Continued fractions of Dirichlet series. Recurrence polynomials study.

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Abstract To prove ζ(3) irrationality, generalized continued fractions have shown their usefulness. Following this way, we give an expression of Dirichlet series continued fractions. For the Riemann and Dirichlet Eta series, we observe the constant value of the real part of the recurrence polynomials roots so that we can propose a new means to evaluate the irrationality of ζ(2n+1) on a case by case basis. We deduce in the same time arguments towards Riemann hypothesis thanks to a new highlight on the source of the Zeta function zeroes with a glance on other elementary functions.

Fractions continuées des séries de Dirichlet.
Etude des racines des polynômes de récurrence.

Résumé Pour la démonstration de l'irrationalité de ζ(3), le recours aux fractions continuées généralisées s'est montré fort utile. Suivant cette voie, nous établissons ici une expression générale des fractions continuées des séries de Dirichlet. Pour la série de Riemann et la série Eta de Dirichlet, nous observons la valeur constante de la partie réelle des racines des polynômes de récurrence constituant ainsi une nouvelle piste pour l'évaluation de l'irrationalité de ζ(2n+1) au cas par cas. Nous apportons ensuite des arguments dans le sens de l'hypothèse de Riemann grâce à un éclairage nouveau sur l'origine des zéros de la fonction Zêta. Se faisant, nous portons ce type de regard à d'autres fonctions plus élémentaires.

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Summary

1 General continued fraction

Let us consider F(s) a function of a complex variable. In the same way as a Taylor limited development forms a series converging towards the initial function, a generalized continued fraction is the limit of the original function. Thus, let us suppose that we have found an expression in the form of a generalized continued fraction for F(s) :

$$F(s) = \left. \frac{a}{B_s(0)} \right| + \left. \frac{A_s(1)}{B_s(1)} \right| + \left. \frac{A_s(2)}{B_s(2)} \right| + \left. \frac{A_s(3)}{B_s(3)} \right| + \ldots + \left. \frac{A_s(m)}{B_s(m)} \right| - \ldots \qquad (1)$$

In these expressions, the polynomials $A_s(n)$ and $B_s(n)$ are called recurrence polynomials.

The signs between fractions may be negative or alternate.

2 General continued fraction of Zeta function

2.1 Riemann series

The Riemann series is defined by :

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} \qquad (2)$$

Lemma 1

Let us write (existence according to [3] confirmed here in a different context) :

$$\zeta(s) = \frac{a\,|}{|\,B_s(0)} - \frac{A_s(1)\,|}{|\,B_s(1)} - \frac{A_s(2)\,|}{|\,B_s(2)} - \frac{A_s(3)\,|}{|\,B_s(3)} - \ldots - \frac{A_s(x)\,|}{|\,B_s(x)} - \ldots \quad (3)$$

Then, the i^th^ development of the series $\zeta(s) = \sum 1/m^s$ is precisely the i^th^ development of the previous generalized continued fraction when :

$$\begin{aligned} &a = 1 \\ &A_s(x) = x^{2s} \qquad\qquad (4) \\ &B_s(x) = x^s+(x+1)^s \end{aligned}$$

The result holds as long as $B_s(x)$ has a meaning in a denominator.
For integer n, the expression $x^s+(x+1)^s$ has singularities in 0 and -1.
We proceed later on to a further study of the domain of definition when s describes the set of complex numbers $\mathscr{C}$.

Proof of lemma 1

We verify by a mere development that the first terms of the continued fraction are equal to the first terms of the series. For example:

Term 1 :

$$1 = 1$$

Term 2 :

$$1/(0^s+1^s-1^{2s}/(1^s+2^s)) = 1/(1^s.2^s)/(1^s+2^s) = (1^s+2^s)/(1^s.2^s) = 1+1/2^s$$

Term 3 :

$$\begin{gathered} 1/(0^s+1^s-1^{2s}/(1^s+2^s)-2^{2s}/(2^s+3^s)) \\ = 1/(1^s-1^{2s}/(1^s.2^s+1^s.3^s+2^s.3^s)/(2^s+3^s)) \\ = 1/(1^s-(1^{2s}.2^s+1^{2s}.3^s)/(1^s.2^s+1^s.3^s+2^s.3^s)) \\ = 1/(1^{2s}.2^s+1^{2s}.3^s+1^s.2^s.3^s -(1^{2s}.2^s+1^{2s}.3^s))/(1^s.2^s+1^s.3^s+2^s.3^s) \\ = (1^s.2^s+1^s.3^s+2^s.3^s)/(1^s.2^s.3^s) \\ = 1+1/2^s+1/3^s \end{gathered}$$

Term 4 :

$$\begin{gathered} 1/(0^s+1^s-1^{2s}/(1^s+2^s)-2^{2s}/(2^s+3^s)-3^{2s}/(3^s+4^s)) \\ = 1/(1^s-1^{2s}/(1^s+2^s)-2^{2s}/((2^s+3^s).(3^s+4^s)-3^{2s})/(3^s+4^s)) \\ = 1/(1^s-1^{2s}/(1^s+2^s)-2^{2s}.(3^s+4^s)/(2^s.3^s+2^s.4^s+3^s.4^s)) \\ = 1/(1^s-1^{2s}.(2^s.3^s +2^s.4^s+3^s.4^s)/((1^s+2^s).(2^s.3^s+2^s.4^s+3^s.4^s)-2^{2s}.(3^s+4^s))) \\ = 1/(1^s-1^{2s}.(2^s.3^s +2^s.4^s+3^s.4^s)/(1^s.(2^s.3^s+2^s.4^s+3^s.4^s)+2^s.(2^s.3^s+2^s.4^s)+2^s.3^s.4^s-2^{2s}.(3^s+4^s))) \\ = 1/(1^s-1^{2s}.(2^s.3^s +2^s.4^s+3^s.4^s)/(1^s.2^s.3^s+1^s.2^s.4^s+1^s.3^s.4^s+2^s.3^s.4^s)) \\ = (1^s.2^s.3^s+1^s.2^s.4^s+1^s.3^s.4^s+2^s.3^s.4^s)/(1^s.(1^s.2^s.3^s+1^s.2^s.4^s+1^s.3^s.4^s+2^s.3^s.4^s)-1^{2s}.(2^s.3^s +2^s.4^s+3^s.4^s)) \\ = (1^s.2^s.3^s+1^s.2^s.4^s+1^s.3^s.4^s+2^s.3^s.4^s)/(1^s.2^s.3^s.4^s) \\ = 1+1/2^s+1/3^s+1/4^s \end{gathered}$$

These examples clearly show the elimination role of the squares x^{2s} at every stage of the calculation routine.
So let us look at the expression :

$$x^s+(x+1)^s- \frac{(x+1)^{2s}}{(x+1)^s+(x+2)^s}$$

It is equal to

$$\begin{gathered} ((x^s+(x+1)^s).((x+1)^s+(x+2)^s)-(x+1)^{2s})/((x+1)^s+(x+2)^s) \\ = ((x^s.(x+1)^s+x^s.(x+2)^s+(x+1)^s.(x+2)^s)/((x+1)^s+(x+2)^s) \\ = R(x) \end{gathered}$$

