867387374857280577311884727097362900202964810604305793986742035732354120936255106364365627742120282256914255266792649793726386652034464397369971920112821291460872267528663671599353204269433550438838091231314822764875102341857119520210542296828375441129298501271664093261002471412454817792920425735813876877574763097132700109815242995833513856689396217640246135277383774484738272970816473999381463964635516193204788970516012645912641624922365924793487295098901837937573009232457465802047724223116189378861844080963673286092187759307863991129374940131226231537328409043523047358089652747975690648635363482029550290996733985324924044349872634006463918985909844347222074962243672124521331491765441365057078880, k1=2587472
i=350, pi=2371,
M1=3639153206754216293159858779234324793239910864276473132609059037199263233146833971405678790951542119441411922391175608348169753440412440901177607275623159046558485464408361432790395306857145078019775390806876213531855112642191828450498465640222615336939793572687994619104784699887595302345206363656991951234393043193125621342900606982381677131228591725451981164821001165495395870811637984910162703119034081094002256085552151885569166511762392380884500857506598764702657650083013232629601968068507479885152708718538174342382088837740295044066537124475457774646604604077401915053281699135446390931354130278168265598212083319406047764470688127387962727566705119078735589135772149451456382180700049182623824006901156682657506159054065489571778865297608405605410797524854920742167479042578258205874451522293629229743155251742460206370130490659215665318016592886801436657575321935831538612574269268801193885923754178107620209515069770615430246294844537970166864080148240763726960143744182855521095602865958040, k1=3615004
i=400, pi=2749,
M1=23700629185613346910172035534986793117953035354347270821431332090752962625539864378390362534674627276134625401023936968080144820681982488451941949239918324404246450462751592622486029351744404769631732258474988810425493070751047781308843143775582754558235040972555539373888527391325122142828238982531865002903260043094223316270923525763144356211361754510358234325716621736933847205790326432704988271150519756134118818449735787759349735075897610964385098610114119310936845941358734036320726701397650129822643515111504284297965472097645389111974836394240156480994413913044488110325420830690882206076544546940965501606851997056661394234852233782877652512222463093887898790946480418891712793052657257062827684399993127160647328588712891631598941901518170535292981987619406714266203790127079051768499823354479215680998269265857699458359398863614791057873109824090157638121874584170165427134839575202249346509359180602081059464246109499982230483384880696132193941073033442088107662525005848803576438035614385695241933703734526499747218103126478235456825408790933834522108297269842159385992675408078240598238118897075694980283383243599250396291208654661034934703577898526143507420, k1=105238
i=450, pi=3187,
M1=266220323900768984403072100244952505560612290652667249035029645013452327471468595106813360545049562208162264884575847102589216756084098053926854521506004691664984765644931369839542647051377017700835514274667671363938256767509818365899103847772159139771037134639559268061083349997806989203579659895654228389397932454234492920147427034840512622053872783078033889970582821915373444771742600719555358477289253547283459288720272867999054136559862497538376169251792527062902265175478487997969748998852005270753718809524394971926831491364007174859293708358045154957092131905352737871951655713206750001262178474992751491835953551499926764435218856894276745074646475043037456168557322887223247087438076907806242384753832333145406437921595604334518123613764003358661929574971123650921778955280470992774974283051500504852935693022692995731394786425606535232115534528139558712936147039251274220694926227654181423870624999050583802199629741391307029379415665309117068723929412087691988536084163932060827783641688606545807150643966501957913072521535234518606673034897203574070965540322254490315348838213630372925810992872752250119887863934061916755501162570484002302991263452063145672664958342155280259564228316450525888361567777442177981830752660961593864951933334459687903325009718570637117049697613516393706461263217203356447219867752291493884022733189296890, k1=4295639
i=500, pi=3581,
M1=33671540806849286766346993131414680749725356916947791441348072938803570552963046895167004359685153140079754240599904642245192987858023530194358454251512790556164406626672483990971943275472259591481958738392724041032668408116723797538285945729385498617977882787409226490678812016741963414348395380580245753493122557596753072956354322451399516044739794559650462611865457611295367302098720863470074886466706709648086262688633593459676153340555976267923440743622390033774614907257895861442463079984987071218922980143351860039271521846276501778206966942437735498612085387828175704185215818043707202812515443537275808064270450152361534631519303107476711459703966304349090980088534167211369935600351805208678813046270741094337430866146718574198745772699924010189544659436711825748476612618342165495352514730477695428883666825410488576687160874375911155879911153884365880648201088278397382226739730294427900536221685880880822900669588007821971706650790734504959309138538715851921938159428858879004620737720591601492975791090997891315583428712558459933129149404973814995115282471917189977259420641249565570644540197301158425584289005163048078118992223479762309820931694768779561157318827018245124937219841344899969939224767827224165643033966709886777741917109455797869661007494376217052956298502414882417751544360586458878613590138254232873765248431313844039264564484163227175236052852374428221355120252820644148217696356632394890148304829238192169897709086272517330922772369308068735560405150126614850232588419329008655773233580170, k1=2156443
i=1000, pi=7927,
M1=631395736506638254508083999999737146532209969866754137669438242579679561467247996229490487333558368514054300493010690530762937900529012430849450426057547405737502354970185040517235535061521588550946887886891312836549402631246979827534592817700976449743324496038207129104273044207416387063952214603192859060813383235932900136608352310777217325802582523370570196275192928021186444835675592633069756588966480253931654813368769520201790953323078751210077995781735760166419535006361846581353840884076853826502101504639469014192025425371216640286466388912969918989687009409133201426862481633757719513358193280468

7833808274271358929001796878833699651519205145459665407863742091946999157772007522823224304841433194131485832822500483224
76769923465891928363599247515238949023550129000829865035273147757110741947328458145720116356908091818592407863204207545863
38649460169551701877850331934902852466733468709592897301090230391141141267605035084334307630544972060415729465192863366999
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11501060149409196296578852780151513630422447571826114331961376977796616925522889150524491882767889275912775371890426773678
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93194046226880141752710859796846442028992660845374076583019792900338320251838855631511128728886037571724652489897173584617
60469114415237159000595645281384547287750650122893794747278742978633425389376186536643792567035193813347380263440770086724
34404955300655874324954065163284247095707347315331282439579218894971341742803407102827592536982693551712342552926076860163
03497100156171529257759717363283129512470777433459723288286351089496730838240027286843577654661691070412109947409617227153
12766908773529948292577541241430253657213648437375394265866157740066048513934646568386469144342917813876992285352656060972
48583818668735897319253739930213145017062426546858661517997507976807310899984018961861338441599106233534526060809849640869
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95183333570732946553809470835704816651830927392443914247549656701074589917898594603728569269440974357326814777877516997659
6287398243799238011180625992422804415184010334246947520602554357744445427931046843172196174230080387372690, k1=7367823
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24693531773488869501953276815916191871590984216817312686487890203483649293212295196978639116705208094380837545659130767009
33586963704605405456525596633735796479923496194347337856290320152318481286942517256999330085661091577857391761736762046833
950171358396996661303942412709583250, k1=27427725

The last number M1 contains 5400 digits that are distributed relatively evenly between the different values from 0 to 9 :

Values	0	1	2	3	4	5	6	7	8	9	All
Quantities	557	544	515	528	546	533	556	523	550	548	5400
Percent	10,3%	10,1%	9,5%	9,8%	10,1%	9,9%	10,3%	9,7%	10,2%	10,1%	

APPENDIX 4

Bijection between related pairs.

Beyond the problem of the bijection, we show here the random behaviour of the depletion which accredits heuristic calculations.
We checked that when two gaps 2n and 2m have same divisors systematically, implementing the Eratosthenes sieve, at the same step i, the same number of elements exist between [p_i+2+2n, p_i+2+2n+2.3… p_i] and [p_i+2+2m, p_i+2+2m+2.3… p_i]. So, there is a bijection at every stage between these elements by matching the numbers in their appearing order. However, at each step's increment, integers in correspondence do not stay the same. The bijection is not sustainable. It has to be redone at each stage.

Let us observe cases 2n = 2 and 2m = 4 and clarify explicitly step i = 2, p_i = 5.

3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43
	2	2			2			2						2						2
		4		4			4			4									4	

We consider here only the entry and the cycle 1, bijection continuing next elementary up to infinity. In the entry, we do not care to have a strict bijection from beginning (5 has no match). Our focus is mainly on the evolution in the cycle 1. Of course, as more numbers are observed and stage i increases, it will match not only primes among these lists. On the contrary, these will become extreme minority. Nevertheless, even if we attest of this minority, we increment i up to infinity and analyse distances among construction.
We are matched to start with the two cycles 1:

	2		2		2
4		4		4	

This gives us an advance or a delay from one to the other, here:

-4	-4	-10

Classifying the differences in ascending order, we get for steps 1 through 3, the following results :

2

-10	-4	-4

-28	-28	-22	-22	-16	-16	-10	-4	-4	2	2	2	8	8	8

Beyond, a graphical representation is more meaningful and we clarify its construction : For i = 4, p_i = 11, (3-1).(5-2).(7-2).(11-2) = 135 gaps are identified, for example, as follows :

Abscissa x'	1	2	3	4	5	6	7	…	129	130	131	132	133	134	135
Ordinate y'	-70	-70	-70	-58	-58	-52	-52	…	26	26	38	38	44	50	56

A new abscissa is chosen using x = -1+2.x'/∏(p_i-2) in order to get x in the interval [-1,1]. Thus for ∏(p_i-2) = 135 :

Abscissa x	-0,99	-0,97	-0,96	-0,94	-0,93	-0,91	-0,9	…	0,91	0,93	0,94	0,96	0,97	0,99	1
Ordinate y	-70	-70	-70	-58	-58	-52	-52	…	26	26	38	38	44	50	56

Then we compare the results in ordinate with the following formula :

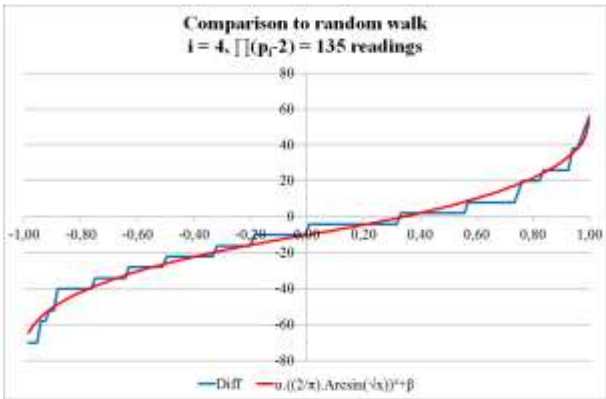
$$y = \alpha.((2/\pi).\text{Arcsine}(|x|^{1/2}))^2+\beta$$

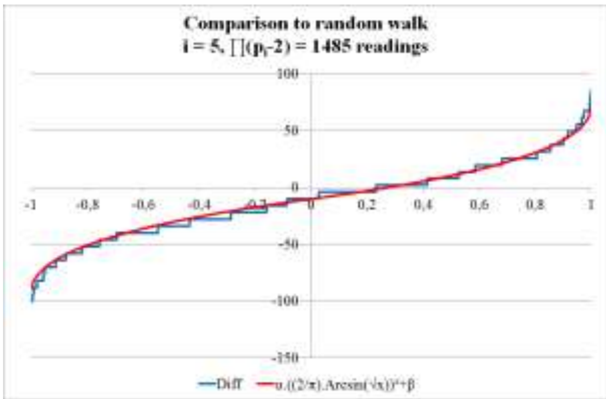
This is done by adjusting the coefficients α and β approximately.

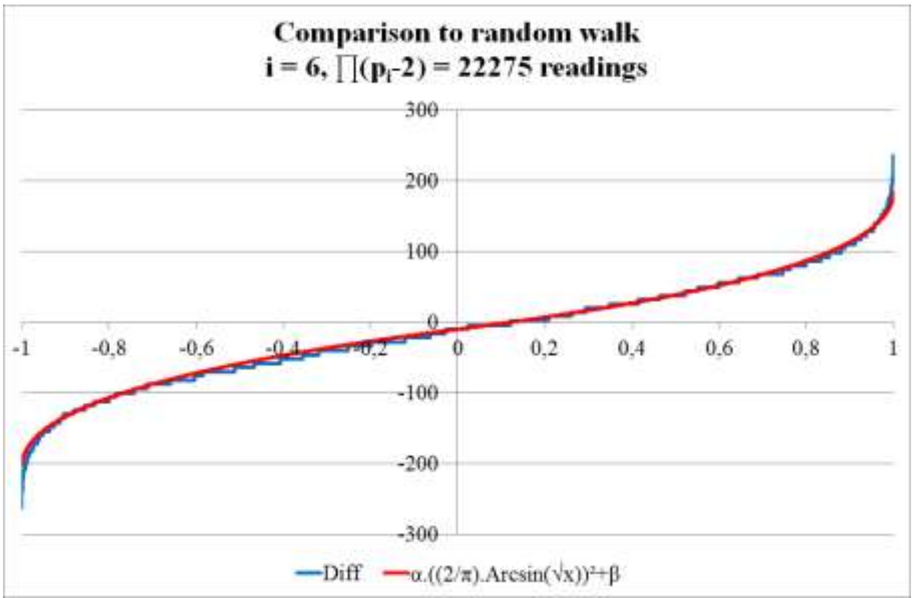
Table 78

i	4	5	6
p_i	11	13	17
$\prod(p_i-2)$	135	1485	22275
α	64	82	195
β	-10	-10	-10
|max(ordinate)/α|	1,086	1,134	1,293

Graphs 24, 25 and 26





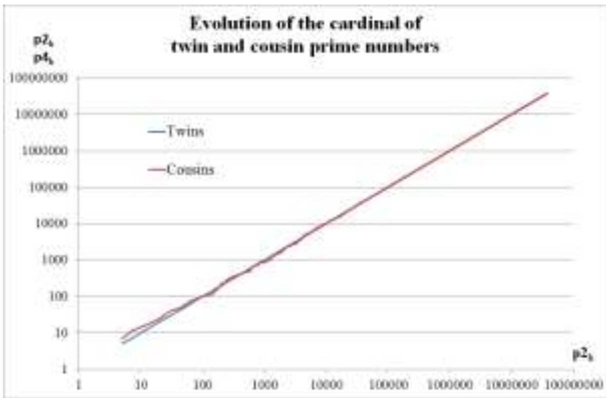


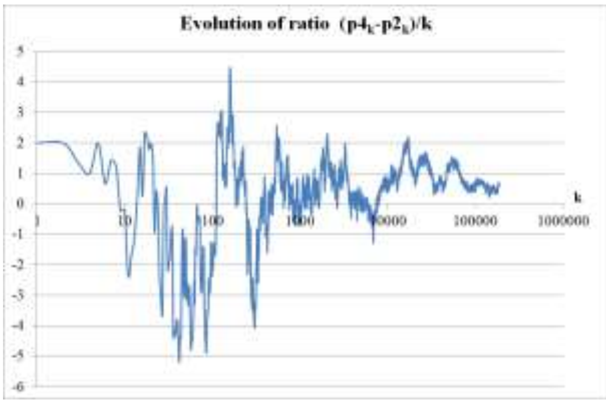
The coincidence of both these curves occurs within two "random" walk, namely that of the integers related by a distance of 2 on the one hand and the integers related by a margin of 4 on the other hand, and illustrates their independence, hence the square of the expression (2/π).Arcsine($|x|^{1/2}$), expression found on the occasion of one random walk only.
The β factor here has only a minor role. It becomes negligible as i increases. Knowledge of the adjustment factor α would on the contrary be valuable, even if the assessment does not give a good approximation of the maximum and minimum values of ordinate (that is, distances at the extreme left and right of the curve), the ratio |max(ordinate)/α| showing here on the rise when we try to match the curves “at best”. Indeed, red curves do not follow correctly the blue when the slope increases quickly at the extremes, the maximum distances being superior to expected values for random walks.

What is the meaning of this type of curves? That the cardinal of small and medium distances is of the same order of magnitude (curve close to a straight line) and that large distances are few (slope towards a vertical). Of course, this is expected!

But now, let us go back to another feature of random walks : If actually, both sets follow such a walk, it is not surprising that one will exceed the other most of the time. Let us check that numerically by recovering the two lists of the twin prime numbers on one side and the cousin prime numbers of the other hand. When we then compare their differences, we find that after many differences’ returns to 0 the twin primes seem to prevail starting at j = 7790 ($p4_k$-$p2_k$ > 0 until at least k = 120000) as the theory of games so provides (cf. [6] p21) : “between two players to equal fortune, one of the two players will stay ahead much longer than the other; in fact, there will be one winning most of the time”. The cousin prime integers are slightly rarer (0.27%) than twin primes in this interval, the $p4_k$-$p2_k$ differences, k^{th} primes cousin and twin respectively, being of the order of magnitude of k between the origin and the last evaluation here.

Graphs 27 and 28





The probability of return to the 0 distance being proportional to $1/(\pi.k)^{1/2}$ (cf. [6] p18) and so becomes increasingly smaller as the random walk progresses. Of course, we are unable to verify this far beyond what is presented here despite our already performing computing resources (Pari GP). Neither are we able to verify that ($p4_k$-$p2_k$) will vary by an order of magnitude of $k \approx p2_k/\ln^2(p2_k)$ up to infinity.

