THE SHORTEST PROOF OF RIEMANN HYPOTHESIS

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Abstract. The purpose of this article is to prove the Riemann Hypothesis using the analytic property of the Zeta function and the symmetry of the zeros to the critical line.

Résumé. *(La plus courte preuve de l'hypothèse de Riemann).*
Le but de cet article est de donner la preuve de l'hypothèse de Riemann en utilisant la nature analytique de la fonction Zêta de Riemann et la symétrie des zéros par rapport à la ligne critique.

Contents

1. Introductional theorems

The Riemann Zeta function is defined over the complex plane $Re(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

This function has an analytic continuation over the whole complex plane except the unique complex point $s = 1 + 0.i$. The Riemann's hypothesis, formulated in 1859 [1], is that the non-trivial zeros of the function are such that $Re(s) = \frac{1}{2}$, the zeros quoted as trivial being $s = -2n$, $n \in N^*$.

A well-established result is that all the non-trivial zeros are located within the critical band $0 \leqslant Re(s) \leqslant 1$. In search of zeros, one can reduce the review to the domain $0 \leqslant \alpha \leqslant 1/2$ thanks to the following fact:

Theorem 1. *Within the critical band, the non-trivial $\zeta-$function zeros are symmetrical to the axis $s = 1/2$.*

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Proof. Using the functional equation (see [2])

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s)\zeta(1-s),$$

Let us write then $\xi(s) = (1/2)\pi^{-s/2} s(s-1)\Gamma(s/2)\zeta(s)$. Referring to [6], we get immediately $\xi(s) = \xi(1-s)$. □

Therefore looking for exception to the Riemann rule is equivalent to examine the $\{0 \leqslant s < 1/2\}$ cases.

As a meromorphic function [6], the Zeta function is infinitely derivable except at its pole. The previous theorem then extends to its derivatives:

Theorem 2. *Within the critical band, the non-trivial systematic multiple zeros of the $\zeta-$function are symmetrical to the axis $s = 1/2$.*

Proof. We refer to [3] which provides the functional equation of the k^{th} derivative of $\zeta(s)$

$$(-1)^k \zeta^{(k)}(1-s) = 2(2\pi)^{-s} \sum_{j=0}^{k} \sum_{m=0}^{k} (a_{jkm} \cos \frac{\pi s}{2} + b_{jkm} \sin \frac{\pi s}{2}) \Gamma^{(j)}(s) \zeta^{(m)}(s).$$

Thus $\zeta^{(k)}(1-s) = 0$ if $\zeta^{(m)}(s) = 0$ for each $m = 0$ to k. Hence the symmetry with respect to the axis $s = 1/2$ at step k for a systematic multiple zero up to k. □

2. Shortest proof

Lemma 1. *The non-trivial zeros of the Zeta function are all located on the critical line except eventually in the case of some double zero s, that is if $\zeta(s) = \zeta'(s) = 0$ (and $\sigma \neq 1/2$).*

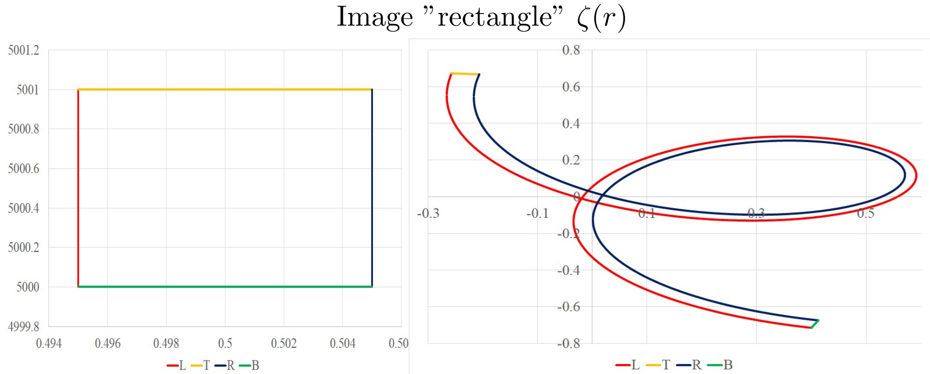
Proof. One of the property of an analytic function is the conservation of angles [4] [7] wherever the derivative doesn't cancel.
So let us consider a rectangle r in the complex plane not encompassing the pole (1,0) of the function, this later case being considered in the next section 3. Applying the function to the rectangle r, the resulting figure $\zeta(r)$ will be a deformed "rectangle". If the rectangle is small, the resulting conformal map [7] is a quasi-rectangle. Local deformation results from non-null scaling factors. The trajectories of the opposite sides of the initial rectangle will give local "parallel" trajectories of the images in the complex plane. Corresponding opposite points of same abscissa, or of same ordinate, don't meet in the image because of the non-null scaling.
A typical example of rectangle deformation is given in figure 1. Of course, as the figure shows, if the rectangle gets long enough, part of the image set can overlap. To get a bijection, an artefact would be to add the depth's dimension (for example attribute to this extra coordinate the value of the length along the red line starting at the bottom left vertex), allowing to unfold the image like a Riemann surface. Going back to the complex plane (without additional dimension), the non-null scaling ensures that the red

line stays always on the same side of the blue line (as does the yellow line in regard of the green one).
Now, let us consider a rectangle r centred on the critical axis. Let us choose the rectangle so that the abscissa of a non-trivial zero identified on the lower half of the critical band is on the left side of the said rectangle. By the functional equation, we know the existence of another zero on the right side of the rectangle exactly at the same height, giving the same image $0 + i.0$, hence a contradiction with the local bijection.
Of course, to be exhaustive, as said in the first phrase of the argumentation, we have to be sure that no point within the initial rectangle corresponds to a zero of the derivative of Zeta causing some eventual havoc to our argument. If so, one will reduce simply the vertical size of the rectangle r. The set of real numbers $\Re$ being dense, this reduction can be as small as needed, the only way to an exception being that this zero of the Zeta function is at the same ordinate as the zero of its derivative. In this peculiar case, one will choose a slightly rounded rectangular shape for the Zeta function's domain, or any bounded domain not including the zero of the derivative, a choice which allows exists because the set of complex numbers is dense. The conformal map's argument can then be applied without dispute except if the zeros to $\zeta-$function and its derivative are the same, the previously mentioned double zero's case. $\square$