We use the result in the following term :

$$(x-1)^s+x^s- \frac{x^{2s}}{}$$

$$\overline{R(x)}$$

This term equals

$$(x-1)^s+x^s-x^{2s}/(x^s+(x+1)^s.(x+2)^s)/((x+1)^s+(x+2)^s)$$
$$= (x-1)^s+x^s-x^{2s}.((x+1)^s+(x+2)^s)/(x^s.(x+1)^s+x^s.(x+2)^s+(x+1)^s.(x+2)^s)$$
$$= (((x-1)^s+x^s).(x^s.(x+1)^s+x^s.(x+2)^s+(x+1)^s.(x+2)^s)-x^{2s}.((x+1)^s+(x+2)^s))/(x^s.(x+1)^s+x^s.(x+2)^s+(x+1)^s.(x+2)^s)$$
$$= (((x-1)^s+x^s).(x^s.(x+1)^s+x^s.(x+2)^s+(x+1)^s.(x+2)^s)-x^{2s}.((x+1)^s+(x+2)^s))/(x^s.(x+1)^s+x^s.(x+2)^s+(x+1)^s.(x+2)^s)$$
$$= ((x-1)^s.x^s.(x+1)^s+(x-1)^s.x^s.(x+2)^s+(x-1)^s.(x+1)^s.(x+2)^s+x^s.(x+1)^s.(x+2)^s)/(x^s.(x+1)^s+x^s.(x+2)^s+(x+1)^s.(x+2)^s)$$

Thus, the terms of type $x^{2s}.(x+i)^s$ will eliminate as we develop the expression because of the square x^{2s} above R(x). By proceeding to the development of generic terms, we get an x-th term consistent with the awaited term. By iterating the process, the last developed term presents, after removal of the squares, a single factor $1^s.2^s.3^s.4^s…r^s$ at the denominator of an r rank development. All terms in the numerator are then composed of r-1 products to the power n (without squares) except 1 term equal to $1^s.2^s.3^s.4^s…r^s$.
Hence the lemma.

2.2 Extension to Zeta function

We have added this paragraph thanks to a remark made by Jonathan Sondow on a previous version of the article concerning the domain of definition of the Riemann series.

The Riemann series merges with the Zeta function on the Re(s) > 1 domain.
The analytical continuation for Re(s) > 0 is given (cf [5]) by the Dirichlet Eta series :

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^s} \qquad (5)$$

Lemma 2

Let us write now :

$$\zeta(s) = \left.\frac{a}{B_s(0)}\right| + \left.\frac{A_s(1)}{B_s(1)}\right| + \left.\frac{A_s(2)}{B_s(2)}\right| + \left.\frac{A_s(3)}{B_s(3)}\right| + \ldots + \left.\frac{A_s(x)}{B_s(x)}\right| + \ldots \quad (6)$$

Then, the i^{th} development of the series ζ(s) is precisely the i^{th} development of the previous generalized continued fraction when :

$$a = -1/(1-2^{1-s})$$
$$A_s(x) = x^{2s} \qquad (7)$$
$$B_s(x) = x^s-(x+1)^s$$

Proof of lemma 2

It suffices to follow the lemma 1 proof procedure.

2.3 Study of the roots of the polynomial of recurrence

2.3.1 Riemann series

Let us have n an integer.
Let us write $B_n(x) = x^n+(x+1)^n = 0$. Then $((x+1)/x)^n = -1 = e^{i.(1+2k).\pi}$. The n-th roots of -1 are given by $e^{i.(1+2k).\pi/n}$, k = 0, 1, 2,..., n-1. It follows $(x+1)/x = e^{i.(1+2k).\pi/n}$, hence $x = 1/(e^{i.(2k+1).\pi/n}-1) = 1/(\cos((2k+1).\pi/n)-1+i.\sin((2k+1).\pi/n)) = (\cos((2k+1).\pi/n)-1-i.\sin((2k+1).\pi/n))/(2-2.\cos((2k+1).\pi/n))$. Using then $1-\cos(x) = 2.\sin^2(x/2)$, it follows immediately :

$$x = -(1/2).(1+i.\mathrm{cotg}((k+1/2).\pi/n)) \qquad (8)$$

We take notice here that the real part of the recurrence polynomials roots is a constant equal to -1/2.

We may also write $B_n(x)$ as a product of real values by gathering conjugated terms. We start with $(x+1/2+i.\mathrm{cotg}((k+1/2).\pi/n)/2).(x+1/2-i.\mathrm{cotg}((k+1/2).\pi/n)/2) = (x+1/2)^2+(1/4).\mathrm{cotg}^2((k+1/2).\pi/n) = x^2+x+(1/4).(1+\mathrm{cotg}^2((k+1/2).\pi/n)) = x^2+x+1/(4.\sin^2((k+1/2).\pi/n))$. Hence the two cases (for n > 1) :

n	$B_n(x)$
0 mod 2	$2.\prod_{k=1}^{n/2-1} x^2+x+\frac{1}{4.\sin^2((k+1/2).\pi/n)}$
1 mod 2	$2.(x+1/2).\prod_{k=1}^{(n-1)/2} x^2+x+\frac{1}{4.\sin^2((k+1/2).\pi/n)}$

(9)

2.3.2 Extension to Zeta function

Let us still have n an integer.
We write now $B_n(x) = x^n-(x+1)^n = 0$. Then $((x+1)/x)^n = 1 = e^{i.2k.\pi}$. The n-th roots of 1 are given by $e^{i.2k.\pi/n}$, k = 0, 1, 2,..., n-1. It follows $(x+1)/x = e^{i.2k.\pi/n}$, hence after a few manipulations :

$$x = -(1/2).(1+i.\mathrm{cotg}(k.\pi/n)) \qquad (10)$$

We take notice again that the real part of the recurrence polynomials roots is a constant equal to -1/2.