Note: We could also choose to compare the positions of the remaining numbers (either twins or cousins) from their positions if they were all equidistant. Here, we would still have distances in arcsine distribution prompting again to favour simple heuristic calculations.

APPENDIX 5

Table of the quantity of spacings Δ in cycle 1 for given 2n.

Case 2n = 4.

Steps i	1	2	3	4	5	6	7
p_i	3	5	7	11	13	17	19
Cycle 1 sizes	6	30	210	2310	30030	510510	9699690
Spacings Δ	Quantity of spacings Δ in cycle 1						
6	1	2	6	42	378	4914	73710
12			2	16	154	2072	31850
18		1	4	32	288	3744	56160
24			2	20	252	3780	62244
30			1	15	214	3636	62988
36				10	126	1934	34010
42					27	601	13572
48					8	224	6160
54					22	528	12624
60					12	544	14308
66					2	160	5146
72					0	4	248
78					0	32	1489
84					2	72	2384
90						12	572
96						18	644
102							158
108							94
114							148
120							120
126							42
132							0
138							0
144							2
150							2
Number of spacings	1	3	15	135	1485	22275	378675
Ratio to the previous		3	5	9	11	15	17

Case 2n = 8.

Steps i	1	2	3	4	5	6	7
p_i	3	5	7	11	13	17	19
Cycle 1 sizes	6	30	210	2310	30030	510510	9699690
Spacings Δ	Quantity of spacings Δ in cycle 1						
6	1	1	4	28	252	3276	49140
12		2	6	42	378	4914	73710
18			2	22	260	3700	59020
24			2	16	154	2072	31850
30			1	24	288	4464	79344
36				0	16	492	10020
42				2	90	1932	35268
48				0	16	494	11836
54				1	19	337	7263
60					4	276	9440
66					2	46	1594
72					4	126	3538
78					2	60	2172
84						44	1782
90						40	1618
96						0	194
102						0	284
108						0	86

Steps i	1	2	3	4	5	6	7
p_i	3	5	7	11	13	17	19
Cycle 1 sizes	6	30	210	2310	30030	510510	9699690
Spacings Δ	Quantity of spacings Δ in cycle 1						
114						2	210
120							200
126							12
132							42
138							12
144							2
150							10
156							10
162							14
168							2
174							0
180							0
186							0
192							0
198							2
Number of spacings	1	3	15	135	1485	22275	378675
Ratio to the previous		3	5	9	11	15	17

Case 2n = 16.

Steps i	1	2	3	4	5	6	7
p_i	3	5	7	11	13	17	19
Cycle 1 sizes	6	30	210	2310	30030	510510	9699690
Spacings Δ	Quantity of spacings Δ in cycle 1						
6	1	2	6	48	432	5616	84240
12		0	2	14	154	2198	35126
18		1	3	21	189	2646	39690
24			4	30	294	3906	60606
30				16	260	4112	72112
36				2	44	1036	21268
42				0	14	418	9782
48				2	44	722	13640
54				2	44	988	21960
60					6	320	9168
66					0	92	2974
72					0	8	484
78					4	165	3793
84						34	2264
90						12	730
96						2	330
102							18
108							190
114							196
120							42
126							18
132							8
138							0
144							4
150							18
156							6

Steps i	1	2	3	4	5	6	7
p_i	3	5	7	11	13	17	19
Cycle 1 sizes	6	30	210	2310	30030	510510	9699690
Spacings Δ	Quantity of spacings Δ in cycle 1						
162							0
168							4
174							0
180							0
186							0
192							0
198							4
Number of spacings	1	3	15	135	1485	22275	378675
Ratio to the previous		3	5	9	11	15	17

<u>Case 2n = 32.</u>

Steps i	1	2	3	4	5	6	7
p_i	3	5	7	11	13	17	19
Cycle 1 sizes	6	30	210	2310	30030	510510	9699690
Spacings Δ	Quantity of spacings Δ in cycle 1						
6	1	1	3	21	210	2730	43680
12		2	6	48	432	5616	84240
18			4	32	312	4440	68712
24			2	14	154	2198	35126
30				10	161	2725	47597
36				4	52	906	15630
42				6	110	2006	38666
48					22	578	12270
54					4	128	3636
60					28	708	18024
66						68	2596
72						50	2312
78						36	2424
84						68	2178
90						10	786
96						2	120
102						6	418
108							98
114							4
120							86
126							12
132							52
138							4
144							4
Number of spacings	1	3	15	135	1485	22275	378675
Ratio to the previous		3	5	9	11	15	17

<u>Case 2n = 64.</u>

Steps i	1	2	3	4	5	6	7
p_i	3	5	7	11	13	17	19
Cycle 1 sizes	6	30	210	2310	30030	510510	9699690
Spacings Δ	Quantity of spacings Δ in cycle 1						
6	1	2	8	56	504	6552	98280
12		0	0	6	90	1410	25200
18		1	3	21	189	2457	36855
24			2	24	264	3768	60216
30			2	18	224	3676	61724
36				6	92	1504	27992
42				0	16	422	9194
48				4	64	1018	18786
54					32	786	16894
60					4	362	10646
66					2	96	3896
72					0	6	376
78					4	132	4316
84						60	2588
90						26	1382
96							48
102							28
108							52
114							64
120							84
126							16
132							0
138							16
144							4
150							12
156							0
162							0
168							6
Number of spacings	1	3	15	135	1485	22275	378675
Ratio to the previous		3	5	9	11	15	17

Case 2n = 6.

Steps i	1	2	3	4	5	6	7
p_i	3	5	7	11	13	17	19
Cycle 1 sizes	6	30	210	2310	30030	510510	9699690
Spacings Δ	Quantity of spacings Δ in cycle 1						
2	1	1	3	21	189	2457	36855
4	1	2	6	42	378	4914	73710
6		2	12	104	1088	15616	254464
8		1	4	28	252	3276	49140
10			2	20	218	3148	51058
12			0	0	0	0	0
14			2	22	246	3582	58338
16			0	4	68	1164	20988
18			0	8	124	2024	35180
20			0	4	88	1672	32088
22			0	2	38	682	12682
24			0	4	80	1540	30092
26			0	0	8	248	6072
28			1	8	92	1548	27128
30				2	56	1138	25122
32				1	14	310	6440
34					4	182	5422
36					8	278	7446
38					4	130	3726
40					9	214	5778

Steps i	1	2	3	4	5	6	7
p_i	3	5	7	11	13	17	19
Cycle 1 sizes	6	30	210	2310	30030	510510	9699690
Spacings Δ	Quantity of spacings Δ in cycle 1						
42					0	86	2612
44					4	132	3686
46					0	16	906
48					0	62	2706
50					0	44	1524
52					0	11	401
54					0	4	568
56					2	30	820
58						2	364
60						32	1096
62						0	40
64						0	226
66						2	152
68						2	96
70						2	184
72						0	16
74						0	28
76						0	16
78						2	84
80							14
82							8
84							44
86							4
88							6
90							10
92							2
94							0
96							2
98							2
100							0
102							0
104							2
106							0
108							0
110							0
112							0
114							2
Number of spacings	2	6	30	270	2970	44550	757350
Ratio to the previous		3	5	9	11	15	17

APPENDIX 6
Evaluation of #S(j,i).
Examples of iterative relationships' systems.

Let us remind that the iterative relationships' systems given below are questionable. We recall that these recursive relationship systems are given for information and have yet to be demonstrated.

Example 2n = 4 :

Table 79

j	Δ	Formulas	Conditions
1	6	#S(1,i) = (p_i-4).#S(1,i-1)	i ≥ 2
2	12	x1(3) = 2 x1(i) = (p_{i-1}-6).x1(i-1) #S(2,2) = 0 #S(2,i) = (p_i-4).#S(2,i-1)+x1(i)	i ≥ 3
3	18	#S(3,i) = (p_i-4).#S(3,i-1)	i ≥ 5
4	24	x1(5) = 72 x1(i) = (p_{i-1}-6).x1(i-1) #S(4,4) = 20 #S(4,i) = (p_i-4).#S(4,i-1)+x1(i)	i ≥ 5

The values below have been checked up to rank i = 8. Beyond that, the values are speculative.
In the table below and thereafter, the values of #S(j,i) in parentheses do not deduce from the iterative formulas.

i	p_i	#S(1,i)	#S(2,i)	#S(3,i)	#S(4,i)
1	3	(1)			
2	5	2	(0)	(1)	
3	7	6	2	(4)	(2)
4	11	42	16	(32)	(20)
5	13	378	154	288	252
6	17	4914	2072	3744	3780
7	19	73710	31850	56160	62244
8	23	1400490	615160	1067040	1254708
9	29	35012250	15549170	26676000	32592924
10	31	945330750	423741500	720252000	908189100
11	37	31195914750	14081317250	23768316000	30674744100
12	41	1154248845750	524042018500	879427692000	1156805149500
13	43	45015704984250	20543803530250	34297679988000	45879787453500

Example 2n = 8 :

Table 80

j	Δ	Formulas	Conditions
1	6	#S(1,i) = (p_i-4).#S(1,i-1)	i ≥ 4
2	12	#S(2,i) = (p_i-4).#S(2,i-1)	i ≥ 3
3	18	x1(6) = 320 x1(i) = (p_{i-1}-6).x1(i-1) #S(3,5) = 260 #S(3,i) = (p_i-4).#S(3,i-1)+x1(i)	i ≥ 6
4	24	x1(3) = 2 x1(i) = (p_{i-1}-6).x1(i-1) #S(4,2) = 0 #S(4,i) = (p_i-4).#S(4,i-1)+x1(i)	i ≥ 3

The values below have been checked up to rank i = 8. Beyond that, the values are speculative.

i	p_i	#S(1,i)	#S(2,i)	#S(3,i)	#S(4,i)
1	3				
2	5	(1)	(2)		(0)
3	7	(4)	6	(2)	2
4	11	28	42	(22)	16
5	13	252	378	(260)	154
6	17	3276	4914	3700	2072
7	19	49140	73710	59020	31850

i	p_i	#S(1,i)	#S(2,i)	#S(3,i)	#S(4,i)
8	23	933660	1400490	1167140	615160
9	29	23341500	35012250	29956420	15549170
10	31	630220500	945330750	826715500	423741500
11	37	20797276500	31195914750	27728915500	14081317250
12	41	769499230500	1154248845750	1039836297500	524042018500
13	43	30010469989500	45015704984250	41038940442500	20543803530250

Example 2n = 6 :

Table 81

j	Δ	Formulas	Conditions
1	2	$\#S(1,i) = (p_i-4).\#S(1,i-1)$	$i \geq 2$
2	4	$\#S(2,i) = (p_i-4).\#S(2,i-1)$	$i \geq 3$
3	6	$x1(4) = 8$ $x1(i) = (p_{i-2}-3).x1(i-1)$ $x2(3) = 6$ $x2(i) = (p_{i-1}-5).x2(i-1)+x1(i)$ $\#S(3,2) = 2$ $\#S(3,i) = (p_i-4).\#S(3,i-1)+x2(i)$	$i \geq 3$
4	8	$\#S(4,i) = (p_i-4).\#S(4,i-1)$	$i \geq 4$
5	10	$x1(4) = 2$ $x1(i) = (p_{i-2}-6).x1(i-1)$ $x2(3) = 2$ $x2(i) = (p_{i-1}-5).x2(i-1)+x1(i)$ $\#S(5,2) = 0$ $\#S(5,i) = (p_i-4).\#S(5,i-1)+x2(i)$	$i \geq 3$
6	12	$\#S(6,i) = 0$	
7	14	$x1(4) = 8$ $x1(i) = (p_{i-1}-5).x1(i-1)$ $\#S(7,3) = 2$ $\#S(7,i) = (p_i-4).\#S(7,i-1)+x1(i)$	$i \geq 4$

The values below have been checked up to rank i = 8. Beyond that, the values are speculative.

i	p_i	#S(1,i)	#S(2,i)	#S(3,i)	#S(4,i)	#S(5,i)	#S(6,i)	#S(7,i)
1	3	(1)	(1)					
2	5	1	(2)	(2)	(1)	(0)		
3	7	3	6	12	(4)	2	0	(2)
4	11	21	42	104	28	20	0	22
5	13	189	378	1088	252	218	0	246
6	17	2457	4914	15616	3276	3148	0	3582
7	19	36855	73710	254464	49140	51058	0	58338
8	23	700245	1400490	5153792	933660	1024604	0	1172934
9	29	17506125	35012250	135159808	23341500	26606146		30484566
10	31	472665375	945330750	3812343808	630220500	742321216		850952466
11	37	15597957375	31195914750	130344288256	20797276500	25123351162		28806030162
12	41	577124422875	1154248845750	4976270114816	769499230500	949717873832		1089010277082
13	43	22507852492125	45015704984250	199885542391808	30010469989500	37767570069866		43306138605366

It becomes difficult to predict lines with n formulas, n given, even if we find the systems here for a larger number of lines than in the case of 2n = 2.

Example 2n = 12 :

Table 82

j	Δ	Formulas	Conditions
1	2	$\#S(1,i) = (p_i-4).\#S(1,i-1)$	$i \geq 4$
2	4	$\#S(2,i) = (p_i-4).\#S(2,i-1)$	$i \geq 2$
3	6	$\#S(3,i) = (p_i-4).\#S(3,i-1)$	$i \geq 3$
4	8	$x1(3) = 8$ $x1(i) = (p_{i-1}-6).x1(i-1)$ $\#S(4,3) = 2$ $\#S(4,i) = (p_i-4).\#S(4,i-1)+x1(i)$	$i \geq 4$

j	Δ	Formulas	Conditions
5	10	x1(5) = 24 x1(i) = $(p_{i-1}$-6).x1(i-1) #S(5,4) = 60 #S(5,i) = $(p_i$-4).#S(5,i-1)+x1(i)	i ≥ 5
6	12	?	
7	14	x1(7) = 288 x1(i) = $(p_{i-2}$-6).x1(i-1) x2(6) = 144 x2(i) = $(p_{i-1}$-5).x2(i-1)+x1(i) #S(7,5) = 62 #S(7,i) = $(p_i$-4).#S(7,i-1)+x2(i)	i ≥ 6
8	16	x1(5) = 48 x1(i) = $(p_{i-1}$-6).x1(i-1) #S(8,4) = 4 #S(8,i) = $(p_i$-4).#S(8,i-1)+x1(i)	i ≥ 5
...	...	…	
12	24	#S(12,i) = 0	

The values below have been checked up to rank i = 8. Beyond that, the values are speculative.

i	p_i	#S(1,i)	#S(2,i)	#S(3,i)	#S(4,i)	#S(5,i)	#S(6,i)	#S(7,i)	#S(8,i)
1	3	(1)	(1)						
2	5	(2)	1	(2)		(1)			
3	7	(8)	3	6	(2)	(7)	(2)		
4	11	56	21	42	22	(60)	(30)	(5)	(4)
5	13	504	189	378	238	564	(476)	(62)	84
6	17	6552	2457	4914	3374	7500	(8152)	950	1428
7	19	98280	36855	73710	53690	114348	(148768)	16266	25116
8	23	1867320	700245	1400490	1060150	2196636	(3236864)	340446	525252
9	29	46683000	17506125	35012250	27184430	55324308	?	9117390	13948116
10	31	1260441000	472665375	945330750	749635250	1503149700	?	261419418	395385900
11	37	41594553000	15597957375	31195914750	25129354250	49838774700	?	9039440826	13517403900
12	41	1538998461000	577124422875	1154248845750	941919228250	1851314536500	?	348065085186	514703689500
13	43	60020939979000	22507852492125	45015704984250	37159509136750	72456062464500	?	14076825990318	20583034972500

APPENDIX 7

Table of pairs of maximum spacings at step $p_i = 17$.

Some series are presented in descending order for the "coherence" of the shadows with the other ones.