As earlier, we may also write (for n > 2) :

n	$B_n(x)$
1 mod 2	$-n.\prod_{k=1}^{(n-1)/2} x^2+x+\frac{1}{4.\sin^2(k.\pi/n)}$
0 mod 2	$-n.(x+1/2).\prod_{k=1}^{n/2-1} x^2+x+\frac{1}{4.\sin^2(k.\pi/n)}$

(11)

2.3.3 Real part conjecture

2.3.3.1 The historical cases

According to [1] and [2], we can write

$$\zeta(2) = \frac{5}{\lvert P_2(0)}\Big| + \frac{1^4}{\lvert P_2(1)}\Big| + \frac{2^4}{\lvert P_2(2)}\Big| + \frac{3^4}{\lvert P_2(3)}\Big| + \ldots + \frac{x^4}{\lvert P_2(x)}\Big| + \ldots \quad \text{where } P_2(x) = 11x^2+11x+3 \qquad (12)$$

and

$$\zeta(3) = \frac{6}{\lvert P_3(0)}\Big| - \frac{1^6}{\lvert P_3(1)}\Big| - \frac{2^6}{\lvert P_3(2)}\Big| - \frac{3^6}{\lvert P_3(3)}\Big| - \ldots - \frac{x^6}{\lvert P_3(x)}\Big| + \ldots \quad \text{where } P_3(x) = 34x^2+51x^2+27x+5 \qquad (13)$$

We observe that :

$$P_2(x) = 11.(x+1/2)^2+1/4$$

and that

$$P_3(x) = 2.(x+1/2).(17.(x+1/2)^2+3/4)$$

These polynomials have following roots :

P_2	$-1/2+i/(2.11^{1/2})$, $-1/2-i/(2.11^{1/2})$
P_3	$-1/2$, $-1/2+i.(3/17)^{1/2}/2$, $-1/2-i.(3/17)^{1/2}/2$

Here again, the real part of the generalized continued fraction recurrence polynomials is equal to -1/2 as in the case of $P_n(x) = x^n \pm (x+1)^n$, n a positive integer.

2.3.3.2 Conjecture statement

Any root of a Riemann Zeta function recurrence polynomial (with integer coefficients and exponents) $P_n(x)$ has real part equal to -1/2 when :

$$\zeta(n) = \frac{a}{\lvert P_n(0)}\Big| + \frac{1^{2n}}{\lvert P_n(1)}\Big| + \frac{2^{2n}}{\lvert P_n(2)}\Big| + \frac{3^{2n}}{\lvert P_n(3)}\Big| + \ldots + \frac{x^{2n}}{\lvert P_n(x)}\Big| + \ldots \qquad (14)$$

Inside the conjecture

In the case of polynomials $P_n(x) = x^n \pm (x+1)^n$, the unique real zeroes part value is simply linked to the unit distance between successive m in the series $\sum 1/m^s$ terms and the exponents n (of $P_n(x)$) and 2n (of x^{2n}) ratio. We will notice that when we focus on the first terms of ζ(s) continued fractions calculation routine. For recurrence polynomials different from $x^n \pm (x+1)^n$ (such as those given for ζ(2) and ζ(3) earlier), the same evidence regarding this link to distance and exponents ratio is more difficult to figure out.

Note 1 :

Coefficient "a" is necessarily an integer or a rational for equality deserves to be stated (in order to prove the irrationality or the transcendence of ζ(n)).

Note 2 :

Other forms as numerator x^{2n} are of course imaginable but improbable.

Note 3 :

For the ζ(s = 1/2+i.t) function, we have $A_s(x) = x^{2s} = x.e^{i.t.\ln(x)}$. Thus, the module of $A_s(x)$ is linear in x (with coefficient 1) in the Riemannian situation.

Note 4 :

The limitation to a constant value -1/2 for the real part of the recurrence polynomials allows a more rapid numerical search of potential polynomials.
Let us assume $P_3(x)$ of form $r.(x+1/2).(s^2(x+1/2)^2+3/4)$, with

$$\zeta(3) = \left|\frac{a}{P_3(0)}\right| + \left|\frac{1^{2n}}{P_3(1)}\right| + \left|\frac{2^{2n}}{P_3(2)}\right| + \left|\frac{3^{2n}}{P_3(3)}\right| + \ldots + \left|\frac{x^{2n}}{P_3(x)}\right| + \ldots$$

We have the following situations :

a	u	w	r	s	
4/3	1	0	3	1	Alternated series η
1	$(-1)^x$	1	1	3	Dirichlet series
6	$(-1)^x$	1	17	3	Apéry series

Are there other solutions?
Our own trials were unsuccessful with simple and compelling examples, except possibly (solution of a numerical approach hence to be proved) :

a	u	w	r	s
8/7	$(-1)^x$	1	3	1

If more of these kind of solutions exist (except the last example), the convergence towards ζ(3) will certainly be faster as in the previous cases, hence a possibility of ζ(3) transcendence exploration.

More generally, one can prove the irrationality or transcendence of such ζ(2n+1) by finding new recurrence polynomials, with faster convergence than $x^n \pm (x+1)^n$, but using the "real value -1/2 of zeroes" hypothesis (much less cases to check). The reader will however note the precisions given in the postscript of [5] and those given in [4] concerning the size of the fraction Z(5) in $\zeta(5) = Z(5).\sum (-1)^{m-1}/m^5/\binom{2m}{m}$ before starting this unlikely enterprise.

Note 5 :

We find plenty of analogies of this kind when studying the continued fractions of the elementary functions and derived functions. The reader will discover this in the appendix.
For a given elementary functions F(s), let us write one of its continued fractions as :

$$F(s) = \left|\frac{a}{1}\right| + \left|\frac{A_s(1)}{B_s(1)}\right| + \left|\frac{A_s(2)}{B_s(2)}\right| + \left|\frac{A_s(3)}{B_s(3)}\right| + \ldots + \left|\frac{A_s(x)}{B_s(x)}\right| + \ldots$$

Then simplifying things, we see that the real part of the $B_s(x)$ recurrence polynomials zeroes equal ±d/n where n is the exponent appearing in $A_s(x)$ and d is the exponent appearing in $B_s(x)$.

This scheme may take varying forms which details may be difficult to completely understand. In peculiar, we will show below the effect of the distance between terms m by varying it.

3 Translation of recurrence polynomials zeroes line

Let us have r a constant. Let us consider the function (ζ translated) defined by infinite series :

$$\zeta T(s,r) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(1-r+r.m)^s} \qquad (15)$$

The series merges with Dirichlet series when r = 1.
Let us write

$$T(s) = \frac{-1|}{|-1} + \frac{A_s(1)|}{|B_s(1)} + \frac{A_s(2)|}{|B_s(2)} + \frac{A_s(3)|}{|B_s(3)} + \ldots + \frac{A_s(x)|}{|B_s(x)} + \ldots \quad (16)$$