		3	5	7	11	13	17
-2	22634	22634	22634	22634	22634	22634	22634
0	22636	22636	22636	22636	22636	22636	22636
2	22638	0	0	0	0	0	0
4	22640	22640	0	0	0	0	0
6	22642	22642	0	0	0	0	0
8	22644	0	0	0	0	0	0
10	22646	22646	22646	22646	22646	0	0
12	22648	22648	22648	22648	22648	0	0
14	22650	0	0	0	0	0	0
16	22652	22652	22652	0	0	0	0
18	22654	22654	22654	0	0	0	0
20	22656	0	0	0	0	0	0
22	22658	22658	0	0	0	0	0
24	22660	22660	0	0	0	0	0
26	22662	0	0	0	0	0	0
28	22664	22664	22664	0	0	0	0
30	22666	22666	22666	0	0	0	0
32	22668	0	0	0	0	0	0
34	22670	22670	0	0	0	0	0
36	22672	22672	0	0	0	0	0
38	22674	0	0	0	0	0	0
40	22676	22676	22676	22676	22676	22676	0
42	22678	22678	22678	22678	22678	22678	0
44	22680	0	0	0	0	0	0
46	22682	22682	22682	22682	0	0	0
48	22684	22684	22684	22684	0	0	0
50	22686	0	0	0	0	0	0
52	22688	22688	0	0	0	0	0
54	22690	22690	0	0	0	0	0
56	22692	0	0	0	0	0	0
58	22694	22694	22694	0	0	0	0
60	22696	22696	22696	0	0	0	0
62	22698	0	0	0	0	0	0
64	22700	22700	0	0	0	0	0
66	22702	22702	0	0	0	0	0
68	22704	0	0	0	0	0	0
70	22706	22706	22706	0	0	0	0
72	22708	22708	22708	0	0	0	0
74	22710	0	0	0	0	0	0
76	22712	22712	22712	22712	22712	22712	0
78	22714	22714	22714	22714	22714	22714	0
80	22716	0	0	0	0	0	0
82	22718	22718	0	0	0	0	0
84	22720	22720	0	0	0	0	0
86	22722	0	0	0	0	0	0
88	22724	22724	22724	22724	0	0	0
90	22726	22726	22726	22726	0	0	0
92	22728	0	0	0	0	0	0
94	22730	22730	0	0	0	0	0
96	22732	22732	0	0	0	0	0
98	22734	0	0	0	0	0	0
100	22736	22736	22736	0	0	0	0
102	22738	22738	22738	0	0	0	0
104	22740	0	0	0	0	0	0
106	22742	22742	22742	22742	22742	22742	22742
108	22744	22744	22744	22744	22744	22744	22744

		3	5	7	11	13	17
108	70186	70186	70186	70186	70186	70186	70186
106	70184	70184	70184	70184	70184	70184	70184
104	70182	0	0	0	0	0	0
102	70180	70180	0	0	0	0	0
100	70178	70178	0	0	0	0	0
98	70176	0	0	0	0	0	0
96	70174	70174	70174	70174	70174	0	0
94	70172	70172	70172	70172	70172	0	0
92	70170	0	0	0	0	0	0
90	70168	70168	70168	0	0	0	0
88	70166	70166	70166	0	0	0	0
86	70164	0	0	0	0	0	0
84	70162	70162	0	0	0	0	0
82	70160	70160	0	0	0	0	0
80	70158	0	0	0	0	0	0
78	70156	70156	70156	0	0	0	0
76	70154	70154	70154	0	0	0	0
74	70152	0	0	0	0	0	0
72	70150	70150	0	0	0	0	0
70	70148	70148	0	0	0	0	0
68	70146	0	0	0	0	0	0
66	70144	70144	70144	70144	70144	70144	0
64	70142	70142	70142	70142	70142	70142	0
62	70140	0	0	0	0	0	0
60	70138	70138	70138	70138	0	0	0
58	70136	70136	70136	70136	0	0	0
56	70134	0	0	0	0	0	0
54	70132	70132	0	0	0	0	0
52	70130	70130	0	0	0	0	0
50	70128	0	0	0	0	0	0
48	70126	70126	70126	0	0	0	0
46	70124	70124	70124	0	0	0	0
44	70122	0	0	0	0	0	0
42	70120	70120	0	0	0	0	0
40	70118	70118	0	0	0	0	0
38	70116	0	0	0	0	0	0
36	70114	70114	70114	0	0	0	0
34	70112	70112	70112	0	0	0	0
32	70110	0	0	0	0	0	0
30	70108	70108	70108	70108	70108	70108	0
28	70106	70106	70106	70106	70106	70106	0
26	70104	0	0	0	0	0	0
24	70102	70102	0	0	0	0	0
22	70100	70100	0	0	0	0	0
20	70098	0	0	0	0	0	0
18	70096	70096	70096	70096	70096	0	0
16	70094	70094	70094	70094	70094	0	0
14	70092	0	0	0	0	0	0
12	70090	70090	0	0	0	0	0
10	70088	70088	0	0	0	0	0
8	70086	0	0	0	0	0	0
6	70084	70084	70084	0	0	0	0
4	70082	70082	70082	0	0	0	0
2	70080	0	0	0	0	0	0
0	70078	70078	70078	70078	70078	70078	70078
-2	70076	70076	70076	70076	70076	70076	70076

		3	5	7	11	13	17
-2	218984	218984	218984	218984	218984	218984	218984
0	218986	218986	218986	218986	218986	218986	218986
2	218988	0	0	0	0	0	0
4	218990	218990	0	0	0	0	0
6	218992	218992	0	0	0	0	0
8	218994	0	0	0	0	0	0
10	218996	218996	218996	218996	218996	0	0
12	218998	218998	218998	218998	218998	0	0
14	219000	0	0	0	0	0	0
16	219002	219002	219002	0	0	0	0
18	219004	219004	219004	0	0	0	0
20	219006	0	0	0	0	0	0
22	219008	219008	0	0	0	0	0
24	219010	219010	0	0	0	0	0
26	219012	0	0	0	0	0	0
28	219014	219014	219014	0	0	0	0
30	219016	219016	219016	0	0	0	0
32	219018	0	0	0	0	0	0
34	219020	219020	0	0	0	0	0
36	219022	219022	0	0	0	0	0
38	219024	0	0	0	0	0	0
40	219026	219026	219026	219026	219026	219026	0
42	219028	219028	219028	219028	219028	219028	0
44	219030	0	0	0	0	0	0
46	219032	219032	219032	219032	0	0	0
48	219034	219034	219034	219034	0	0	0
50	219036	0	0	0	0	0	0
52	219038	219038	0	0	0	0	0
54	219040	219040	0	0	0	0	0
56	219042	0	0	0	0	0	0
58	219044	219044	219044	0	0	0	0
60	219046	219046	219046	0	0	0	0
62	219048	0	0	0	0	0	0
64	219050	219050	0	0	0	0	0
66	219052	219052	0	0	0	0	0
68	219054	0	0	0	0	0	0
70	219056	219056	219056	0	0	0	0
72	219058	219058	219058	0	0	0	0
74	219060	0	0	0	0	0	0
76	219062	219062	219062	219062	219062	219062	0
78	219064	219064	219064	219064	219064	219064	0
80	219066	0	0	0	0	0	0
82	219068	219068	0	0	0	0	0
84	219070	219070	0	0	0	0	0
86	219072	0	0	0	0	0	0
88	219074	219074	219074	219074	0	0	0
90	219076	219076	219076	219076	0	0	0
92	219078	0	0	0	0	0	0
94	219080	219080	0	0	0	0	0
96	219082	219082	0	0	0	0	0
98	219084	0	0	0	0	0	0
100	219086	219086	219086	0	0	0	0
102	219088	219088	219088	0	0	0	0
104	219090	0	0	0	0	0	0
106	219092	219092	219092	219092	219092	219092	219092
108	219094	219094	219094	219094	219094	219094	219094

		3	5	7	11	13	17
-2	126164	126164	126164	126164	126164	126164	126164
0	126166	126166	126166	126166	126166	126166	126166
2	126168	0	0	0	0	0	0
4	126170	126170	0	0	0	0	0
6	126172	126172	0	0	0	0	0
8	126174	0	0	0	0	0	0
10	126176	126176	126176	126176	126176	0	0
12	126178	126178	126178	126178	126178	0	0
14	126180	0	0	0	0	0	0
16	126182	126182	126182	0	0	0	0
18	126184	126184	126184	0	0	0	0
20	126186	0	0	0	0	0	0
22	126188	126188	0	0	0	0	0
24	126190	126190	0	0	0	0	0
26	126192	0	0	0	0	0	0
28	126194	126194	126194	0	0	0	0
30	126196	126196	126196	0	0	0	0
32	126198	0	0	0	0	0	0
34	126200	126200	0	0	0	0	0
36	126202	126202	0	0	0	0	0
38	126204	0	0	0	0	0	0
40	126206	126206	126206	126206	126206	126206	0
42	126208	126208	126208	126208	126208	126208	0
44	126210	0	0	0	0	0	0
46	126212	126212	126212	126212	0	0	0
48	126214	126214	126214	126214	0	0	0
50	126216	0	0	0	0	0	0
52	126218	126218	0	0	0	0	0
54	126220	126220	0	0	0	0	0
56	126222	0	0	0	0	0	0
58	126224	126224	126224	0	0	0	0
60	126226	126226	126226	0	0	0	0
62	126228	0	0	0	0	0	0
64	126230	126230	0	0	0	0	0
66	126232	126232	0	0	0	0	0
68	126234	0	0	0	0	0	0
70	126236	126236	126236	0	0	0	0
72	126238	126238	126238	0	0	0	0
74	126240	0	0	0	0	0	0
76	126242	126242	126242	126242	126242	126242	0
78	126244	126244	126244	126244	126244	126244	0
80	126246	0	0	0	0	0	0
82	126248	126248	0	0	0	0	0
84	126250	126250	0	0	0	0	0
86	126252	0	0	0	0	0	0
88	126254	126254	126254	126254	126254	0	0
90	126256	126256	126256	126256	126256	0	0
92	126258	0	0	0	0	0	0
94	126260	126260	0	0	0	0	0
96	126262	126262	0	0	0	0	0
98	126264	0	0	0	0	0	0
100	126266	126266	126266	0	0	0	0
102	126268	126268	126268	0	0	0	0
104	126270	0	0	0	0	0	0
106	126272	126272	126272	126272	126272	126272	126272
108	126274	126274	126274	126274	126274	126274	126274

		3	5	7	11	13	17
-2	55784	55784	55784	55784	55784	55784	55784
0	55786	55786	55786	55786	55786	55786	55786
2	55788	0	0	0	0	0	0
4	55790	55790	0	0	0	0	0
6	55792	55792	0	0	0	0	0
8	55794	0	0	0	0	0	0
10	55796	55796	55796	55796	55796	0	0
12	55798	55798	55798	55798	55798	0	0
14	55800	0	0	0	0	0	0
16	55802	55802	55802	0	0	0	0
18	55804	55804	55804	0	0	0	0
20	55806	0	0	0	0	0	0
22	55808	55808	0	0	0	0	0
24	55810	55810	0	0	0	0	0
26	55812	0	0	0	0	0	0
28	55814	55814	55814	55814	0	0	0
30	55816	55816	55816	55816	0	0	0
32	55818	0	0	0	0	0	0
34	55820	55820	0	0	0	0	0
36	55822	55822	0	0	0	0	0
38	55824	0	0	0	0	0	0
40	55826	55826	55826	55826	55826	55826	0
42	55828	55828	55828	55828	55828	55828	0
44	55830	0	0	0	0	0	0
46	55832	55832	55832	0	0	0	0
48	55834	55834	55834	0	0	0	0
50	55836	0	0	0	0	0	0
52	55838	55838	0	0	0	0	0
54	55840	55840	0	0	0	0	0
56	55842	0	0	0	0	0	0
58	55844	55844	55844	0	0	0	0
60	55846	55846	55846	0	0	0	0
62	55848	0	0	0	0	0	0
64	55850	55850	0	0	0	0	0
66	55852	55852	0	0	0	0	0
68	55854	0	0	0	0	0	0
70	55856	55856	55856	55856	0	0	0
72	55858	55858	55858	55858	0	0	0
74	55860	0	0	0	0	0	0
76	55862	55862	55862	55862	55862	55862	0
78	55864	55864	55864	55864	55864	55864	0
80	55866	0	0	0	0	0	0
82	55868	55868	0	0	0	0	0
84	55870	55870	0	0	0	0	0
86	55872	0	0	0	0	0	0
88	55874	55874	55874	0	0	0	0
90	55876	55876	55876	0	0	0	0
92	55878	0	0	0	0	0	0
94	55880	55880	0	0	0	0	0
96	55882	55882	0	0	0	0	0
98	55884	0	0	0	0	0	0
100	55886	55886	55886	0	0	0	0
102	55888	55888	55888	0	0	0	0
104	55890	0	0	0	0	0	0
106	55892	55892	55892	55892	55892	55892	55892
108	55894	55894	55894	55894	55894	55894	55894

		3	5	7	11	13	17
-2	252134	252134	252134	252134	252134	252134	252134
0	252136	252136	252136	252136	252136	252136	252136
2	252138	0	0	0	0	0	0
4	252140	252140	0	0	0	0	0
6	252142	252142	0	0	0	0	0
8	252144	0	0	0	0	0	0
10	252146	252146	252146	252146	252146	0	0
12	252148	252148	252148	252148	252148	0	0
14	252150	0	0	0	0	0	0
16	252152	252152	252152	0	0	0	0
18	252154	252154	252154	0	0	0	0
20	252156	0	0	0	0	0	0
22	252158	252158	0	0	0	0	0
24	252160	252160	0	0	0	0	0
26	252162	0	0	0	0	0	0
28	252164	252164	252164	252164	0	0	0
30	252166	252166	252166	252166	0	0	0
32	252168	0	0	0	0	0	0
34	252170	252170	0	0	0	0	0
36	252172	252172	0	0	0	0	0
38	252174	0	0	0	0	0	0
40	252176	252176	252176	252176	252176	252176	0
42	252178	252178	252178	252178	252178	252178	0
44	252180	0	0	0	0	0	0
46	252182	252182	252182	0	0	0	0
48	252184	252184	252184	0	0	0	0
50	252186	0	0	0	0	0	0
52	252188	252188	0	0	0	0	0
54	252190	252190	0	0	0	0	0
56	252192	0	0	0	0	0	0
58	252194	252194	252194	0	0	0	0
60	252196	252196	252196	0	0	0	0
62	252198	0	0	0	0	0	0
64	252200	252200	0	0	0	0	0
66	252202	252202	0	0	0	0	0
68	252204	0	0	0	0	0	0
70	252206	252206	252206	252206	0	0	0
72	252208	252208	252208	252208	0	0	0
74	252210	0	0	0	0	0	0
76	252212	252212	252212	252212	252212	252212	0
78	252214	252214	252214	252214	252214	252214	0
80	252216	0	0	0	0	0	0
82	252218	252218	0	0	0	0	0
84	252220	252220	0	0	0	0	0
86	252222	0	0	0	0	0	0
88	252224	252224	252224	0	0	0	0
90	252226	252226	252226	0	0	0	0
92	252228	0	0	0	0	0	0
94	252230	252230	0	0	0	0	0
96	252232	252232	0	0	0	0	0
98	252234	0	0	0	0	0	0
100	252236	252236	252236	0	0	0	0
102	252238	252238	252238	0	0	0	0
104	252240	0	0	0	0	0	0
106	252242	252242	252242	252242	252242	252242	252242
108	252244	252244	252244	252244	252244	252244	252244

		3	5	7	11	13	17
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0	24946	24946	24946	24946	24946	24946	24946
2	24948	0	0	0	0	0	0
4	24950	24950	0	0	0	0	0
6	24952	24952	0	0	0	0	0
8	24954	0	0	0	0	0	0
10	24956	24956	24956	24956	24956	24956	0
12	24958	24958	24958	24958	24958	24958	0
14	24960	0	0	0	0	0	0
16	24962	24962	24962	0	0	0	0
18	24964	24964	24964	0	0	0	0
20	24966	0	0	0	0	0	0
22	24968	24968	0	0	0	0	0
24	24970	24970	0	0	0	0	0
26	24972	0	0	0	0	0	0
28	24974	24974	24974	0	0	0	0
30	24976	24976	24976	0	0	0	0
32	24978	0	0	0	0	0	0
34	24980	24980	0	0	0	0	0
36	24982	24982	0	0	0	0	0
38	24984	0	0	0	0	0	0
40	24986	24986	24986	24986	24986	0	0
42	24988	24988	24988	24988	24988	0	0
44	24990	0	0	0	0	0	0
46	24992	24992	24992	24992	0	0	0
48	24994	24994	24994	24994	0	0	0
50	24996	0	0	0	0	0	0
52	24998	24998	0	0	0	0	0
54	25000	25000	0	0	0	0	0
56	25002	0	0	0	0	0	0
58	25004	25004	25004	0	0	0	0
60	25006	25006	25006	0	0	0	0
62	25008	0	0	0	0	0	0
64	25010	25010	0	0	0	0	0
66	25012	25012	0	0	0	0	0
68	25014	0	0	0	0	0	0
70	25016	25016	25016	0	0	0	0
72	25018	25018	25018	0	0	0	0
74	25020	0	0	0	0	0	0
76	25022	25022	25022	25022	25022	25022	0
78	25024	25024	25024	25024	25024	25024	0
80	25026	0	0	0	0	0	0
82	25028	25028	0	0	0	0	0
84	25030	25030	0	0	0	0	0
86	25032	0	0	0	0	0	0
88	25034	25034	25034	25034	0	0	0
90	25036	25036	25036	25036	0	0	0
92	25038	0	0	0	0	0	0
94	25040	25040	0	0	0	0	0
96	25042	25042	0	0	0	0	0
98	25044	0	0	0	0	0	0
100	25046	25046	25046	0	0	0	0
102	25048	25048	25048	0	0	0	0
104	25050	0	0	0	0	0	0
106	25052	25052	25052	25052	25052	25052	25052
108	25054	25054	25054	25054	25054	25054	25054

		3	5	7	11	13	17
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0	58096	58096	58096	58096	58096	58096	58096
2	58098	0	0	0	0	0	0
4	58100	58100	0	0	0	0	0
6	58102	58102	0	0	0	0	0
8	58104	0	0	0	0	0	0
10	58106	58106	58106	58106	58106	58106	0
12	58108	58108	58108	58108	58108	58108	0
14	58110	0	0	0	0	0	0
16	58112	58112	58112	0	0	0	0
18	58114	58114	58114	0	0	0	0
20	58116	0	0	0	0	0	0
22	58118	58118	0	0	0	0	0
24	58120	58120	0	0	0	0	0
26	58122	0	0	0	0	0	0
28	58124	58124	58124	58124	0	0	0
30	58126	58126	58126	58126	0	0	0
32	58128	0	0	0	0	0	0
34	58130	58130	0	0	0	0	0
36	58132	58132	0	0	0	0	0
38	58134	0	0	0	0	0	0
40	58136	58136	58136	58136	58136	0	0
42	58138	58138	58138	58138	58138	0	0
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52	58148	58148	0	0	0	0	0
54	58150	58150	0	0	0	0	0
56	58152	0	0	0	0	0	0
58	58154	58154	58154	0	0	0	0
60	58156	58156	58156	0	0	0	0
62	58158	0	0	0	0	0	0
64	58160	58160	0	0	0	0	0
66	58162	58162	0	0	0	0	0
68	58164	0	0	0	0	0	0
70	58166	58166	58166	58166	0	0	0
72	58168	58168	58168	58168	0	0	0
74	58170	0	0	0	0	0	0
76	58172	58172	58172	58172	58172	58172	0
78	58174	58174	58174	58174	58174	58174	0
80	58176	0	0	0	0	0	0
82	58178	58178	0	0	0	0	0
84	58180	58180	0	0	0	0	0
86	58182	0	0	0	0	0	0
88	58184	58184	58184	0	0	0	0
90	58186	58186	58186	0	0	0	0
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94	58190	58190	0	0	0	0	0
96	58192	58192	0	0	0	0	0
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100	58196	58196	58196	0	0	0	0
102	58198	58198	58198	0	0	0	0
104	58200	0	0	0	0	0	0
106	58202	58202	58202	58202	58202	58202	58202
108	58204	58204	58204	58204	58204	58204	58204

		3	5	7	11	13	17
-2	221294	221294	221294	221294	221294	221294	221294
0	221296	221296	221296	221296	221296	221296	221296
2	221298	0	0	0	0	0	0
4	221300	221300	0	0	0	0	0
6	221302	221302	0	0	0	0	0
8	221304	0	0	0	0	0	0
10	221306	221306	221306	221306	221306	221306	0
12	221308	221308	221308	221308	221308	221308	0
14	221310	0	0	0	0	0	0
16	221312	221312	221312	0	0	0	0
18	221314	221314	221314	0	0	0	0
20	221316	0	0	0	0	0	0
22	221318	221318	0	0	0	0	0
24	221320	221320	0	0	0	0	0
26	221322	0	0	0	0	0	0
28	221324	221324	221324	0	0	0	0
30	221326	221326	221326	0	0	0	0
32	221328	0	0	0	0	0	0
34	221330	221330	0	0	0	0	0
36	221332	221332	0	0	0	0	0
38	221334	0	0	0	0	0	0
40	221336	221336	221336	221336	221336	0	0
42	221338	221338	221338	221338	221338	0	0
44	221340	0	0	0	0	0	0
46	221342	221342	221342	221342	0	0	0
48	221344	221344	221344	221344	0	0	0
50	221346	0	0	0	0	0	0
52	221348	221348	0	0	0	0	0
54	221350	221350	0	0	0	0	0
56	221352	0	0	0	0	0	0
58	221354	221354	221354	0	0	0	0
60	221356	221356	221356	0	0	0	0
62	221358	0	0	0	0	0	0
64	221360	221360	0	0	0	0	0
66	221362	221362	0	0	0	0	0
68	221364	0	0	0	0	0	0
70	221366	221366	221366	0	0	0	0
72	221368	221368	221368	0	0	0	0
74	221370	0	0	0	0	0	0
76	221372	221372	221372	221372	221372	221372	0
78	221374	221374	221374	221374	221374	221374	0
80	221376	0	0	0	0	0	0
82	221378	221378	0	0	0	0	0
84	221380	221380	0	0	0	0	0
86	221382	0	0	0	0	0	0
88	221384	221384	221384	221384	0	0	0
90	221386	221386	221386	221386	0	0	0
92	221388	0	0	0	0	0	0
94	221390	221390	0	0	0	0	0
96	221392	221392	0	0	0	0	0
98	221394	0	0	0	0	0	0
100	221396	221396	221396	0	0	0	0
102	221398	221398	221398	0	0	0	0
104	221400	0	0	0	0	0	0
106	221402	221402	221402	221402	221402	221402	221402
108	221404	221404	221404	221404	221404	221404	221404

		3	5	7	11	13	17
-2	254444	254444	254444	254444	254444	254444	254444
0	254446	254446	254446	254446	254446	254446	254446
2	254448	0	0	0	0	0	0
4	254450	254450	0	0	0	0	0
6	254452	254452	0	0	0	0	0
8	254454	0	0	0	0	0	0
10	254456	254456	254456	254456	254456	254456	0
12	254458	254458	254458	254458	254458	254458	0
14	254460	0	0	0	0	0	0
16	254462	254462	254462	0	0	0	0
18	254464	254464	254464	0	0	0	0
20	254466	0	0	0	0	0	0
22	254468	254468	0	0	0	0	0
24	254470	254470	0	0	0	0	0
26	254472	0	0	0	0	0	0
28	254474	254474	254474	254474	0	0	0
30	254476	254476	254476	254476	0	0	0
32	254478	0	0	0	0	0	0
34	254480	254480	0	0	0	0	0
36	254482	254482	0	0	0	0	0
38	254484	0	0	0	0	0	0
40	254486	254486	254486	254486	254486	0	0
42	254488	254488	254488	254488	254488	0	0
44	254490	0	0	0	0	0	0
46	254492	254492	254492	0	0	0	0
48	254494	254494	254494	0	0	0	0
50	254496	0	0	0	0	0	0
52	254498	254498	0	0	0	0	0
54	254500	254500	0	0	0	0	0
56	254502	0	0	0	0	0	0
58	254504	254504	254504	0	0	0	0
60	254506	254506	254506	0	0	0	0
62	254508	0	0	0	0	0	0
64	254510	254510	0	0	0	0	0
66	254512	254512	0	0	0	0	0
68	254514	0	0	0	0	0	0
70	254516	254516	254516	254516	0	0	0
72	254518	254518	254518	254518	0	0	0
74	254520	0	0	0	0	0	0
76	254522	254522	254522	254522	254522	254522	0
78	254524	254524	254524	254524	254524	254524	0
80	254526	0	0	0	0	0	0
82	254528	254528	0	0	0	0	0
84	254530	254530	0	0	0	0	0
86	254532	0	0	0	0	0	0
88	254534	254534	254534	0	0	0	0
90	254536	254536	254536	0	0	0	0
92	254538	0	0	0	0	0	0
94	254540	254540	0	0	0	0	0
96	254542	254542	0	0	0	0	0
98	254544	0	0	0	0	0	0
100	254546	254546	254546	0	0	0	0
102	254548	254548	254548	0	0	0	0
104	254550	0	0	0	0	0	0
106	254552	254552	254552	254552	254552	254552	254552
108	254554	254554	254554	254554	254554	254554	254554

Appendix 8

Preparatory table to links at step $p_i = 11$.

0	6 0 0 0 0					
1		12 0 1 0 0		30 0 0 1 0	36 0 0 0 1	
2		12 0 1 0 1	18 0 0 2 0	30 0 0 1 1	36 0 0 0 2	
2		12 0 1 1 0				
2		12 0 2 0 0				
3	6 0 0 0 3	12 0 1 0 2	18 0 0 2 1		36 0 0 1 2	
3	6 0 0 3 0	12 0 1 1 1				
3		12 0 1 2 0				
3		12 0 2 0 1				
3		12 0 2 1 0				
3		12 0 3 0 0				
4	6 0 0 0 4	12 0 1 0 3	18 0 0 2 2		36 0 0 1 3	
4	6 0 0 4 0	12 0 1 1 2	18 0 0 3 1			
4		12 0 1 2 1	18 0 3 0 1			
4		12 0 1 3 0	18 0 3 1 0			
4		12 0 2 0 2				
4		12 0 2 1 1				
4		12 0 2 2 0				
4		12 0 4 0 0				
5	6 0 0 0 5	12 0 1 1 3	18 0 0 2 3	30 0 0 1 4	36 0 3 2 0	42 0 1 0 4
5		12 0 1 2 2	18 0 0 3 2	30 0 4 0 1		42 0 4 1 0
5		12 0 1 3 1	18 0 0 4 1			
5		12 0 2 0 3	18 0 0 5 0			
5		12 0 2 1 2	18 0 1 4 0			
5		12 0 2 2 1	18 0 3 0 2			
5		12 0 2 3 0	18 0 3 1 1			
6	6 0 0 0 6	12 0 1 2 3	18 0 0 2 4	30 0 0 1 5	36 0 3 2 1	42 0 1 0 5
6	6 0 0 3 3	12 0 1 3 2	18 0 0 4 2	30 0 0 5 1		
6		12 0 2 1 3	18 0 1 4 1	30 0 0 6 0		
6		12 0 2 2 2	18 0 3 1 2	30 0 1 1 4		
6		12 0 2 3 1		30 0 1 5 0		
6		12 0 3 0 3		30 0 2 4 0		
6				30 0 3 3 0		
6				30 0 4 0 2		
6				30 0 4 1 1		
6				30 0 4 2 0		
7	6 0 0 0 7	12 0 1 0 6	18 0 0 2 5	30 0 0 1 6	36 0 2 5 0	42 0 0 6 1
7	6 0 0 3 4	12 0 1 3 3	18 0 1 2 4	30 0 0 5 2	36 0 3 3 1	42 0 1 6 0
7	6 0 0 4 3	12 0 2 2 3	18 0 1 4 2	30 0 1 1 5		
7		12 0 2 3 2	18 0 3 1 3	30 0 1 5 1		
7		12 0 3 0 4	18 0 4 3 0	30 0 2 4 1		
7		12 0 3 4 0		30 0 3 2 2		
7		12 0 4 0 3		30 0 4 1 2		
7				30 0 4 2 1		
8	6 0 0 0 8	12 0 1 0 7	18 0 0 2 6	30 0 0 1 7	36 0 3 3 2	42 0 0 6 2
8	6 0 0 3 5	12 0 1 1 6	18 0 1 2 5	30 0 0 5 3		42 0 1 6 1
8	6 0 0 4 4	12 0 2 0 6	18 0 1 3 4	30 0 1 5 2		42 0 2 4 2
8		12 0 2 3 3	18 0 1 4 3	30 0 2 5 1		42 0 3 4 1
8		12 0 3 0 5	18 0 2 6 0	30 0 3 1 4		
8		12 0 3 5 0		30 0 3 2 3		
8		12 0 4 0 4		30 0 4 1 3		
8		12 0 4 4 0		30 0 4 2 2		
8				30 0 4 3 1		

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6	0	0	3	6
6	0	0	4	5

12	0	1	0	8
12	0	1	1	7
12	0	1	2	6
12	0	2	0	7
12	0	2	1	6
12	0	3	6	0
12	0	4	0	5
12	0	4	5	0

18	0	1	3	5
18	0	1	4	4
18	0	2	6	1

30	0	0	1	8
30	0	0	2	7
30	0	0	5	4
30	0	0	6	3
30	0	1	6	2
30	0	2	3	4
30	0	2	5	2
30	0	3	1	5
30	0	3	3	3
30	0	3	5	1
30	0	4	1	4
30	0	4	3	2

36	0	0	0	9
36	0	1	5	3

42	0	2	4	3
42	0	3	2	4
42	0	3	4	2
42	0	4	2	3

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6	0	0	3	7
6	0	0	4	6

12	0	1	1	8
12	0	1	2	7
12	0	1	3	6
12	0	2	0	8
12	0	2	1	7
12	0	2	2	6
12	0	3	4	3
12	0	4	6	0

18	0	0	5	5
18	0	1	4	5
18	0	2	6	2
18	0	3	6	1

30	0	0	2	8
30	0	0	6	4
30	0	1	0	9
30	0	1	6	3
30	0	2	3	5
30	0	2	4	4
30	0	2	5	3
30	0	3	1	6
30	0	3	3	4
30	0	3	5	2
30	0	4	1	5
30	0	4	3	3

36	0	0	0	10
36	0	1	5	4

42	0	0	1	9
42	0	3	2	5
42	0	4	0	6
42	0	4	2	4

11
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6	0	0	3	8
6	0	0	4	7

12	0	1	2	8
12	0	1	3	7
12	0	2	1	8
12	0	2	2	7
12	0	2	3	6
12	0	3	0	8
12	0	3	4	4
12	0	3	5	3
12	0	4	4	3

18	0	0	5	6
18	0	1	4	6
18	0	2	0	9
18	0	2	6	3
18	0	3	6	2

30	0	0	2	9
30	0	0	6	5
30	0	1	0	10
30	0	1	5	5
30	0	2	4	5
30	0	2	5	4
30	0	3	1	7
30	0	3	2	6
30	0	3	3	5
30	0	4	2	5
30	0	4	3	4

36	0	1	6	4
36	0	4	1	6

42	0	0	1	10
42	0	4	0	7

12
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6	0	0	4	8

12	0	1	3	8
12	0	2	2	8
12	0	2	3	7
12	0	3	0	9
12	0	3	4	5
12	0	3	5	4
12	0	3	6	3
12	0	4	0	8
12	0	4	4	4
12	0	4	5	3

18	0	0	3	9
18	0	0	5	7
18	0	2	0	10
18	0	2	1	9
18	0	3	1	8
18	0	4	3	5

30	0	0	2	10
30	0	0	6	6
30	0	1	4	7
30	0	1	5	6
30	0	2	4	6
30	0	2	5	5
30	0	2	6	4
30	0	3	2	7
30	0	3	3	6
30	0	4	2	6

36	0	1	6	5
36	0	4	1	7

13
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12	0	2	3	8
12	0	3	0	10
12	0	3	5	5
12	0	3	6	4
12	0	4	0	9
12	0	4	4	5
12	0	4	5	4
12	0	4	6	3

18	0	0	3	10
18	0	0	4	9
18	0	0	5	8
18	0	2	1	10
18	0	3	1	9
18	0	4	3	6

30	0	0	6	7
30	0	1	4	8
30	0	1	6	6
30	0	2	2	9
30	0	2	4	7
30	0	2	6	5
30	0	3	2	8
30	0	3	3	7
30	0	3	4	6
30	0	4	1	8

36	0	4	2	7

42	0	1	5	7
42	0	2	5	6

14
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12	0	3	6	5
12	0	4	0	10
12	0	4	5	5
12	0	4	6	4

18	0	0	4	10
18	0	0	5	9
18	0	3	1	10
18	0	4	3	7
18	0	4	4	6

30	0	1	4	9
30	0	1	6	7
30	0	2	2	10
30	0	2	4	8
30	0	2	6	6
30	0	3	2	9
30	0	3	4	7
30	0	4	1	9

36	0	0	6	8
36	0	4	2	8

42	0	1	5	8
42	0	2	3	9
42	0	2	5	7
42	0	3	3	8

15
15
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12	0	3	4	8
12	0	4	6	5

18	0	0	5	10
18	0	4	3	8
18	0	4	4	7
18	0	4	5	6

30	0	1	4	10
30	0	1	5	9
30	0	1	6	8
30	0	2	5	8
30	0	2	6	7
30	0	3	2	10
30	0	4	2	9

36	0	0	6	9
36	0	2	4	9

42	0	2	3	10
42	0	3	3	9
42	0	4	1	10

16
16
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16

12	0	3	4	9
12	0	3	5	8
12	0	4	4	8

18	0	2	6	8
18	0	4	3	9
18	0	4	5	7

30	0	0	6	10
30	0	1	5	10
30	0	1	6	9
30	0	2	5	9
30	0	3	3	10
30	0	4	2	10
30	0	4	6	6

36	0	2	4	10

17	12	0	3	4	10	18	0	2	6	9	30	0	1	6	10	36	0	2	5	10
17	12	0	3	5	9	18	0	4	3	10	30	0	4	6	7					
17	12	0	3	6	8															
17	12	0	4	4	9															
17	12	0	4	5	8															

18	12	0	3	5	10	18	0	2	6	10
18	12	0	3	6	9					
18	12	0	4	4	10					
18	12	0	4	5	9					
18	12	0	4	6	8					

19	12	0	3	6	10
19	12	0	4	5	10
19	12	0	4	6	9

20	12	0	4	6	10

APPENDIX 9

Table of positive progressions configurations at step $p_i = 11$.