T(s) merges with ζT(s,r) when :

$$A_s(x) = (1-r+r.x)^{2s}$$
$$B_s(x) = (1-r+r.x)^s-(1+r.x)^s \qquad (17)$$

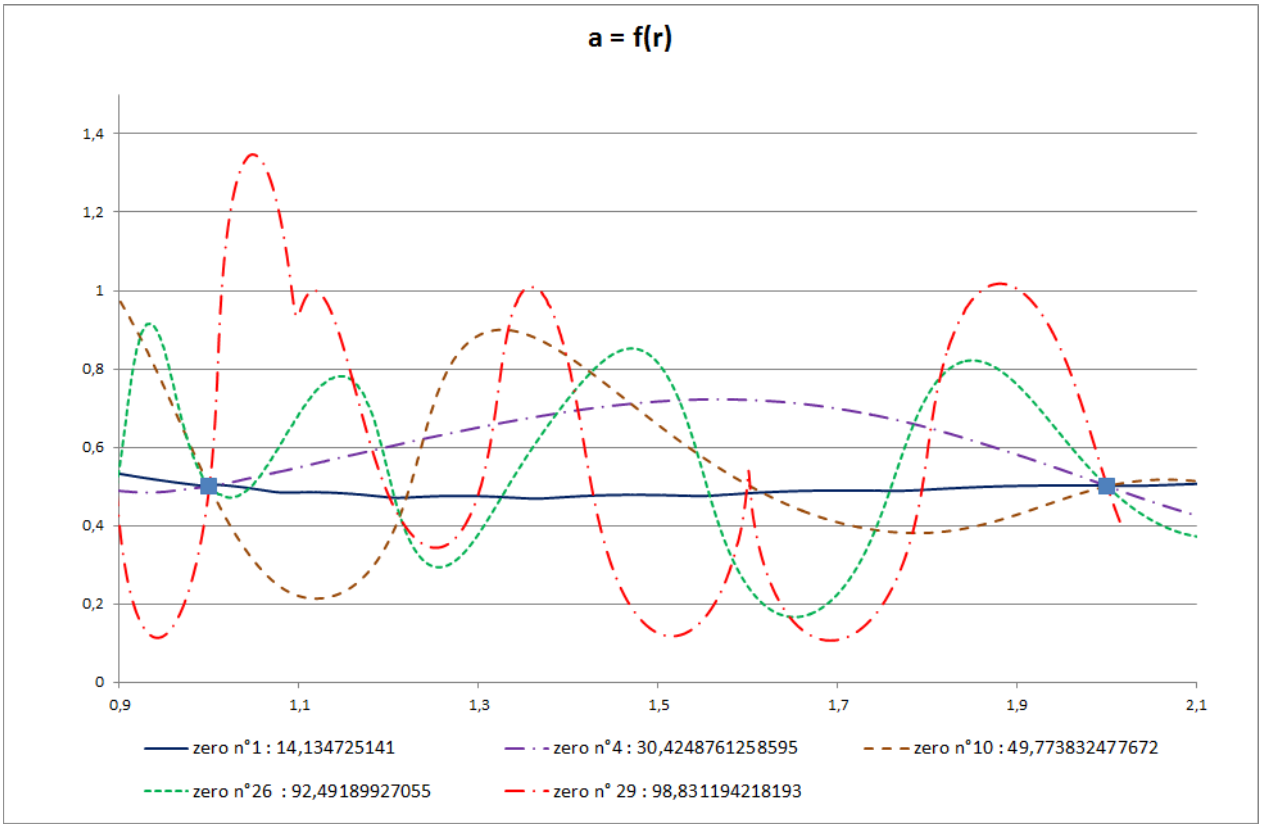
Here, it is necessary to consider two polynomials of recurrence in the approach by a continued fraction of the function ζT(s,r). For r = 1, we get of course the elements of the usual Zeta function continued fractions.

In this configuration, the zeroes of $A_s(x)$ have multiplicity 2s (if s = n integer) and are equal to 1-1/r. The zeroes of $B_s(x)$ have multiplicity 1 (if s = n integer) and are equal to :

$$x = -\frac{1}{r} + \frac{1}{2} - i.\frac{\text{cotg}(k.\pi/n)}{2} \qquad (18)$$

The distance within the ζT(s,r) denominators is now equal to r instead of 1 and imposes a translation equal to -1/r instead of -1 in the real part of recurrence polynomial zeroes. This real part remains constant at constant specified distance.

We can ask ourselves the question of the evolution of the zeroes of the function ζT(s,r) when r varies. For that, we have simply chosen some numerical examples, the general problem being out of reach at this stage. The term "zero" is in fact not totally correct as we do not prove it as such. Let us write s = a+i.b. We follow the minima of ζT(s,r) varying r to begin with and varying a and b after that to get back to the minima. We get points on a tree dimensional curve a = f(r), b = g(r), b = h(a) for each of the chosen zeroes. We have represented the underneath results only for a = f(r), the other curve being of little interest.



The real part of the minima (that is a priori zeroes) is no more constant. We only verify the excepted intersection at r = 1

(and a = 0,5) by Riemann hypothesis. By following the evolution of a few zeroes of this function, we observe changes in sinusoids for which some rule of calculations appears to be difficult to obtain. The Riemann hypothesis is indeed a very peculiar case.
This shows that the (multiple) roots of $A_s(x)$ have an immediate impact on the zeroes. The little variation of the roots in 1-1/r (instead of 0), that is without imaginary part, hides completely the obvious message send at r = 1.

Moreover, the generalized Riemann hypothesis provides a real part of the zeroes of Dirichlet L-series equal to 1/2. We find an example in the graph at r = 2, that is the Beta Dirichlet series $\zeta T(s, r = 2) = 1-1/3^s+1/5^s-1/7^s+1/9^s-\dots$

4 Dirichlet series

Let us have

$$Z(s) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}, \ a_1 = 1, \ a_m \neq 0 \qquad (19)$$

All a_i being non-null constants, we write $c_m = 1/a_m$.
Then, we get :

$$Z(s) = \left.\frac{1}{1}\right|\!-\!\left|\frac{A_s(1)}{B_s(1)}\right|\!-\!\left|\frac{A_s(2)}{B_s(2)}\right|\!-\!\left|\frac{A_s(3)}{B_s(3)}\right|\!-\dots-\!\left|\frac{A_s(x)}{B_s(x)}\right|\!-\dots \qquad (20)$$

where :

$$\begin{aligned} A_s(n) &= c_n^{\,2}.n^{2s} \\ B_s(n) &= c_n.n^s+c_{n+1}.(n+1)^s \end{aligned} \qquad (21)$$

We cannot use this expression for the Mobius M(s) function because its $a_i = 0$ coefficients.

5 Two recurrence polynomials, two lines of zeroes

To address the Riemann hypothesis problem, we now will work in a more fragile framework on a mathematical point of view.

From relation (1), we get immediately :

$$F(s) = F(A_s(1), A_s(2), \dots, A_s(\infty), B_s(0), B_s(1), \dots, B_s(\infty)) \qquad (22)$$

Let us try to solve :

$$F(s) = 0 \qquad (23)$$

The F(s) function zeroes are necessarily linked to the two polynomials of recurrence $A_s(n)$ and $B_s(n)$. These two polynomials feed then two lines of zeroes. The first line depends on the $A_s(n)$ polynomials in the presence of $B_s(n)$ and we will write this line $L_1(A_s\backslash B_s)$, the second line depends on the $B_s(n)$ polynomial in the presence of $A_s(n)$ and we will write it $L_2(B_s\backslash A_s)$. Thus :

$$F(L_1(A_s\backslash B_s), L_2(B_s\backslash A_s)) = 0 \qquad (24)$$

Hence, the zeroes of F(s) are the images of the recurrence polynomials by two transformations :

$$\begin{aligned} A_s\backslash B_s &\rightarrow L_1(A_s\backslash B_s) \\ B_s\backslash A_s &\rightarrow L_2(B_s\backslash A_s) \end{aligned} \qquad (25)$$

Thus, to such geometrical figure may correspond such other geometrical figure (likewise mappings). In the same way, if $A_s(n)$ and $B_s(n)$ are rather simple, simplicity may also appear in the geometrical transformations.