p_i	3	5	7	11
6	0	0	0	5
+	0	1	1	1
12	0	1	1	6
+	0	0	3	0
18	0	1	4	6
+	0	2	1	0
24	0	3	5	6
+	0	1	1	1
30	0	4	6	7
+	0	1	0	2
36	0	0	6	9
+	0	0	0	3
42	0	0	6	1
∑	+0	+5	+6	+7

p_i	3	5	7	11
6	0	0	0	5
+	0	1	1	1
12	0	1	1	6
+	0	0	3	0
18	0	1	4	6
+	0	2	1	0
24	0	3	5	6
+	0	2	1	0
30	0	0	6	6
+	0	0	0	3
36	0	0	6	9
+	0	0	0	3
42	0	0	6	1
∑	+0	+5	+6	+7

p_i	3	5	7	11
6	0	0	0	5
+	0	1	1	1
12	0	1	1	6
+	0	0	3	0
18	0	1	4	6
+	0	2	1	0
24	0	3	5	6
+	0	1	1	1
30	0	4	6	7
+	0	1	0	2
36	0	0	6	9
+	0	0	2	1
42	0	0	1	10
∑	+0	+5	+8	+5

p_i	3	5	7	11
6	0	0	0	5
+	0	1	1	1
12	0	1	1	6
+	0	0	3	0
18	0	1	4	6
+	0	2	1	0
24	0	3	5	6
+	0	2	1	0
30	0	0	6	6
+	0	0	0	3
36	0	0	6	9
+	0	0	2	1
42	0	0	1	10
∑	+0	+5	+8	+5

p_i	3	5	7	11
6	0	0	0	0
+	0	1	2	0
12	0	1	2	0
+	0	0	2	1
18	0	1	4	1
+	0	3	0	0
24	0	4	4	1
+	0	1	1	1
30	0	0	5	2
+	0	1	0	2
36	0	1	5	4
+	0	0	0	3
42	0	1	5	7
∑	+0	+6	+5	+7

p_i	3	5	7	11
6	0	0	0	0
+	0	1	1	1
12	0	1	1	1
+	0	0	3	0
18	0	1	4	1
+	0	3	0	0
24	0	4	4	1
+	0	1	1	1
30	0	0	5	2
+	0	1	0	2
36	0	1	5	4
+	0	0	0	3
42	0	1	5	7
∑	+0	+6	+5	+7

p_i	3	5	7	11
6	0	0	0	5
+	0	1	1	1
12	0	1	1	6
+	0	0	3	0
18	0	1	4	6
+	0	2	1	0
24	0	3	5	6
+	0	1	1	1
30	0	4	6	7
+	0	1	0	2
36	0	0	6	9
+	0	1	0	2
42	0	1	6	0
∑	+0	+6	+6	+6

p_i	3	5	7	11
6	0	0	0	5
+	0	1	1	1
12	0	1	1	6
+	0	0	3	0
18	0	1	4	6
+	0	2	1	0
24	0	3	5	6
+	0	2	1	0
30	0	0	6	6
+	0	0	0	3
36	0	0	6	9
+	0	1	0	2
42	0	1	6	0
∑	+0	+6	+6	+6

p_i	3	5	7	11
6	0	0	0	0
+	0	1	2	0
12	0	1	2	0
+	0	0	2	1
18	0	1	4	1
+	0	3	0	0
24	0	4	4	1
+	0	1	1	1
30	0	0	5	2
+	0	1	0	2
36	0	1	5	4
+	0	0	2	1
42	0	1	0	5
∑	+0	+6	+7	+5

p_i	3	5	7	11
6	0	0	0	0
+	0	1	1	1
12	0	1	1	1
+	0	0	3	0
18	0	1	4	1
+	0	3	0	0
24	0	4	4	1
+	0	1	1	1
30	0	0	5	2
+	0	1	0	2
36	0	1	5	4
+	0	0	2	1
42	0	1	0	5
∑	+0	+6	+7	+5

p_i	3	5	7	11
6	0	0	0	0
+	0	1	2	0
12	0	1	2	0
+	0	0	2	1
18	0	1	4	1
+	0	3	0	0
24	0	4	4	1
+	0	1	1	1
30	0	0	5	2
+	0	1	0	2
36	0	1	5	4
+	0	1	0	2
42	0	2	5	6
∑	+0	+7	+5	+6

p_i	3	5	7	11
6	0	0	0	0
+	0	1	1	1
12	0	1	1	1
+	0	0	3	0
18	0	1	4	1
+	0	3	0	0
24	0	4	4	1
+	0	1	1	1
30	0	0	5	2
+	0	1	0	2
36	0	1	5	4
+	0	1	0	2
42	0	2	5	6
∑	+0	+7	+5	+6

APPENDIX 10

Table of positive progressions configurations at step p_i = 13.

All of 3341 configurations are not represented here but only that, almost ideal, where the column guides is not reached except for guide 5.

p_i	3	5	7	11	13
6	0	0	0	7	3
+	0	2	1	0	0
12	0	2	1	7	3
+	0	0	0	3	0
18	0	2	1	10	3
+	0	0	0	2	1
24	0	2	1	1	4
+	0	2	1	0	0
30	0	4	2	1	4
+	0	0	0	0	3
36	0	4	2	1	7
+	0	0	2	1	0
42	0	4	4	2	7
+	0	1	1	0	1
48	0	0	5	2	8
+	0	0	0	1	5
60	0	0	5	3	0
+	0	0	1	2	0
66	0	0	6	5	0
∑	0	5	6	9	10

p_i	3	5	7	11	13
6	0	0	0	7	3
+	0	2	1	0	0
12	0	2	1	7	3
+	0	0	0	3	0
18	0	2	1	10	3
+	0	0	0	2	1
24	0	2	1	1	4
+	0	2	0	0	1
30	0	4	1	1	5
+	0	0	1	0	2
36	0	4	2	1	7
+	0	0	2	1	0
42	0	4	4	2	7
+	0	1	1	0	1
48	0	0	5	2	8
+	0	0	0	1	5
60	0	0	5	3	0
+	0	0	1	2	0
66	0	0	6	5	0
∑	0	5	6	9	10

p_i	3	5	7	11	13
6	0	0	0	7	3
+	0	2	1	0	0
12	0	2	1	7	3
+	0	0	0	3	0
18	0	2	1	10	3
+	0	0	0	2	1
24	0	2	1	1	4
+	0	2	0	0	1
30	0	4	1	1	5
+	0	0	0	0	3
36	0	4	1	1	8
+	0	0	0	1	2
42	0	4	1	2	10
+	0	1	0	0	2
48	0	0	1	2	12
+	0	0	4	1	1
60	0	0	5	3	0
+	0	0	1	2	0
66	0	0	6	5	0
∑	0	5	6	9	10

p_i	3	5	7	11	13
6	0	0	0	7	3
+	0	2	1	0	0
12	0	2	1	7	3
+	0	0	0	3	0
18	0	2	1	10	3
+	0	0	0	2	1
24	0	2	1	1	4
+	0	2	0	0	1
30	0	4	1	1	5
+	0	0	0	0	3
36	0	4	1	1	8
+	0	0	0	0	3
42	0	4	1	1	11
+	0	1	0	1	1
48	0	0	1	2	12
+	0	0	4	1	1
60	0	0	5	3	0
+	0	0	1	2	0
66	0	0	6	5	0
∑	0	5	6	9	10

p_i	3	5	7	11	13
6	0	0	0	7	3
+	0	2	1	0	0
12	0	2	1	7	3
+	0	0	0	3	0
18	0	2	1	10	3
+	0	0	0	2	1
24	0	2	1	1	4
+	0	1	0	1	1
30	0	3	1	2	5
+	0	0	2	0	1
36	0	3	3	2	6
+	0	1	1	0	1
42	0	4	4	2	7
+	0	1	1	0	1
48	0	0	5	2	8
+	0	0	0	1	5
60	0	0	5	3	0
+	0	0	1	2	0
66	0	0	6	5	0
∑	0	5	6	9	10

p_i	3	5	7	11	13
6	0	0	0	7	3
+	0	2	1	0	0
12	0	2	1	7	3
+	0	0	0	3	0
18	0	2	1	10	3
+	0	0	0	2	1
24	0	2	1	1	4
+	0	0	2	0	1
30	0	2	3	1	5
+	0	1	0	1	1
36	0	3	3	2	6
+	0	1	1	0	1
42	0	4	4	2	7
+	0	1	1	0	1
48	0	0	5	2	8
+	0	0	0	1	5
60	0	0	5	3	0
+	0	0	1	2	0
66	0	0	6	5	0
∑	0	5	6	9	10

p_i	3	5	7	11	13
6	0	0	0	7	3
+	0	2	1	0	0
12	0	2	1	7	3
+	0	0	0	3	0
18	0	2	1	10	3
+	0	0	0	2	1
24	0	2	1	1	4
+	0	0	2	0	1
30	0	2	3	1	5
+	0	1	0	0	2
36	0	3	3	1	7
+	0	1	1	1	0
42	0	4	4	2	7
+	0	1	1	0	1
48	0	0	5	2	8
+	0	0	0	1	5
60	0	0	5	3	0
+	0	0	1	2	0
66	0	0	6	5	0
∑	0	5	6	9	10

p_i	3	5	7	11	13
6	0	0	0	7	3
+	0	1	1	1	0
12	0	1	1	8	3
+	0	1	0	2	0
18	0	2	1	10	3
+	0	0	0	2	1
24	0	2	1	1	4
+	0	2	1	0	0
30	0	4	2	1	4
+	0	0	0	0	3
36	0	4	2	1	7
+	0	0	2	1	0
42	0	4	4	2	7
+	0	1	1	0	1
48	0	0	5	2	8
+	0	0	0	1	5
60	0	0	5	3	0
+	0	0	1	2	0
66	0	0	6	5	0
∑	0	5	6	9	10

p_i	3	5	7	11	13
6	0	0	0	7	3
+	0	1	1	1	0
12	0	1	1	8	3
+	0	1	0	2	0
18	0	2	1	10	3
+	0	0	0	2	1
24	0	2	1	1	4
+	0	2	0	0	1
30	0	4	1	1	5
+	0	0	1	0	2
36	0	4	2	1	7
+	0	0	2	1	0
42	0	4	4	2	7
+	0	1	1	0	1
48	0	0	5	2	8
+	0	0	0	1	5
60	0	0	5	3	0
+	0	0	1	2	0
66	0	0	6	5	0
∑	0	5	6	9	10

p_i	3	5	7	11	13
6	0	0	0	7	3
+	0	1	1	1	0
12	0	1	1	8	3
+	0	1	0	2	0
18	0	2	1	10	3
+	0	0	0	2	1
24	0	2	1	1	4
+	0	2	0	0	1
30	0	4	1	1	5
+	0	0	0	0	3
36	0	4	1	1	8
+	0	0	0	1	2
42	0	4	1	2	10
+	0	1	0	0	2
48	0	0	1	2	12
+	0	0	4	1	1
60	0	0	5	3	0
+	0	0	1	2	0
66	0	0	6	5	0
∑	0	5	6	9	10

p_i	3	5	7	11	13
6	0	0	0	7	3
+	0	1	1	1	0
12	0	1	1	8	3
+	0	1	0	2	0
18	0	2	1	10	3
+	0	0	0	2	1
24	0	2	1	1	4
+	0	2	0	0	1
30	0	4	1	1	5
+	0	0	0	0	3
36	0	4	1	1	8
+	0	0	0	0	3
42	0	4	1	1	11
+	0	1	0	1	1
48	0	0	1	2	12
+	0	0	4	1	1
60	0	0	5	3	0
+	0	0	1	2	0
66	0	0	6	5	0
∑	0	5	6	9	10

p_i	3	5	7	11	13
6	0	0	0	7	3
+	0	1	1	1	0
12	0	1	1	8	3
+	0	1	0	2	0
18	0	2	1	10	3
+	0	0	0	2	1
24	0	2	1	1	4
+	0	1	0	1	1
30	0	3	1	2	5
+	0	0	2	0	1
36	0	3	3	2	6
+	0	1	1	0	1
42	0	4	4	2	7
+	0	1	1	0	1
48	0	0	5	2	8
+	0	0	0	1	5
60	0	0	5	3	0
+	0	0	1	2	0
66	0	0	6	5	0
∑	0	5	6	9	10

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6	0	0	0	7	3
+	0	1	1	1	0
12	0	1	1	8	3
+	0	1	0	2	0
18	0	2	1	10	3
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APPENDIX 11

Diverse arguments upon gap 2n = 2.

Additional argument in favour of a proof

Let us come back to table 34. We got three relations :

$$\sum_{j = jmin}^{jmax} \#S(j,i) = \prod_{k = 1}^{i} (p_k-2) \qquad (111)$$

$$\sum_{j = jmin}^{jmax} \Delta(j).\#S(j,i) = \prod_{k = 1}^{i} p_k \qquad (112)$$

$$\#R(j,i) \geq p_i-4 \qquad (113)$$

The first two relationships confine #S(j,i) statistically in a tunnel of values all the more limited since these values must be integers and consistent with the third relationship. Examples of compatible results are easily obtained by taking 2n = 4, 2n = 8, 2n = 16, etc. instead of the twin prime numbers case 2n = 2.
What we are concerned about here is to demonstrate that #S(j,i) becomes zero around an approximate value Δ(j) greater than $\sum_i 2p_k$.

We propose to evaluate the expression #S(j,i) assuming that the values of that expression roughly espouse the form of certain functions when j (and i) vary. Examples of functions examined are constant function, monomial function (the previous of which is a sub-case) and exponential function. Beyond a certain value j, which we note jmax(i), #S(j,i) becomes zero. For values regularly spaced out by a j value, #S(j,i) is supposed to follow the function taken as an example. The rest of the argument is in no way affected by assuming the 1 spaced j for our modelling.

Case 1 : #S(j,i) = if(j = 1 to jmax(i), c(i), 0), c(i) constant versus j.
Then $\sum\#S(j,i) = \prod(p_k-2) = jmax.c(i)$ and $\sum\Delta(j).\#S(j,i) = \sum j.\#S(j,i) = (1/2).jmax^2.c(i) = \prod p_k$.
We deduce from the relationship 4 for the first equation below :

$$jmax(i) = 2 \prod_{k = 1}^{i} p_k/(p_k-2) \ \rightarrow\approx 2.(1/c_2).e^{2\gamma}.\ln^2(p_i) \qquad (114)$$

and

$$c(i) = (1/2) \prod_{k = 1}^{i} (p_k-2)^2/p_k \qquad (115)$$

Thus asymptotically using $\ln(p_{i+1})-\ln(p_i) = \ln(p_{i+1}/p_i) \rightarrow \ln(1) = 0$:

$$jmax(i+1)-jmax(i) \rightarrow 2.(1/c_2).e^{2\gamma}.(\ln^2(p_{i+1})-\ln^2(p_i)) \approx 2.(1/c_2).e^{2\gamma}.(\ln(p_{i+1})-\ln(p_i)).2.\ln(p_i) << 2.\ln(p_i) << 2.p_i$$

This shows that with such a model the increase of jmax(i) with i is much slower than that observed in the facts remaining thus consistent with the needs of the previous demonstration (only a growth faster than $2p_i$ is detrimental).
It remains to be noted in simple remark that asymptotically $c(i+1)/c(i) \rightarrow (p_{i+1}-2)^2/p_{i+1} \rightarrow p_{i+1}-4$ which is the order of magnitude in relation 113.

Case 2 : $\#S(j,i) = a.j^{-b}$ where a = a(i) and b = b(i) constant versus j (and one supposes b ≠ 1, b ≠ 2).
Then

$$\sum\#S(j,i) = \prod(p_k-2) = \sum a.j^{-b} \approx \int a.j^{-b} \approx a/(-b+1).jmax^{(-b+1)}$$

and

$$\sum\Delta(j).\#S(j,i) = \prod p_k = = \sum a.j^{-b+1} \approx \int a.j^{-b+1} \approx a/(-b+2).jmax^{(-b+2)}$$

We deduce (according to relation 4) :

$$jmax(i) = (-b+2)/(-b+1) \prod_{k = 1}^{i} p_k/(p_k-2) \ \rightarrow\approx (-b+2)/(-b+1).(1/c_2).e^{2\gamma}.\ln^2(p_i) \qquad (116)$$

Thus asymptotically using $\ln(p_{i+1})-\ln(p_i) = \ln(p_{i+1}/p_i) \rightarrow \ln(1) = 0$, we get :

$$jmax(i+1)-jmax(i) \rightarrow$$

$$(-b+2)/(-b+1).(1/c_2).e^{2\gamma}.(\ln^2(p_{i+1})-\ln^2(p_i))$$
$$\approx (-b+2)/(-b+1).(1/c_2).e^{2\gamma}.(\ln(p_{i+1})-\ln(p_i)).2.\ln(p_i)$$
$$<< 2.\ln(p_i)$$
$$<< 2.p_i$$

For $b \neq 1$, the result is again in line with our need. For $b = 1$, it results in

$$jmax(i+1)/\ln(jmax(i+1))-jmax(i)/\ln(jmax(i) \rightarrow$$
$$\approx (1/c_2).e^{2\gamma}.(\ln(p_{i+1})-\ln(p_i)).2.\ln(p_i)$$

thus again using $\ln(jmax(i+1)) \approx \ln(jmax(i))$

$$jmax(i+1)-jmax(i) \rightarrow$$
$$\approx (1/c_2).e^{2\gamma}.\ln(jmax(i).(\ln(p_{i+1})-\ln(p_i)).2.\ln(p_i) << 2.p_i$$

Hence again the same conclusion.