Here, of course, we cannot explicit the source sets $(A_s\backslash B_s)$ and $(B_s\backslash A_s)$ but only the characteristics of $A_s(m)$ and $B_s(m)$, thus again two sets. These characteristics may be the zeroes of $A_s(m)$ and $B_s(m)$. However, as neither $A_s(m)$ nor $B_s(m)$ are numerators or denominators, we may consider in the same time zeroes and poles of $A_s(m)$ and $B_s(m)$. Concerning the images, to foster the symmetries of the observed correlations, we may also include both zeroes and poles of F(s). Accessorily, in the same symmetry preoccupation, it can be interesting to take into account or not the infinite and other (un-reached) limits.

The images by L_1 and L_2 are (trivially) among the following choice :

- an infinite set
- in finite set (possibly)
- a unique point, finite or infinite
- a void set

The concept of trivial and non-trivial zeroes of F(s) makes no sense in the elementary cases even if two sets remain (however possibly void).

Note
The previous argument would remain identical, in every point, if $A_s(n)$ and $B_s(n)$ are no more polynomials, but more general functions.

5.1 Application to Riemann Zeta function

The Zeta function has actually a generalized continued fraction with two recurrence polynomials :

$$A_s(n) = n^{2s}$$
$$B_s(n) = n^s+\varepsilon.(n+1)^s$$

Here ε equals 1 when you consider the Dirichlet series and -1 for the analytical extension to Riemann Zeta function. This last case is, of course, your ultimate aim.
We get for the two transformations L_1 and L_2 :

$$A_s\backslash B_s \rightarrow L_1(A_s\backslash B_s) = \{\text{trivial zeroes}\}+\{S_1\}$$
$$B_s\backslash A_s \rightarrow L_2(B_s\backslash A_s) = \{1/2+i.b_{k=1 \text{ à } \infty}\}+\{S_2\}$$

The trivial zeroes {-∞, …, -6, -4, -2} are not this concerned here. As we do not use the functional equation $\zeta(s) = 2^s\pi^{s-1} \sin(\pi.s/2)\Gamma(1-s)\zeta(1-s)$, these zeroes reduce in fact to the void set. Nevertheless, if we consider the Eta function η(s), the trivial zeros are i.n.2π/ln(2) (see our other article).

The polynomials $A_s(n)$ and $B_s(n)$ can be seen alternatively as « numerator » or as « denominator » in the continued fraction, hence the interest to seek both zeroes and poles.

Searching zeroes of $A_s(n)$ and $B_s(n)$, with s a function of n (instead of n a function of s as earlier), we get with $\eta = (1-(-1)^{(\varepsilon+1)/2})/2$ (that is η = 1 if ε = 1 and η = 0 if ε = -1) :

$n^{2s} = 0$	=>	$s \rightarrow -\infty$	Point rejected to infinite
$n^s+\varepsilon.(n+1)^s = 0$	=>	$s = 0+i.\pi.(\eta+2k)/\ln((n+1)/n)$ $s \rightarrow -\infty$	Points with real part equal to 0 and 1 point rejected to infinite

Concerning the poles, we have :

$n^{2s} \rightarrow \infty$	=>	$s \rightarrow +\infty$	Point rejected to infinite
$n^s+\varepsilon.(n+1)^s \rightarrow \pm\infty$	=>	$s \rightarrow +\infty$	Point rejected to infinite

Gathering zeroes and poles, it follows :

Poles and zeroes of $n^s+\varepsilon.(n+1)^s$ and n^{2s}	=>	$s \rightarrow \pm\infty$ $s = 0+i.\pi.(\eta+2k)/\ln((n+1)/n)$	P_1 : Points at infinite P_2 : Points with real part equal to 0

It is easy to imagine a correspondence between the $\{P_1, P_2\}$ and $\{L_1, L_2\}$ sets when the $\{S_1\}$ and $\{S_2\}$ sets are empty. This is more difficult otherwise as we loose harmony.

5.2 Application to other functions

5.2.1 Example of the Logarithm function

Let us observe the logarithm function behaviour (under a new form derived of the existing mathematic literature)

$$\mathrm{Ln}(s) = \frac{2}{|\,((s+1)/(s-1)).1}\Big| - \frac{1^2}{|\,((s+1)/(s-1)).3}\Big| - \frac{2^2}{|\,((s+1)/(s-1)).5}\Big| - \ldots - \frac{m^2}{|\,((s+1)/(s-1)).(2m+1)}\Big| - \ldots \quad (26)$$

We have thus

$$A_s(m) = m^2$$
$$B_s(m) = (2m+1).((s+1)/(s-1)) \quad (27)$$

Hence the table :

Roots of initial function	Recurrence polynomials	Roots for s	Poles for s
s = 1, s = -1 (Ln(-1) = k.i.π = 0 if k = 0)	m^2 (=m^2+0.s by ex.)	Void set if m ≠ 0 otherwise any s	Void set if m ≠ ±∞ otherwise any s
	(2m+1).((s+1)/(s-1))	s = -1	s = 1

5.2.2 Example of the Tangent function

For the Tangent function

$$\mathrm{Tan}(s)/s = \frac{1\,|}{|\,1} - \frac{s^2\,|}{|\,3} - \frac{s^2\,|}{|\,5} - \ldots - \frac{s^2\,|}{|\,2m+1} - \ldots \qquad (28)$$

We have thus :

$$\begin{aligned} A_s(m) &= s^2 \\ B_s(m) &= 2m+1 \end{aligned} \qquad (29)$$

Hence the table :

Roots of initial function	Recurrence polynomials	Roots for s	Poles for s
Void set if $k = 0$ in $s = k.\pi$ (otherwise $s = k.\pi$, $k\neq 0$)	s^2	0	$s \to \pm\infty$
	$2m+1$	Void set if $m \neq -1/2$ otherwise any s	Void set (if $m \neq \pm\infty$ otherwise any s)

5.2.3 Recapitulative table

The reader will find more in the appendix where we note the patterns between the zeroes of the elementary functions and the recurrence polynomials zeroes and poles. To get there, we use elementary functions, with well known generalized continued fractions [5], which we transform to obtain the form of the second member of (1). This means usually a basic division or multiplication of the variable (hence tan(s) gives tan(s)) but sometimes is a bit longer (hence $\sec^{-1}(s)$ is now $(\sec^{-1}(s)- \pi/2) /(1-s^{-2})^{1/2}$).