Case 3 : $\#S(j,i) = a.e^{-bj}$ where $a = a(i) = a_i$ and $b = b(i) = b_i$ positive constants versus j.
The first two cases are very far from the actual case and the condition $<< 2.p_i$ is easily met. Here we are much better configured.
At the origin (j = 1 or rather j = 0), the value of #S(j,i) is $\prod(p_k-4)$, thus

$$a(i) = \prod_{k=1}^{i} (p_k-4) \qquad (117)$$

With this type of profile, #S(j,i) takes a priori zero values after reaching #S(j,i) = 1 (and therefore j = jmax here). This is the case when $a.e^{-bjmax} = 1$, that is $jmax = (1/b).\ln(a)$.
Moving from i to i-1, we get:

$$jmax(i+1)-jmax(i)$$
$$\approx$$
$$(1/b_{i+1}).\ln(a_{i+1})-(1/b_i).\ln(a_i)$$
$$= (1/b_{i+1}).\ln(a_{i+1})-(1/b_{i+1}).\ln(a_i)+(1/b_{i+1}).\ln(a_i)-(1/b_i).\ln(a_i)$$
$$= (1/b_{i+1}).\ln(a_{i+1}/a_i)+(1/b_{i+1}-1/b_i).\ln(a_i)$$
$$= (1/b_{i+1}).\ln(p_{i+1}-4)+(1/b_{i+1}-1/b_i).\sum_k \ln(p_k-4)$$

Here the last sum is on k = 1 to k = i.
Now, according to the fundamental theorem of prime numbers, on average the distance between prime numbers is $\ln(p_k)$. . An asymptotic approximate value of p_i is therefore $\sum_k \ln(p_k)$, k describing 1 to i, and besides asymptotically $\ln(p_k-4) \approx \ln(p_k)$.
Therefore :

$$jmax(i+1)-jmax(i)$$
$$\approx< (1/b_{i+1}).\ln(p_{i+1}-4)+(1/b_{i+1}-1/b_i).p_i$$
$$\approx (1/b_{i+1}).\ln(p_{i+1})+(1/b_{i+1}-1/b_i).p_i$$
$$= if(b_i = b_{i+1}, (1/b_{i+1}).\ln(p_{i+1}), (1/b_{i+1}-1/b_i).p_i)$$

This corresponds effectively to the increase of jmax(i) that matters to us. In fact, in the $b_i = b_{i+1}$ case, the result $(1/b_{i+1}).\ln(p_{i+1}) << 2.p_i$ is trivial asymptotically (b_{i+1} can be considered a constant), otherwise the values of b_i and b_{i+1} being close, we still have $(1/b_{i+1}-1/b_i).p_i < 2.p_i$.

Case 4 : $\#S(j,i) = a.e^{-b.(j\wedge r)}$ where $a = a(i) = a_i$ and $b = b(i) = b_i$, $r = r(i) = r_i$, positive constants versus j.
This is the case that is closest to the real case. Asymptotically, the r value varies little between i and i+1 and we repeat the previous calculations assuming $r(i+1) \approx r(i) \approx r$ when i increases and besides $r > 1$.
At the origin (for j = 0), the value of #S(j,i) is $\prod(p_k-4)$, then $a.e^{-b.(jmax\wedge r)} = 1$ gives $jmax^r = (1/b).\ln(a)$.
Moving from i to i+1, we get:

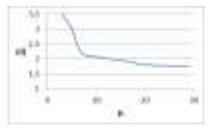
$$jmax(i+1)-jmax(i)$$
$$\approx$$
$$(1/b_{i+1})^{1/r}.\ln^{1/r}(a_{i+1})-(1/b_i)^{1/r}.\ln^{1/r}(a_i)$$
$$= (1/b_{i+1})^{1/r}.\ln^{1/r}(a_{i+1})-(1/b_{i+1})^{1/r}.\ln^{1/r}(a_i)+(1/b_{i+1})^{1/r}.\ln^{1/r}(a_i)-(1/b_i)^{1/r}.\ln^{1/r}(a_i)$$
$$= (1/b_{i+1})^{1/r}.\ln^{1/r}(a_{i+1}/a_i)+((1/b_{i+1})^{1/r} - (1/b_i)^{1/r}).(\ln(a_i))^{1/r}$$
$$= (1/b_{i+1})^{1/r}.\ln^{1/r}(p_{i+1}-4)+((1/b_{i+1})^{1/r} - (1/b_i)^{1/r}).(\sum_k \ln(p_k-4))^{1/r}$$

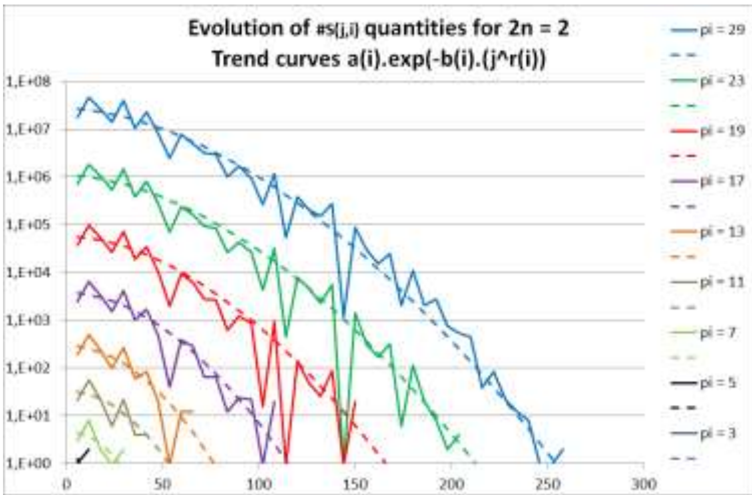
Then :

$$jmax(i+1)-jmax(i)$$
$$\approx if(b_i = b_{i+1}, (1/b_{i+1})^{1/r}.\ln^{1/r}(p_{i+1}), ((1/b_{i+1})^{1/r} - (1/b_i)^{1/r}).p_i^{1/r})$$

This corresponds effectively to the increase of jmax(i) that matters to us. Indeed, in the $b_i = b_{i+1}$ case, the result $(1/b_{i+1})^{1/r}.\ln^{1/r}(p_{i+1}) << 2.p_i$ is trivial asymptotically, otherwise the values of b_i and b_{i+1} being close and r > 1, one still has $((1/b_{i+1})^{1/r} - (1/b_i)^{1/r}).p_i^{1/r} < 2.p_i$ asymptotically.

Note: It is the condition $\#R(j,i) \geq p_i-4$ that is certainly the source of the value r > 1 that neighbours the said coefficient on graphic trend curves.
Here $a(i) \approx 1{,}57.\prod_k (p_k-4)$, $b(i) \approx 0{,}001$ and r(i) according to the following table:

p_i	3	5	7	11	13	17	19	23	29	
r(i)	3,5	3	2,2	2,05	1,99	1,9	1,82	1,78	1,76	



All cases show that it is very difficult (if not impossible) to find a non-discrete simulation leading to a range of values as large as that really observed.
Which goes again in favour of the theorem.

Argument against the proof

To remain impartial, we propose a counter-example in the form of a discrete simulation. The construction is relatively trivial and $\Delta max/2\sum_i p_k \rightarrow +\infty$ while responding to the known constraints for the problem.
Let us give this counter-example first.

Steps i	1	2	3	4	5	6
p_i	3	5	7	11	13	17
Ranges of cycle 1	6	30	210	2310	30030	510510
Numbers of spacings	1	3	15	135	1485	22275
Spacings Δ(j)	Quantities #Sn(j,i) of spacings Δ(j) in the cycle 1					
6	1	1	3	21	189	2457
12		2	8	56	504	6552
18			0	12	192	3252
24			4	38	412	5986
30				0	0	0
36				0	24	696
42				0	0	0
48				8	148	2748
54					0	0
60					0	0
66					0	0
72					0	48
78					0	0
84					0	0
90					0	0
96					16	504
102						0
108						0

114						0
120						0
126						0
132						0
138						0
144						0
150						0
156						0
162						0
168						0
174						0
180						0
186						0
192						32

The algorithm from step i-1 to step i is as follows (from i = 4 on):

j	$\Delta(j)$	$T_{i-1}(j)$	$M_i(j) = T_{i-1}(j).(p_i-4)$	$N_i(2j-1) = 0$ and $N_i(2j) = 2.T_{i-1}(2j-1)$	$P_i(2) = -N_i(2)$ and $P_i(3) = 2N_i(2)$ and $P_i(4) = -N_i(2)$	$T_i(j) = M_{i-1}(j)+N_{i-1}(j)+P_{i-1}(j)$
1	6	21	189			189
2	12	56	504	42	-42	504
3	18	12	108		84	192
4	24	38	342	112	-42	412
5	30	0	0			0
6	36	0	0	24		24
7	42	0	0			0
8	48	8	72	76		148
9	54					0
10	60			0		0
11	66					0
12	72			0		0
13	78					0
14	84			0		0
15	90					
16	96			16		16

In addition to respecting the total number of spacings, the size of cycle 1, the three relationships (two ties and one inequality), the table is also consistent for these first two lines in Table 34 (i.e. $\Delta(1) = 6$ and $\Delta(2) = 12$).

Nevertheless, we have:

Steps i	1	2	3	4	5	6	7	8	9	10	…	i
p_i	3	5	7	11	13	17	19	23	29	31	…	p_i
3.2^i	6	12	24	48	96	192	384	768	1536	3072	…	3.2^i
$2\sum_i p_k$	6	16	30	52	78	112	150	196	254	316	…	$2\sum_i p_k$
$3.2^i/2\sum_i p_k$	1,00	0,75	0,80	0,92	1,23	1,71	2,56	3,92	6,05	9,72		$\rightarrow 3.2^i/(\ln(p_i).i^2) \rightarrow +\infty$

If this discrepancy were effective, it would not be able to respond to the desired theorem.

As a final note, however, we note that the iterative formulas at work here are not in the mould observed for the effective tables of populations. They have the property of being all based on multiplication p_i-4 and not p_i-4, p_i-6, p_i-8, p_i-10, p_i-12, and so on. The initial data are those which follows. The only non-zero lines j(n) are such that $j(n) = j(n-1)+2^{ent((k-3)/2)}$ for $n \geq 3$, k being incremented starting from k = 3 and n = 3.

			i	1	2	3	4	5	6	7	8	9	10	…
			p_i	3	5	7	11	13	17	19	23	29	31	…
k	j(n)	Δ(j)	Dif Δ(j)											
1	1	6		1										…
2	2	12	3.2^1		(2)	8								…
3	3	18	3.2^1			0	**12**							…
4	4	24	3.2^1			(4)	**38**	**70**						…
5	6	36	3.2^2					24	**168**					…
6	8	48	3.2^2				**8**	76	140					…
7	12	72	3.2^3						48	336				…
8	16	96	3.2^3					16	152	280				…
9	24	144	3.2^4							96	672			…
10	32	192	3.2^4						32	304	560			…
11	48	288	3.2^5								192	1344		…
12	64	384	3.2^5							64	608	1120		…
13	96	576	3.2^6									384	2688	…
14	128	768	3.2^6								128	1216	2240	…
…	…	…	…	…	…	…	…	…	…	…	…	…	…	…

Starting from j(n) ≥ 3, the number of initial values, excluding the initial values at 0, alternates between 2 and 3 values, values which double by pairs of k (8 = 2.4, 76 = 2.38 = 140 = 2.70, 48 = 2.24, 336 = 2.168, etc.), while the number of recursive equations increases by one equation after each pair. The following table, which gives the first samples, is to be read with the k index instead of j in #Sn(k,i).

k	Formulas
1	#Sn(1,1) = 1 #Sn(1,i) = (p_i-4).#Sn(1,i-1)
2	#Sn(2,3) = 8 #Sn(2,i) = (p_i-4).#Sn(2,i-1)
3	x1(4) = 12 x1(i) = (p_{i-1}-4).x1(i-1) #Sn(3,3) = 0 #Sn(3,i) = (p_i-4).#Sn(3,i-1)+x1(i)
4	x1(5) = 70 x1(i) = (p_{i-1}-4).x1(i-1) #Sn(4,4) = 38 #Sn(4,i) = (p_i-4).#Sn(4,i-1)+x1(i)
5	x1(6) = 168 x1(i) = (p_{i-2}-4).x1(i-1) x2(5) = 24 x2(i) = (p_{i-1}-4).x2(i-1)+x1(i) #Sn(5,4) = 0 #Sn(5,i) = (p_i-4).#Sn(5,i-1)+x2(i)
6	x1(6) = 140 x1(i) = (p_{i-2}-4).x1(i-1) x2(5) = 76 x2(i) = (p_{i-1}-4).x2(i-1)+x1(i) #Sn(6,4) = 8 #Sn(6,i) = (p_i-4).#Sn(6,i-1)+x2(i)
7	x1(7) = 336 x1(i) = (p_{i-3}-4).x1(i-1) x2(6) = 46 x2(i) = (p_{i-2}-4).x2(i-1)+x1(i) x3(5) = 0 x3(i) = (p_{i-1}-4).x3(i-1)+x2(i) #Sn(7,4) = 0 #Sn(7,i) = (p_i-4).#Sn(7,i-1)+x3(i)
8	x1(7) = 280 x1(i) = (p_{i-3}-4).x1(i-1) x2(6) = 152 x2(i) = (p_{i-2}-4).x2(i-1)+x1(i) x3(5) = 16 x3(i) = (p_{i-1}-4).x3(i-1)+x2(i) #Sn(8,4) = 0 #Sn(8,i) = (p_i-4).#Sn(8,i-1)+x3(i)

9	x1(8) = 672 x1(i) = (p_{i-4}-4).x1(i-1) x2(7) = 96 x2(i) = (p_{i-3}-4).x2(i-1) +x1(i) x3(6) = 0 x3(i) = (p_{i-2}-4).x3(i-1) +x2(i) x4(5) = 0 x4(i) = (p_{i-1}-4).x4(i-1) +x3(i) #Sn(9,4) = 0 #Sn(9,i) = (p_i-4).#Sn(9,i-1)+x4(i)
10	x1(8) = 560 x1(i) = (p_{i-4}-4).x1(i-1) x2(7) = 304 x2(i) = (p_{i-3}-4).x2(i-1) +x1(i) x3(6) = 32 x3(i) = (p_{i-2}-4).x3(i-1) +x2(i) x4(5) = 0 x4(i) = (p_{i-1}-4).x4(i-1) +x3(i) #Sn(10,4) = 0 #Sn(10,i) = (p_i-4).#Sn(10,i-1)+x4(i)
…	...

APPENDIX 12

Table of 2^m-gaps' enumeration for.

Step 4 : $p_i = 11$. Periodicity = 30.

Δ \ 2n	2^{1}	2^{2}	2^{3}	2^{4}	2^{5}	2^{6}	2^{7}	2^{8}	2^{9}	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}	2^{15}	2^{16}	2^{17}	2^{18}	2^{19}	2^{20}	2^{21}	2^{22}	2^{23}	2^{24}	2^{25}	2^{26}	2^{27}	2^{28}	2^{29}	2^{30}	2^{31}	…
6	21	42	28	48	21	56	21	42	32	42	21	56	21	48	28	42	21	56	24	42	28	42	21	64	21	42	28	42	24	56	21	…
12	56	16	42	14	48	6	56	16	42	20	42	6	56	14	48	18	42	6	56	18	42	18	42	0	64	16	42	18	42	6	56	…
18	22	32	22	21	32	21	24	28	18	21	36	24	22	28	18	21	40	21	18	28	22	24	36	21	18	28	24	21	32	21	22	…
24	6	20	16	30	14	24	6	26	16	32	20	18	6	20	14	36	18	22	6	26	18	30	18	18	0	28	16	32	18	22	6	…
30	22	15	24	16	10	18	18	16	19	16	10	23	24	15	19	14	8	24	24	13	19	16	12	20	22	13	19	20	11	22	22	…
36	4	10	0	2	4	6	4	6	0	0	2	4	2	6	2	2	0	2	2	6	2	2	2	4	4	6	0	0	2	4	4	…
42	4	0	2	0	6	0	6	1	8	0	2	0	2	0	4	0	4	0	4	0	4	1	4	2	4	2	6	0	6	0	4	…
48	0	0	0	2	0	4	0	0	0	4	2	2	2	2	2	0	0	2	0	0	0	2	0	6	0	0	0	2	0	4	0	…
54	0	0	1	2	0	0	0	0	0	0	0	2	0	2	0	2	0	2	0	2	0	0	0	0	0	0	0	0	0	0	0	…
60	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	2	0	0	0	0	0	0	…
66	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	…

Step 5 : $p_i = 13$. Periodicity = 30.

Δ \ 2n	2^{1}	2^{2}	2^{3}	2^{4}	2^{5}	2^{6}	2^{7}	2^{8}	2^{9}	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}	2^{15}	2^{16}	2^{17}	2^{18}	2^{19}	2^{20}	2^{21}	2^{22}	2^{23}	2^{24}	2^{25}	2^{26}	2^{27}	2^{28}	2^{29}	2^{30}	2^{31}	…
6	189	378	252	432	210	504	189	378	288	378	210	504	189	432	252	378	210	504	216	378	252	378	210	576	189	378	252	378	240	504	189	…
12	504	154	378	154	432	90	504	154	378	204	378	90	504	140	432	182	378	90	504	174	378	182	378	42	576	154	378	182	378	96	504	…
18	238	288	260	189	312	189	264	252	228	189	344	216	238	252	228	189	384	189	210	252	260	216	344	189	210	252	288	189	312	189	238	…
24	96	252	154	294	154	264	90	294	154	308	204	210	90	252	140	348	182	238	90	294	174	294	182	210	42	324	154	308	182	238	96	…
30	270	214	288	260	161	224	238	240	239	264	161	259	304	206	235	250	149	284	280	202	239	264	192	236	262	194	235	316	167	254	270	…
36	60	126	16	44	52	92	56	100	16	40	40	88	48	94	38	52	22	64	44	98	34	60	38	76	60	98	16	32	36	80	60	…
42	84	27	90	14	110	16	94	21	138	4	100	12	60	21	102	14	106	8	92	19	108	16	98	26	80	38	114	8	130	8	84	…
48	20	8	16	44	22	64	20	8	20	54	24	58	24	28	30	30	12	54	16	14	20	38	14	82	20	8	16	36	8	72	20	…
54	0	22	19	44	4	32	4	12	12	40	0	30	2	38	8	30	4	38	0	36	6	30	5	28	4	18	16	24	0	32	0	…
60	12	12	4	6	28	4	14	22	4	4	14	14	10	20	12	6	28	12	16	18	10	6	22	16	26	16	8	8	32	12	12	…
66	12	2	2	0	0	2	8	2	0	0	8	0	8	2	0	2	2	2	13	0	0	0	0	0	12	4	0	0	0	0	12	…
72	0	0	4	0	0	0	4	0	8	0	0	0	4	0	8	0	2	0	0	0	2	0	0	0	4	0	8	0	0	0	0	…
78	0	0	2	4	0	4	0	2	0	0	2	4	0	0	0	4	2	2	4	0	0	1	0	4	0	1	0	4	0	0	0	…
84	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	0	0	0	2	0	2	0	0	0	0	0	0	0	0	…
90	0	0	0	0	0	0	0	0	0	0	0	0	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	…

Step 6 : $p_i = 17$. Periodicity = 60.

Δ \ 2n	2^{1}	2^{2}	2^{3}	2^{4}	2^{5}	2^{6}	2^{7}	2^{8}	2^{9}	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}	2^{15}	2^{16}	2^{17}	2^{18}	2^{19}	2^{20}	2^{21}	2^{22}	2^{23}	2^{24}	2^{25}	2^{26}	2^{27}	2^{28}	2^{29}	2^{30}
6	2457	4914	3276	5616	2730	6552	2457	4914	3744	4914	2730	6552	2457	5616	3276	4914	2730	6552	2808	4914	3276	4914	2730	7488	2457	4914	3276	4914	3120	6552
12	6552	2072	4914	2198	5616	1410	6552	2072	4914	2820	4914	1410	6552	1918	5616	2506	4914	1410	6552	2346	4914	2506	4914	882	7488	2072	4914	2506	4914	1536
18	3374	3744	3700	2646	4440	2457	3768	3528	3348	2457	4792	3024	3374	3276	3348	2646	5376	2457	3066	3528	3700	2808	4792	2646	3066	3276	4128	2646	4440	2457
24	1536	3780	2072	3906	2198	3768	1410	4032	2072	4144	2820	2898	1410	3780	1918	4608	2506	3374	1410	4032	2346	3990	2506	2898	882	4716	2072	4074	2506	3374
30	4230	3636	4464	4112	2725	3676	3838	3792	3763	4600	2773	3819	4800	3492	3719	4012	2617	4568	4320	3206	3763	4600	3264	3565	4054	3372	3719	4912	2827	4056
36	1022	1934	492	1036	906	1504	906	1904	484	864	746	1712	900	1550	750	1172	522	1152	770	1786	748	1096	730	1452	1030	1626	480	988	714	1356
42	1716	601	1932	418	2006	422	1860	479	2442	196	2136	408	1332	495	2052	484	2010	244	1904	535	2172	404	1980	494	1544	718	2244	306	2322	260
48	474	224	494	722	578	1018	468	216	580	832	468	968	472	472	652	568	428	890	396	334	556	654	386	1226	492	224	488	606	364	1126
54	40	528	337	988	128	786	126	400	320	888	34	702	110	730	186	762	164	824	64	776	202	720	131	738	136	444	298	680	112	872
60	380	544	276	320	708	362	338	582	280	294	472	438	302	704	380	298	636	486	408	602	314	296	578	510	534	584	284	320	748	422
66	286	160	46	92	68	96	304	192	14	70	220	92	218	130	40	70	96	126	345	148	34	74	88	120	296	190	34	84	56	114
72	64	4	126	8	50	6	108	4	244	10	10	0	122	8	216	8	66	6	44	2	138	12	32	2	132	4	208	10	36	6
78	66	32	60	165	36	132	64	66	30	120	84	172	66	40	56	147	56	102	140	22	44	107	30	198	96	65	42	155	34	90
84	12	72	44	34	68	60	32	58	28	38	4	64	24	32	46	62	112	54	8	28	48	74	46	28	24	42	62	54	50	34
90	24	12	40	12	10	26	14	16	10	14	44	16	92	0	18	14	16	30	12	6	18	14	26	20	16	4	18	16	8	16
96	22	18	0	2	2		0	6	0	4	20		8	8	0	0	6		0	2	0	2	6	0	4	14	2	0	4	0
102	0		0		6		2	0	2	0	4		0	0	0	2	16		0	0	0	0	28	0	0	0	0	0	18	2
108	20		0				24	6		6	4		16	8	0	0	2		20	4	0	4	0	4	20	2	0	4	0	2
114			2				0	4		4			0	12	2	2	0		0	4	0		0	4	0	0	6		0	
120							4	4					20	4			2		4		2		4		4	2			2	
126																			0				0			4				
132																			4				0			0				
138																							4			0				
144																										0				
150																										2				

Δ \ 2n	2^{31}	2^{32}	2^{33}	2^{34}	2^{35}	2^{36}	2^{37}	2^{38}	2^{39}	2^{40}	2^{41}	2^{42}	2^{43}	2^{44}	2^{45}	2^{46}
6	2457	4914	3276	5616	2730	6552	2457	4914	3744	4914	2730	6552	2457	5616	3276	4914
12	6552	2072	4914	2198	5616	1410	6552	2072	4914	2820	4914	1410	6552	1918	5616	2506
18	3374	4032	3700	2457	4440	2646	3768	3276	3348	2646	4792	2808	3374	3528	3348	2457
24	1536	3528	2072	3990	2198	3600	1410	4242	2072	4074	2820	3066	1410	3528	1918	4692
30	4230	3326	4464	4532	2725	3445	3838	4128	3763	4124	2773	4106	4800	3246	3719	4432
36	962	2094	508	832	854	1692	962	1680	468	1064	786	1468	852	1742	788	996
42	1800	679	1842	358	2116	472	1766	437	2580	296	2036	369	1392	537	1950	386
48	460	224	520	722	526	1018	476	216	508	832	524	968	482	472	664	568
54	28	576	387	904	98	782	160	344	250	1024	52	752	98	778	260	644
60	340	560	280	314	736	316	366	588	276	286	486	472	290	664	360	328
66	368	172	38	90	102	92	228	178	38	60	176	92	276	140	30	78
72	32	4	172	10	26	4	156	8	198	2	22	6	92	8	256	8
78	64	10	46	174	40	170	76	110	58	86	74	124	48	28	42	162
84	8	62	32	52	18	58	12	38	30	38	30	68	20	42	32	76
90	28	8	24	14	32	18	24	12	14	3	40	10	78	2	16	22
96	20	6		0	6		4	18	0	0	6	2	2	8		2
102	0	4		0	6		0	4	10	0	12	0	2	0		0
108	16	2		6	2		12	0	0	4	2	0	16	10		0
114		2		6	0		0	4	4	2		0	0	8		4
120					4		4	6				0	26			
126							0					2	0			
132							0						4			
138							4						4			
144																
150																

Δ \ 2n	2^{47}	2^{48}	2^{49}	2^{50}	2^{51}	2^{52}	2^{53}	2^{54}	2^{55}	2^{56}	2^{57}	2^{58}	2^{59}	2^{60}	2^{61}	…
6	2730	6552	2808	4914	3276	4914	2730	7488	2457	4914	3276	4914	3120	6552	2457	…
12	4914	1410	6552	2346	4914	2506	4914	882	7488	2072	4914	2506	4914	1536	6552	…
18	5376	2646	3066	3276	3700	3024	4792	2457	3066	3528	4128	2457	4440	2646	3374	…
24	2506	3234	1410	4242	2346	3906	2506	3066	882	4464	2072	4144	2506	3234	1536	…
30	2617	4256	4320	3516	3763	4124	3264	3796	4054	3126	3719	5440	2827	3769	4230	…
36	500	1412	814	1626	714	1308	768	1236	970	1786	496	744	678	1576	1022	…
42	2116	283	1812	449	2280	480	1882	486	1624	788	2130	228	2452	316	1716	…
48	372	890	422	334	532	654	424	1226	492	224	536	606	292	1126	474	…
54	124	808	96	704	146	804	149	760	106	526	372	576	72	844	40	…
60	630	454	430	630	304	296	584	524	532	528	272	220	748	406	380	…
66	116	110	288	136	64	96	56	112	348	194	22	114	88	90	286	…
72	54	0	80	6	92	14	46	4	98	4	248	16	22	0	64	…
78	66	150	120	48	48	103	32	160	102	43	30	166	48	128	66	…
84	82	48	0	30	60	46	96	52	22	48	52	92	12	36	12	…
90	40	18	29	6	28		12	16	10	10	4	32	26	12	24	…
96	6	2	8	10	4		2	0	0	10	0	12	6	0	22	…
102	20	0	0	2	2		16	0	0	0	0	0	12	0	0	…
108	2	0	12		0		0	6	12	4	2	4	4	4	20	…
114	0	0	0		2		0	4	0	0	0	2	0			…
120	0	2	8				2		2	2	2	0	0			…
126	4								6	4		0	4			…
132									4			0	0			…
138												2	4			…
144																…
150																…

At the next step $p_i = 19$, there are exactly 180 cases.

The enumerations, at a given step and within the corresponding period, are all different one from each other without exception for all of the examples that we examined exhaustively (i.e. up to $p_i = 19$) and we except it to be so in general.

APPENDIX 13
Horizons of spacings.

Table 83

fac / Δ	1	3
2	0	1
4	0	1
6	1	

Table 84

fac / Δ	1	3	15	1	5	15
2	0	1	3	0	0	3
4	0	2	3	0	0	3
6	1	2	2	1	3	2
8	0	1		0	0	
10	0			0	0	
12	2			2	1	

Table 85

fac / Δ	1	3	15	105	1	7	35	385
2	0	3	12	15	0	0	0	15
4	0	6	9	15	0	0	0	15
6	3	12	10	14	3	10	15	14
8	0	4	2	2	0	0	0	2
10	0	2	6	2	0	0	0	2
12	8	0	0		8	1	7	
14	0	2	1		0	0	0	
16	0	0			0	0	0	
18	2	0			2	5	2	
20	0	0			0	0		
22	0	0			0	0		
24	0	0			0	2		
26	0	0			0			
28	0	1			0			
30	2				2			

Table 86

fac / Δ	1	3	15	105	1155	1	11	77	385	1155
2	0	21	84	105	135	0	0	0	0	135
4	0	42	63	105	135	0	0	0	0	135
6	21	104	86	130	142	21	36	90	135	142
8	0	28	28	34	28	0	0	0	0	28
10	0	20	54	40	30	0	0	0	0	30
12	56	0	26	12	8	56	54	13	71	8
14	0	22	10	6	2	0	0	0	0	2
16	0	4	4			0	0	0	0	
18	22	8	4			22	22	45	28	
20	0	4	0			0	0	0	0	
22	0	2	1			0	0	0	0	
24	6	4				6	19	26	6	
26	0	0				0	0	0		
28	0	8				0	0	0		
30	22	2				22	17	6		
32	0	1				0	0			
34	0					0	0			
36	4					4	0			

Δ \ fac	1	3	15	105	1155	1	11	77	385	1155
38	0					0	0			
40	0					0	0			
42	4					4	2			

Table 87

Δ \ fac	1	3	15	105	1155	15015	1	13	143	1001	5005	15015
2	0	189	756	1050	1215	1485	0	0	0	0	0	1485
4	0	378	630	945	1350	1485	0	0	0	0	0	1485
6	189	1088	814	1250	1406	1690	189	462	792	495	1485	1690
8	0	252	336	368	445	394	0	0	0	0	0	394
10	0	218	516	576	378	438	0	0	0	0	0	438
12	504	0	434	276	306	188	504	208	57	990	845	188
14	0	246	196	146	110	58	0	0	0	0	0	58
16	0	68	108	42	40	12	0	0	0	0	0	12
18	238	124	76	66	22	8	238	264	297	350	394	8
20	0	88	10	10	2	0	0	0	0	0	0	0
22	0	38	36	15	4	2	0	0	0	0	0	2
24	96	80	28	4	2		96	342	274	175	132	
26	0	8	6	4			0	0	0	0	0	
28	0	92	13				0	0	0	0	0	
30	270	56	0				270	236	280	132	24	
32	0	14	0				0	0	0	0		
34	0	4	1				0	0	0	0		
36	60	8					60	42	46	6		
38	0	4					0	0	0	0		
40	0	9					0	0	0	0		
42	84	0					84	15	4	12		
44	0	4					0	0	0			
46	0	0					0	0	0			
48	20	0					20	32	42			
50	0	0					0	0	0			
52	0	0					0	0	0			
54	0	0					0	16	8			
56	0	2					0	0				
58	0						0	0				
60	12						12	3				
62	0						0					
64	0						0					
66	12						12					

Table 88

Δ \ fac	1	3	15	105	1155	15015	255255	1	17	221	2431	17017	85085	255255
2	0	2457	9828	13650	17010	19305	22275	0	0	0	0	0	0	22275
4	0	4914	8820	12285	17550	19305	22275	0	0	0	0	0	0	22275
6	2457	15616	10902	18780	19302	24530	26630	2457	7560	4620	4455	14850	22275	26630
8	0	3276	4968	5298	6429	7320	6812	0	0	0	0	0	0	6812
10	0	3148	6948	8208	7104	8022	7734	0	0	0	0	0	0	7734
12	6552	0	7198	5712	5862	4658	4096	6552	1476	6930	8910	2945	13315	4096
14	0	3582	3708	2550	2538	1450	1406	0	0	0	0	0	0	1406
16	0	1164	2044	1072	1308	692	432	0	0	0	0	0	0	432
18	3374	2024	1692	1956	1254	766	376	3374	3150	4130	7400	7425	6812	376
20	0	1672	422	536	292	116	24	0	0	0	0	0	0	24
22	0	682	1034	585	324	174	78	0	0	0	0	0	0	78
24	1536	1540	846	350	164	54	20	1536	3430	3120	3700	5890	2766	20
26	0	248	350	164	37	4	2	0	0	0	0	0	0	2
28	0	1548	379	38	20	2		0	0	0	0	0	0	
30	4230	1138	186	76	2	2		4230	4099	4235	2382	2766	816	
32	0	310	0	2	0			0	0	0	0	0	0	
34	0	182	49	10	0			0	0	0	0	0	0	

Δ \ fac	1	3	15	105	1155	15015	255255	1	17	221	2431	17017	85085	255255
36	1022	278	10	2	4			1022	1580	658	493	408	72	
38	0	130	4	0				0	0	0	0	0	0	
40	0	214	6	2				0	0	0	0	0	0	
42	1716	86	2	4				1716	298	1686	1126	24	24	
44	0	132	0					0	0	0	0	0		
46	0	16	0					0	0	0	0	0		
48	474	62	2					474	1028	310	152	204		
50	0	44	0					0	0	0	0	0		
52	0	11	0					0	0	0	0	0		
54	40	4	2					40	736	72	12	48		
56	0	30						0	0	0	0			
58	0	2						0	0	0	0			
60	380	32						380	248	115	152			
62	0	0						0	0	0	0			
64	0	0						0	0	0	0			
66	286	2						286	30	12	12			
68	0	2						0	0	0	0			
70	0	2						0	0	0	0			
72	64	0						64	1	20	0			
74	0	0						0	0	0	0			
76	0	0						0	0	0	0			
78	66	2						66	78	8	0			
80	0							0	0	0	0			
82	0							0	0	0	0			
84	12							12	40	4	6			
86	0							0	0					
88	0							0	0					
90	24							24	4					
92	0							0	0					
94	0							0	0					
96	22							22	0					
98	0							0	0					
100	0							0	0					
102	0							0	0					
104	0							0	0					
106	0							0	0					
108	20							20	0					
110									0					
112									0					
114									2					