We can still enhance the symmetries of the correlations on one hand by considering also the poles of the studied functions.and on the other hand by omitting infinites which are not effective solutions. We then get the following table :

Zeroes and poles of the function	Zeroes and poles of continued fraction	Examples
Void	0	$\mathrm{Ln}(1+s)/s$ $\mathrm{Ln}((1+s)/(1-s))/s$ $\mathrm{Tan}^{-1}(s)/s$ $\mathrm{Tanh}^{-1}(s) /s$ $s/\tan^{-1}(s)$ $s/\tanh^{-1}(s)$ $s.\cot^{-1}(s)$ $s.\coth^{-1}(s)$ $1/(s.\coth^{-1}(s))$ $s.\csc^{-1}(s)/(1-s^2)^{1/2}$ $\mathrm{Sin}^{-1}(s)/(s.(1-s^2)^{1/2})$ $\mathrm{Sinh}^{-1}(s)/(s.(1+s^2)^{1/2})$ $(\mathrm{Cos}^{-1}(s)- \pi/2) /(s.(1-s^2)^{1/2})$ $s.(\sec^{-1}(s)- \pi/2) /(1-s^{-2})^{1/2}$ $(1+s)^{-a}$ $(a > 0)$
Void	Periodic, real ($k.\pi/2$, $k \neq 0$)	$(\mathrm{Cot}(s)-1/s)/s$
Void	Périodic, imaginary ($i.k.\pi/2$, $k \neq 0$)	$(\mathrm{Coth}(s)-1/s)/s$
Periodic, real ($k.\pi$, $k \neq 0$)	0	$\mathrm{Tan}(s)/s$
Périodic, imaginary ($i.k.\pi$, $k \neq 0$)	0	$(e^s-1)/s$ $\mathrm{Tanh}(s)/s$
$\to 0$	0	$s.\mathrm{csch}^{-1}(s) /(1+s^2)^{1/2}$
± 1	± 1	$\mathrm{Ln}(s)$
-1	0	$(1+s)^{-a}$ $(a < 0)$
$1/2+i.t$ $\{-\infty, \ldots -2n, \ldots, -6, -4, -2\}$ or {void} {1}+{other poles ?}	$-1/2+i.t'$ / No analog	$\zeta(s)$

The two last lines are here to recall that symmetries (around 0) are sometimes settled away.
We may deplore here the lack of generality drawn from these elementary functions to address the present problem (for $\zeta(s)$). In general, it remains not even two distinct sets on the source side and/or on the image side. However, the major point to retain here is the sobriety of the solutions on the two sides of the table.

5.2.4 Final note

Coincidences of the same kind can be dressed from series, instead of continued fractions, when considering for the first line of zeroes one current term of the sum and for the second line of zeroes two consecutive terms for which we seek both poles and zeroes.
Thus for the Riemann Zeta function $\zeta(s) = (1/(1-2^{1-s})).\sum (-1)^{m-1}/m^s$, excepting trivial zeroes and poles like 0 and $\pm\infty$ in what follows, the sum of two consecutive terms that we equal to 0 will give us $m^s-(m+1)^s = 0$, bringing us back exactly to the afore study of the zeroes of $B_s(m)$ with the same results. (And so also for the Dirichlet series).

To stop at two consecutives terms is however, with series, somewhat more artificial as nothing justifies such prohibition except the exceptional nature of 1 and 2 which didn't escape to any civilisation even in their every day language.

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Appendix

(Source : http://functions.wolfram.com/)

Function		Recurrence polynomials			Continued fractions
Function	Roots for z (*)	$B_z(m)$	Roots in m for $B_z(m)$	Roots and poles in z for $A_z(m)$	
$(e^z-1)/z$	∞ $i.2\pi.k$, $k\neq 0$	$B_z(m=0 \bmod 2) = m+1$ $B_z(m=1 \bmod 2) = 2$	-1 /	0 $\pm\infty$	$e^z = 1 + \cfrac{z}{1 - \cfrac{z}{2 + \cfrac{z}{3 - \cfrac{z}{2 + \cfrac{z}{5 - \cfrac{z}{2 + \cfrac{z}{7 - \ldots}}}}}}}$
$\mathrm{Ln}(1+z)/z$	$\pm\infty$ none	$B_z(m=0 \bmod 2) = m+1$ $B_z(m=1 \bmod 2) = 2$	-1 /	0 $\pm\infty$	$\log(1+z) = \cfrac{z}{1 + \cfrac{z}{2 + \cfrac{z}{3 + \cfrac{2z}{2 + \cfrac{2z}{5 + \cfrac{3z}{2 + \cfrac{3z}{7 + \ldots}}}}}}} \;/;\; z \notin (-\infty, -1)$
$\mathrm{Ln}(1+z)/z$	$\pm\infty$ none	$m+1$	-1	0 $\pm\infty$	$\log(1+z) = \cfrac{z}{1 + \cfrac{z}{2 + \cfrac{z}{3 + \cfrac{4z}{4 + \cfrac{4z}{5 + \cfrac{9z}{6 + \cfrac{9z}{7 + \ldots}}}}}}} \;/;\; z \notin (-\infty, -1)$
$\mathrm{Ln}((1+z)/(1-z))/z$	$\pm\infty$ none	$2m+1$	-1/2	0 $\pm\infty$	$\log\left(\frac{1+z}{1-z}\right) = \cfrac{2z}{1 - \cfrac{z^2}{3 - \cfrac{4z^2}{5 - \cfrac{9z^2}{7 - \cfrac{16z^2}{9 - \cfrac{25z^2}{11 - \cfrac{36z^2}{13 - \ldots}}}}}}} \;/;\; z \notin (-\infty, -1) \land z \notin (1, \infty)$
$\mathrm{Tan}(z)/z$	$k.\pi$, $k \neq 0$?	$2m+1$	-1/2	0 $\pm\infty$	$\tan(z) = \cfrac{z}{1 - \cfrac{z^2}{3 - \cfrac{z^2}{5 - \cfrac{z^2}{7 - \cfrac{z^2}{9 - \cfrac{z^2}{11 - \ldots}}}}}} \;/;\; \frac{z}{\pi} - \frac{1}{2} \notin \mathbb{Z}$
$\mathrm{Tanh}(z)/z$	$i.k.\pi$, $k\neq 0$ none	$2m+1$	-1/2	0 $\pm\infty$	$\tanh(z) = \cfrac{z}{1 + \cfrac{z^2}{3 + \cfrac{z^2}{5 + \cfrac{z^2}{7 + \cfrac{z^2}{9 + \cfrac{z^2}{11 + \ldots}}}}}} \;/;\; \frac{iz}{\pi} - \frac{1}{2} \notin \mathbb{Z}$

Function		Recurrence polynomials			Continued fractions
Function	Roots for z (*)	$B_z(m)$	Roots in m for $B_z(m)$	Roots and poles in z for $A_z(m)$	
$Tan^{-1}(z)/z$	$\pm\infty$ none	2m+1	-1/2	0 $\pm\infty$	$\tan^{-1}(z) = \cfrac{z}{1+\cfrac{z^2}{3+\cfrac{4z^2}{5+\cfrac{9z^2}{7+\cfrac{16z^2}{9+\cfrac{25z^2}{11+\cfrac{36z^2}{13+\ldots}}}}}}} \;/;\; i\,z \notin (-\infty,-1) \wedge i\,z \notin (1,\infty)$
$Tanh^{-1}(z)/z$	$\pm\infty$ none	2m+1	-1/2	0 $\pm\infty$	$\tanh^{-1}(z) = \cfrac{z}{1-\cfrac{z^2}{3-\cfrac{4z^2}{5-\cfrac{9z^2}{7-\cfrac{16z^2}{9-\cfrac{25z^2}{11-\cfrac{36z^2}{13-\ldots}}}}}}} \;/;\; z \notin (-\infty,-1) \wedge z \notin (1,\infty)$
$z/tan^{-1}(z)$ = $1/((tan^{-1}(z)-z)/z+1)$	none none	2m+1	-1/2	0 $\pm\infty$	$\tan^{-1}(z) = z - \cfrac{z^3}{3+\cfrac{9z^2}{5+\cfrac{4z^2}{7+\cfrac{25z^2}{9+\cfrac{16z^2}{11+\cfrac{49z^2}{13+\cfrac{36z^2}{15+\ldots}}}}}}} \;/;\; i\,z \notin (-\infty,-1) \wedge i\,z \notin (1,\infty)$
$z/tanh^{-1}(z)$ = $1/((tanh^{-1}(z)-z)/z+1)$	none none	2m+1	-1/2	0 $\pm\infty$	$\tanh^{-1}(z) = z + \cfrac{z^3}{3-\cfrac{9z^2}{5-\cfrac{4z^2}{7-\cfrac{25z^2}{9-\cfrac{16z^2}{11-\cfrac{49z^2}{13-\cfrac{36z^2}{15-\ldots}}}}}}} \;/;\; z \notin (-\infty,-1) \wedge z \notin (1,\infty)$
$Tan^{-1}(z)/z$	$\pm\infty$ none	$B_m(z)$ = (2m+1). si(m=0 mod 2, (1+z²), 1)	-1/2	0 $\pm\infty$	$\tan^{-1}(z) = \cfrac{z}{1+z^2-\cfrac{2z^2}{3-\cfrac{2z^2}{5(1+z^2)-\cfrac{12z^2}{7-\cfrac{12z^2}{9(1+z^2)-\cfrac{30z^2}{11-\cfrac{30z^2}{13(1+z^2)-\ldots}}}}}}} \;/;\; i\,z \notin (-\infty,-1) \wedge i\,z \notin (1,\infty)$
$Tanh^{-1}(z)/z$	$\pm\infty$ none	$B_m(z)$ = (2m+1).si(m = 0 mod 2,(1-z²),1)	-1/2	0 $\pm\infty$	$\tanh^{-1}(z) = \cfrac{z}{1-z^2+\cfrac{2z^2}{3+\cfrac{2z^2}{5(1-z^2)+\cfrac{12z^2}{7+\cfrac{12z^2}{9(1-z^2)+\cfrac{30z^2}{11+\cfrac{30z^2}{13(1-z^2)+\ldots}}}}}}} \;/;\; z \notin (-\infty,-1) \wedge z \notin (1,\infty)$

Function		Recurrence polynomials			Continued fractions
Function	Roots for z (*)	$B_z(m)$	Roots in m for $B_z(m)$	Roots and poles in z for $A_z(m)$	
$z.\cot^{-1}(z)$	none none	2m+1	-1/2	±∞ 0	$\cot^{-1}(z) = \cfrac{z^{-1}}{1+\cfrac{z^{-2}}{3+\cfrac{4z^{-2}}{5+\cfrac{9z^{-2}}{7+\cfrac{16z^{-2}}{9+\cfrac{25z^{-2}}{11+\cfrac{36z^{-2}}{13+\ldots}}}}}}} \;/;\; i\,z \notin (-1,1)$
$z.\coth^{-1}(z)$	none none	2m+1	-1/2	±∞ 0	$\coth^{-1}(z) = \cfrac{z^{-1}}{1-\cfrac{z^{-2}}{3-\cfrac{4z^{-2}}{5-\cfrac{9z^{-2}}{7-\cfrac{16z^{-2}}{9-\cfrac{25z^{-2}}{11-\cfrac{36z^{-2}}{13-\ldots}}}}}}} \;/;\; z \notin (-1,1)$
(Cot(z)-1/z)/z	±∞ none	2m+1	-1/2	m.π/2 ±∞	$\cot(z) = \frac{1}{z} - \cfrac{4\pi^{-2}z}{1+\cfrac{1-4\pi^{-2}z^2}{3+\cfrac{4(4-4\pi^{-2}z^2)}{5+\cfrac{9(9-4\pi^{-2}z^2)}{7+\cfrac{16(16-4\pi^{-2}z^2)}{9+\cfrac{25(25-4\pi^{-2}z^2)}{11+\cfrac{36(36-4\pi^{-2}z^2)}{13+\ldots}}}}}}}$
(Coth(z)-1/z)/z	±∞ none	2m+1	-1/2	i.m.π/2 ±∞	$\coth(z) = \frac{1}{z} + \cfrac{4\pi^{-2}z}{1+\cfrac{1+4\pi^{-2}z^2}{3+\cfrac{4(4+4\pi^{-2}z^2)}{5+\cfrac{9(9+4\pi^{-2}z^2)}{7+\cfrac{16(16+4\pi^{-2}z^2)}{9+\cfrac{25(25+4\pi^{-2}z^2)}{11+\cfrac{36(36+4\pi^{-2}z^2)}{13+\ldots}}}}}}}$
Tan(z)/z = 1/(cot(z)-1/z)*z+1)	k.π, k ≠ 0 ?	2m+1	-1/2	0 ±∞	$\cot(z) = \frac{1}{z} - \cfrac{z}{3-\cfrac{z^2}{5-\cfrac{z^2}{7-\cfrac{z^2}{9-\cfrac{z^2}{11-\ldots}}}}}$
Tanh(z)/z = 1/(coth(z)-1/z).z+1)	i.k.π, k≠0 none	2m+1	-1/2	0 ±∞	$\coth(z) = \frac{1}{z} + \cfrac{z}{3+\cfrac{z^2}{5+\cfrac{z^2}{7+\cfrac{z^2}{9+\cfrac{z^2}{11+\ldots}}}}}$