Table 89

Δ \ fac	1	3	15	105	1155	15015	255255	4849845	1	19	323	4199	46189	323323	1616615	4849845
2	0	36855	147420	204750	255150	289575	334125	378675	0	0	0	0	0	0	0	378675
4	0	73710	132300	184275	263250	289575	334125	378675	0	0	0	0	0	0	0	378675
6	36855	254464	171210	297060	314106	396110	435290	470630	36855	47736	96390	67320	100980	252450	378675	470630
8	0	49140	85920	92058	118824	132960	128192	128810	0	0	0	0	0	0	0	128810
10	0	51058	109980	140976	128592	156984	150114	148530	0	0	0	0	0	0	0	148530
12	98280	0	125398	114528	113722	99338	102424	90124	98280	92820	48580	157080	151470	54545	235315	90124
14	0	58338	66132	51258	50150	41854	39698	33206	0	0	0	0	0	0	0	33206
16	0	20988	41832	25344	31175	18242	16536	12372	0	0	0	0	0	0	0	12372
18	53690	35180	38680	47792	36720	24742	18080	12424	53690	77592	48195	64050	79790	126225	128810	12424
20	0	32088	11528	15112	10414	6338	2224	1440	0	0	0	0	0	0	0	1440
22	0	12682	24246	14735	11712	6110	3450	2622	0	0	0	0	0	0	0	2622
24	26208	30092	20856	10930	7560	4732	1844	1136	26208	39852	80710	25305	57815	109090	59160	1136
26	0	6072	9812	4684	1917	1008	258	142	0	0	0	0	0	0	0	142
28	0	27128	10917	2304	1714	812	268	72	0	0	0	0	0	0	0	72
30	72378	25122	9390	3756	1060	302	82	20	72378	53850	103632	89580	83845	59160	22488	20
32	0	6440	154	488	70	32	2	0	0	0	0	0	0	0	0	0
34	0	5422	1591	662	90	40	6	2	0	0	0	0	0	0	0	2
36	18776	7446	898	490	132	38	2		18776	14361	11956	12728	6760	11244	3384	
38	0	3726	304	114	22	2			0	0	0	0	0	0	0	

fac / Δ	1	3	15	105	1155	15015	255255	4849845	1	19	323	4199	46189	323323	1616615	4849845
40	0	5778	428	130	14	6			0	0	0	0	0	0	0	
42	34812	2612	332	208	6				34812	35626	4056	30458	28170	1152	1392	
44	0	3686	64	14					0	0	0	0	0	0	0	
46	0	906	32	44					0	0	0	0	0	0	0	
48	10462	2706	176	26					10462	11246	11806	5074	5036	5622	192	
50	0	1524	16	6					0	0	0	0	0	0	0	
52	0	401	20	8					0	0	0	0	0	0	0	
54	1968	568	90	2					1968	3255	9130	1204	2242	2256	24	
56	0	820	14	0					0	0	0	0	0	0		
58	0	364	14	6					0	0	0	0	0	0		
60	9452	1096	0						9452	17584	3267	5500	1476	312		
62	0	40	6						0	0	0	0	0	0		
64	0	226	14						0	0	0	0	0	0		
66	6322	152	10						6322	2170	3512	5263	72	0		
68	0	96	8						0	0	0	0	0	0		
70	0	184	4						0	0	0	0	0	0		
72	2816	16	2						2816	1342	242	860	648	0		
74	0	28	0						0	0	0	0	0	0		
76	0	16	0						0	0	0	0	0	0		
78	2620	84	2						2620	1214	3144	1616	48	24		
80	0	14							0	0	0	0	0			
82	0	8							0	0	0	0	0			
84	632	44							632	1142	1502	72	48			
86	0	4							0	0	0	0				
88	0	6							0	0	0	0				
90	1236	10							1236	344	1126	198				
92	0	2							0	0	0	0				
94	0	0							0	0	0	0				
96	876	2							876	108	188	32				
98	0	2							0	0	0	0				
100	0	0							0	0	0	0				
102	16	0							16	560	12	0				
104	0	2							0	0	0	0				
106	0	0							0	0	0	0				
108	954	0							954	46	48	138				
110	0	0							0	0	0	0				
112	0	0							0	0	0	0				
114	0	2							0	0	92	0				
116	0								0	0	0	0				
118	0								0	0	0	0				
120	142								142	62	70	74				
122	0								0	0	0	0				
124	0								0	0	0	0				
126	48								48	4	8	0				
128	0								0	0	0	0				
130	0								0	0	0	0				
132	26								26	12	0	4				
134	0								0	0	0	0				
136	0								0	0	0	0				
138	86								86	6	0	0				
140	0								0	0	0	0				
142	0								0	0	0	0				
144	0								0	6	12	0				
146	0								0	0	0	0				
148	0								0	0	0	0				
150	20								20	8	0	4				
152										0	0					
154										0	0					
156										4	0					
158											0					
160											0					
162											0					
164											0					
166											0					
168											0					
170											0					
172											0					
174											2					

Example of evolution :
Column fac = 3.
Populations #SP3(j,i).

Numbers in parentheses are not deduced from the iterative formulas.

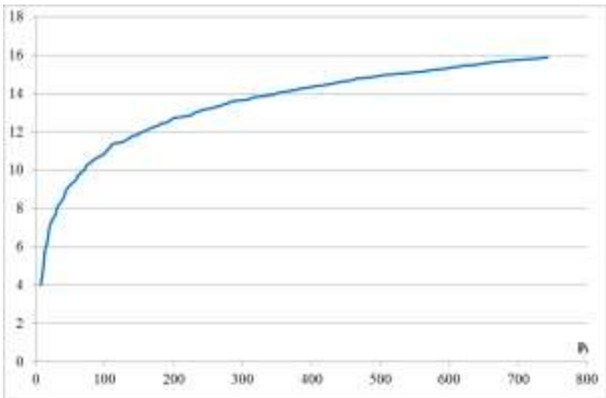
i	1	2	3	4	5	6	7	…
p_i	3	5	7	11	13	17	19	…
j = 1	(1)	1	3	21	189	2457	36855	…
j = 2	(1)	(2)	6	42	378	4914	73710	…
j = 3		(2)	12	104	1088	15616	254464	…
j = 4		(1)	(4)	28	252	3276	49140	…
j = 5		(0)	2	20	218	3148	51058	…
j = 6			(0)	0	0	0	0	…
j = 7			(2)	22	246	3582	58338	…
j = 8			…	…	…	…	…	…

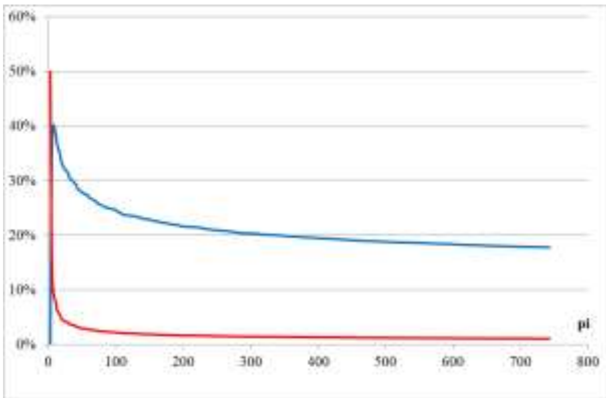
j	Formulas
1	#SP3(1,1) = 1 #SP3(1,i) = (p_i-4).#SP3(1,i-1)
2	#SP3(2,2) = 2 #SP3(2,i) = (p_i-4).#SP3(2,i-1)
3	x1(3) = 4 x1(i) = (p_{i-1}-5).x1(i-1) #SP3(3,2) = 2 #SP3(3,i) = (p_i-3).#SP3(3,i-1)+x1(i) Indistinguishable from x1(4) = 32 x1(i) = (p_{i-1}-3).x1(i-1) #SP3(3,3) = 12 #SP3(3,i) = (p_i-5).#SP3(3,i-1)+x1(i)
4	#SP3(4,3) = 4 #SP3(4,i) = (p_i-4).#SP3(4,i-1)
5	x1(4) = 2 x1(i) = (p_{i-2}-6).x1(i-1) x2(3) = 2 x2(i) = (p_{i-1}-5).x2(i-1)+x1(i) #SP3(5,2) = 0 #SP3(5,i) = (p_i-4).#SP3(5,i-1)+x2(i)
6	#SP3(6,i) = 0
7	x1(4) = 8 x1(i) = (p_{i-3}-5).x1(i-1) #SP3(7,3) = 2 #SP3(7,i) = (p_i-4).#SP3(7,i-1)+x1(i)
8	x1(6) = 24 x1(i) = (p_{i-2}-6).x1(i-1) x2(5) = 32 x2(i) = (p_{i-1}-5).x2(i-1)+x1(i) #SP3(8,4) = 0 #SP3(8,i) = (p_i-4).#SP3(8,i-1)+x2(i)
9	x1(5) = 12 x1(i) = (p_{i-2}-7).x1(i-1) x2(4) = 8 x2(i) = (p_{i-1}-6).x2(i-1)+x1(i) #SP3(9,3) = 0 #SP3(9,i) = (p_i-4).#SP3(9,i-1)+x2(i)
10	x1(5) = 28 x1(i) = (p_{i-2}-7).x1(i-1) x2(4) = 4 x2(i) = (p_{i-1}-5).x2(i-1)+x1(i) #SP3(10,3) = 0 #SP3(10,i) = (p_i-4).#SP3(10,i-1)+x2(i)

11	$x1(6) = 28$ $x1(i) = (p_{i-2}-6).x1(i-1)$ $x2(5) = 20$ $x2(i) = (p_{i-1}-5).x2(i-1)+x1(i)$ $\#SP3(10,4) = 2$ $\#SP3(10,i) = (p_i-4).\#SP3(10,i-1)+x2(i)$
…	?

We note above the unique case $j = 3$ where the multiplier factor of $\#SP3(j,i)$ is no more p_i-4.
However, the p_i-3 and p_i-5 multiplier factors of the two indistinguishable iterative formula systems (always constructible according to our commentary below Table 10) produce an average of p_i-4. Just a coincidence ?
Of course, numerically, we find that the ratios $\#SP3(3,i)/\#SP3(3,i-1)$ are tending towards p_i-3. We know that the most left column of the tables sees the ratios $\#SP(j,i)/\#SP(j,i-1)$ tending towards p_i-4 (pseudo-twins) and the most right of them tending towards p_i-2 (pseudo-primes). Is the strategy for reconciling the two trends to move gradually from p_i-4 to p_i-3 and then to p_i-2 ? We could not verify this for lack of sufficient numerical data from which we would be able to deduce routines by successive Euclidian divisions. This failure may also be due to the fact that this assumption may be totally false.
As a result of the p_i-3 leading ratio (instead of p_i-4), the populations of the $j = 3$ line are growing faster than they do on the other lines (with logarithmic gain). In the first graph below, we show the ratio of the populations in this line ($j = 3$) compared to that of the first line populations ($j = 1$). In addition, the proportion of the populations of this line to the overall populations (which grows as p_i-2) remains significant for a relatively long time (although ultimately tending as all the other lines to 0) as shown in the second graph where the curve in blue is the one relating to $j = 3$ and the curve in red the one relating to $j = 1$.

Graphics 29 and 30





APPENDIX 14
Programming of #S(i,j) evaluations.

Code PARI
https ://pari.math.u-bordeaux.fr/

Recursive method (steps 3, 4, 5, etc.)
Only for the 2n = 2^m cases.
Large memory space needed (fast saturation of memory space)

```
{expo = 1; ec = 2^expo; \\ choose the gap by choosing the exponent
kk0 = 4001; kk1 = kk0; \\ to choose so that kk0 and kk0+ec are prime numbers
rg = 3; prodt = 1; pp1 = primes(100)[rg];
for(i = 2, rg-1, prodt = prodt*(primes(100)[i]-2));
sizm = (pp1-2)*prodt;
siz = pp1*prodt;
nbb = vector(siz+1,i,0);
nbf = vector(siz+1,i,0);
nbb[1] =6;
for(i = 1, pp1-1, for(j = 1, prodt, nbb[i*prodt+j] = nbb[j]));
for(i = 1, siz, kk1 = kk1+nbb[i]; kk2 = kk1+2;
if(Mod(kk1, pp1) == 0, nbf[i] = 1);
if(Mod(kk2, pp1) == 0, nbf[i] = 1));
for(i = 1, siz, if(nbf[i] == 1, nbb[i+1] = nbb[i] + nbb[i+1]; nbb[i] =0));
k = 0;
for(i = 1, siz, if(nbb[i] <> 0, k = k+1; nbb[k] = nbb[i]));
\\ for(i = 1, sizm, print(nbb[i]));

print("/");
nb = vecmax(nbb);
print(nb);
print("/");
nz = vector(nb/6,i,0);
for(i = 1, sizm, nz[nbb[i]/6] = nz[nbb[i]/6]+1);
for(i = 1, nb/6, print(nz[i]));
print("/");

for(rg = 4, 11, \\ choose 6 or more
prodt = 1; pp1 = primes(100)[rg]; kk1 = kk0;
for(i = 2, rg-1, prodt = prodt*(primes(100)[i]-2));
sizm = (pp1-2)*prodt;
siz = pp1*prodt;
nba = vector(siz+1,i,0);
nbg = vector(siz+1,i,0);
for(i = 0, pp1-1, for(j = 1, prodt, nba[i*prodt+j] = nbb[j]));
for(i = 1, siz, kk1 = kk1+nba[i]; kk2 = kk1+2;
if(Mod(kk1, pp1) == 0, nbg[i] = 1);
if(Mod(kk2, pp1) == 0, nbg[i] = 1));
for(i = 1, siz, if(nbg[i] == 1, nba[i+1] = nba[i] + nba[i+1]; nba[i] =0));
k = 0;
for(i = 1, siz, if(nba[i] <> 0, k = k+1; nba[k] = nba[i]));
\\ for(i = 1, sizm, print(nba[i]));

nbb = vector(sizm,i,0);
for(i = 1, sizm, nbb[i]= nba[i]);
print("/");
nb = vecmax(nbb);
print(nb);
print("/");
nz = vector(nb/6,i,0);
for(i = 1, sizm, nz[nbb[i]/6] = nz[nbb[i]/6]+1);
for(i = 1, nb/6, print(nz[i]));
print("/"););}
```

Direct evaluation method (step i)
Low memory space needed

```
{siz = 33; \\ to be ajusted
fac = 1; \\ to choose
expo = 1; \\ to choose
qtpr = 6; \\ to choose
ec = fac*(2^expo); ec2 = ec/2; nb = vector(siz,i,0); prodt = 1;
for(i = 2, qtpr, prodt = prodt * primes(qtpr)[i]);
for(c = 2001+ec2, 2001+ec2+prodt, a = 2*c+1 ; ac = a-ec;
if(Mod(ac, 3) <> 0,
if(Mod(a, 3) <> 0,
if(Mod(ac, 5) <> 0,
if(Mod(a, 5) <> 0,
if(Mod(ac, 7) <> 0,
if(Mod(a, 7) <> 0,
if(Mod(ac, 11) <> 0,
if(Mod(a, 11) <> 0,
if(Mod(ac, 13) <> 0,
if(Mod(a, 13) <> 0,
anc = a; canc = (anc-1)/2;
))))) ))))) );
for(c = canc+1, canc+1+prodt, a = 2*c+1 ; ac = a-ec;
if(Mod(ac, 3) <> 0,
if(Mod(a, 3) <> 0,
if(Mod(ac, 5) <> 0,
if(Mod(a, 5) <> 0,
if(Mod(ac, 7) <> 0,
if(Mod(a, 7) <> 0,
if(Mod(ac, 11) <> 0,
if(Mod(a, 11) <> 0,
if(Mod(ac, 13) <> 0,
if(Mod(a, 13) <> 0,
nouv = a; dif = nouv-anc; dif2 = dif/2; nb[dif2] = nb[dif2]+1; anc = nouv
))))) ))))) );
for(i = 1, siz, print(nb[i]))}
```

Direct evaluation method (step i)
Large memory space needed (fast saturation in memory space, may miss also an item in the final count)

```
{reserve  = 1005;
siz = 50; \\ to be adjusted
nb = vector(siz,i,0);
qtpr = 7; \\ to choose
prodt = 1;
for(i = 1, qtpr, prodt = prodt * primes(qtpr)[i]);
base = vector(prodt+reserve,i,i+100);
for(j = 1,qtpr,
prem = primes(qtpr)[j]-100%primes(qtpr)[j];
\\ print(prem);
for(i = 1, prodt/primes(qtpr)[j], base[primes(qtpr)[j]*i+prem] = 0));
for(i = 1, prodt/2, c = 2*i+1;
if(base[c]-base[c-2] == 2, anc = c; break));
for(i = anc, prodt/2, c = 2*i+1;
if(base[c]-base[c-2] == 2, dif = c-anc ; dif6 = dif/6; nb[dif6] = nb[dif6]+1; anc = c));
for(i = 1, siz, print(nb[i]))}
```