Function		Recurrence polynomials			Continued fractions
Function	Roots for z (*)	$B_z(m)$	Roots in m for $B_z(m)$	Roots and poles in z for $A_z(m)$	
$1/(z.\coth^{-1}(z))$ = $1/((\coth^{-1}(z)-1/z).z+1)$	none none	2m+1	-1/2	±∞ 0	$\coth^{-1}(z) = \frac{1}{z} + \cfrac{z^{-3}}{3 - \cfrac{9z^{-2}}{5 - \cfrac{4z^{-2}}{7 - \cfrac{25z^{-2}}{9 - \cfrac{16z^{-2}}{11 - \cfrac{49z^{-2}}{13 - \cfrac{36z^{-2}}{15 - \ldots}}}}}}} \;/;\; z \notin (-1,1)$
Tan(z)/z = $1/((z/2).\cot(z)-1/z)-1/2)$	k.π, k ≠ 0 ?	-(2m+1)/2	-1/2	0 ±∞	$\cot(z) = \frac{1}{z} + \cfrac{z/2}{-\frac{3}{2} - \cfrac{z^2/4}{-\frac{5}{2} - \cfrac{z^2/4}{-\frac{7}{2} - \cfrac{z^2/4}{-\frac{9}{2} - \cfrac{z^2/4}{-\frac{11}{2} - \cfrac{z^2/4}{-\frac{13}{2} - \cfrac{z^2/4}{-\frac{15}{2} - \cfrac{z^2/4}{-\frac{17}{2} - \cfrac{z^2/4}{\ldots}}}}}}}}}$
Tanh(z)/z = $1/((z/2).\coth(z)-1/z)-1/2)$	i.2π.k, k≠0 none	2m+1	-1/2	0 ±∞	$\coth(z) = \frac{1}{z} - \cfrac{z/2}{-\frac{3}{2} + \cfrac{z^2/4}{-\frac{5}{2} + \cfrac{z^2/4}{-\frac{7}{2} + \cfrac{z^2/4}{-\frac{9}{2} + \cfrac{z^2/4}{-\frac{11}{2} + \cfrac{z^2/4}{-\frac{13}{2} + \cfrac{z^2/4}{-\frac{15}{2} + \cfrac{z^2/4}{-\frac{17}{2} + \cfrac{z^2/4}{\ldots}}}}}}}}}$
$z.\csc^{-1}(z)/(1-z^2)^{1/2}$	none none	2m+1	-1/2	±∞ 0	$\csc^{-1}(z) = \cfrac{z^{-1}\sqrt{1-z^{-2}}}{1 - \cfrac{1\times 2z^{-2}}{3 - \cfrac{1\times 2z^{-2}}{5 - \cfrac{3\times 4z^{-2}}{7 - \cfrac{3\times 4z^{-2}}{9 - \cfrac{5\times 6z^{-2}}{11 - \ldots}}}}}} \;/;\; z \notin (-1,1)$
$z.\operatorname{csch}^{-1}(z)/(1+z^2)^{1/2}$	$\rightarrow 0^{\pm}$ none	2m+1	-1/2	±∞ 0	$\operatorname{csch}^{-1}(z) = \cfrac{z^{-1}\sqrt{1+z^{-2}}}{1 + \cfrac{1\times 2z^{-2}}{3 + \cfrac{1\times 2z^{-2}}{5 + \cfrac{3\times 4z^{-2}}{7 + \cfrac{3\times 4z^{-2}}{9 + \cfrac{5\times 6z^{-2}}{11 + \ldots}}}}}} \;/;\; i\,z \notin (-1,1)$
$\sin^{-1}(z)/(z.(1-z^2)^{1/2})$	none none	2m+1	-1/2	0 ±∞	$\sin^{-1}(z) = \cfrac{z\sqrt{1-z^2}}{1 - \cfrac{1\times 2z^2}{3 - \cfrac{1\times 2z^2}{5 - \cfrac{3\times 4z^2}{7 - \cfrac{3\times 4z^2}{9 - \cfrac{5\times 6z^2}{11 - \ldots}}}}}} \;/;\; z \notin (-\infty,-1) \wedge z \notin (1,\infty)$

Function		Recurrence polynomials			Continued fractions
Function	Roots for z (*)	$B_z(m)$	Roots in m for $B_z(m)$	Roots and poles in z for $A_z(m)$	
$\mathrm{Sinh}^{-1}(z)$ $/(z.(1+z^2)^{1/2})$	$\pm\infty$ none	2m+1	-1/2	0 $\pm\infty$	$\sinh^{-1}(z) = \cfrac{z\sqrt{1+z^2}}{1+\cfrac{1\times 2z^2}{3+\cfrac{1\times 2z^2}{5+\cfrac{3\times 4z^2}{7+\cfrac{3\times 4z^2}{9+\cfrac{5\times 6z^2}{11+\dots}}}}}}$ $/;\ iz \notin (-\infty,-1) \bigwedge iz \notin (1,\infty)$
$(\mathrm{Cos}^{-1}(z)-\pi/2)$ $/(z.(1-z^2)^{1/2})$	none none	2m+1	-1/2	0 $\pm\infty$	$\cos^{-1}(z) = \frac{\pi}{2} - \cfrac{z\sqrt{1-z^2}}{1-\cfrac{1\times 2z^2}{3-\cfrac{1\times 2z^2}{5-\cfrac{3\times 4z^2}{7-\cfrac{3\times 4z^2}{9-\cfrac{5\times 6z^2}{11-\dots}}}}}}$ $/;\ z \notin (-\infty,-1) \bigwedge z \notin (1,\infty)$
$z.(\sec^{-1}(z)-\pi/2)$ $/(1-z^{-2})^{1/2}$	none none	2m+1	-1/2	$\pm\infty$ 0	$\sec^{-1}(z) = \frac{\pi}{2} - \cfrac{z^{-1}\sqrt{1-z^{-2}}}{1-\cfrac{1\times 2z^{-2}}{3-\cfrac{1\times 2z^{-2}}{5-\cfrac{3\times 4z^{-2}}{7-\cfrac{3\times 4z^{-2}}{9-\cfrac{5\times 6z^{-2}}{11-\dots}}}}}}$ $/;\ z \notin (-1,1)$
$(1+z)^{-a}$	∞ none (a>0) -1 none (a<0)	B_z(m=0 mod2) = m+1 B_z(m=1 mod2) = 2m+1	-1 (-1/2)	0 $\pm\infty$	$(1+z)^a = 1 + \cfrac{az}{1+\cfrac{(1-a)z}{3+\cfrac{(1+a)z}{5+\cfrac{(2-a)z}{5+\cfrac{(2+a)z}{9+\cfrac{(3-a)z}{7+\dots}}}}}}$ $/;\ z \notin (-\infty,-1)$

* Trivial and non-trivial roots (by analogy) not necessarily in this order

Sec(z) = 1/cos(z) Csc(z) = 1/sin(z). Sech(z) = 1/cosh(z). Csch(z) = 1/sinh(z)
If(a = 0, a, b) is equivalent to if a = 0 then a otherwise b.