In figure 1, as the chosen initial rectangle is encompassing two solutions s to $\zeta(s) = 0$ (the 4521^{th} zero equal approximatively to $1/2 + i.5000.2343169$ and the 4522^{th} zero near the complex value $1/2 + i.5000.8343814$, we get two transits around the axis intersection.

Remarkable cases like the Zeta function's pole (1,0) and the two first of its trivial zeros (-2,0) and (-4,0), the other trivial zeros being similar, are displayed in the next section 3.

Figure 1.
Initial rectangle r delimiting
$\sigma = [0.495, 0.505]$, $t = [5000, 5001]$
Image "rectangle" $\zeta(r)$



Note. Although not crucial to our argument, we give some information in the next section 3 figures 8 *a* and *b* about the pattern in the vicinity of a first derivative's cancelling. In a sufficient close-up, there is no difference to the standard behaviour, a rectangle is still transformed in an almost rectangular surface. But bereft of a proof to generalize this observation, we resolve to the following further analysis.

Lemma 2. *The non-trivial zeros of the Zeta function are all located on the critical line except if* $\zeta(s) = \zeta'(s) = \zeta''(s) = \ldots\ \zeta^{(n)}(s)\ \ldots = 0$ *for any* $n \in N^*$ *(and* $\sigma \neq 1/2$*).*

Proof. The Zeta function is an analytic complex meromorphic function. So it is indefinitely derivable and each derivative is also analytic. The functional expression of the n^{th} derivative of $\zeta(s)$ is provided in the proof of theorem 2. According to that general functional equation, if $\zeta(s) = 0$ and $\zeta'(s) = 0$ then $\zeta'(1 - s) = 0$. Then we apply the previous local conformal map's argument to $\zeta'(s)$ instead of $\zeta(s)$. A potential exception is thereafter the case $\zeta(s) = \zeta'(s) = \zeta''(s) = 0$. Recalling that the functional equation implies the symmetry with respect to the axis $s = 1/2$ for systematic multiple zeros and that the conformal map's argument applies to any analytic function, the procedure can be repeated as long as the derivative of the current derivative is null. If not, we get a contradiction to the possibility to have a non-trivial zero outside the critical line.

Hence the non-trivial zeros of the Zeta function not located on the critical line are infinite multiple zeros. □

Figures 2 and 3 show the examples of deformation of rectangles by the first and second derivatives of the Zeta function. In order to collect these graphics, we use the quotient difference's and the second symmetric derivative's approximations $(\zeta(s+\Delta\epsilon)-\zeta(s))/\Delta\epsilon$ and $(\zeta(s+\Delta\epsilon)-2\zeta(s)+\zeta(s-\Delta\epsilon))/\Delta\epsilon^2$. The choice of $\Delta\epsilon$ for these developments is arbitrary as long as small enough. It can be purely real or purely imaginary or a mix. Here the data collection was done with $\Delta\epsilon = 0.00001$ except for the red curves where we use $\Delta\epsilon = 0.00001i$ (to show that it has no damaging effect and that the curves still stay "parallel" and meet at their vertices).

In these figures, we observe, here and there, places where the red and blue curves get, for corresponding equal ordinates in the initial rectangle, very near one to each other. However, as we checked on close-ups, there is no meeting (nor therefore crossing) of the two curves.

Note. The reader will find an example of second derivative's cancellation's effect on the first derivative of $\zeta(s)$ in the next section 3 figures 9 *a* and *b*. It shows the same pattern as the first derivative's cancellation's effect on $\zeta(s)$.

Theorem 3. *The non-trivial zeros of the Zeta function are all located on the critical line.*

FIGURE 2.
Initial rectangle r delimiting
$\sigma = [0.495, 0.505]$, $t = [0.2, 21]$
Image "rectangle" $\zeta'(r)$

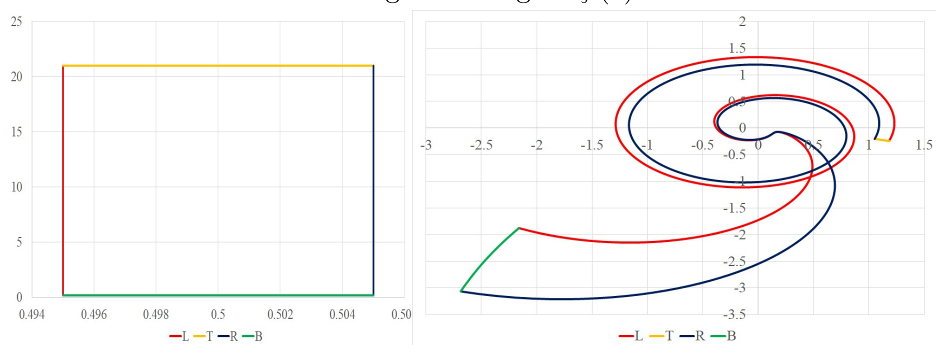
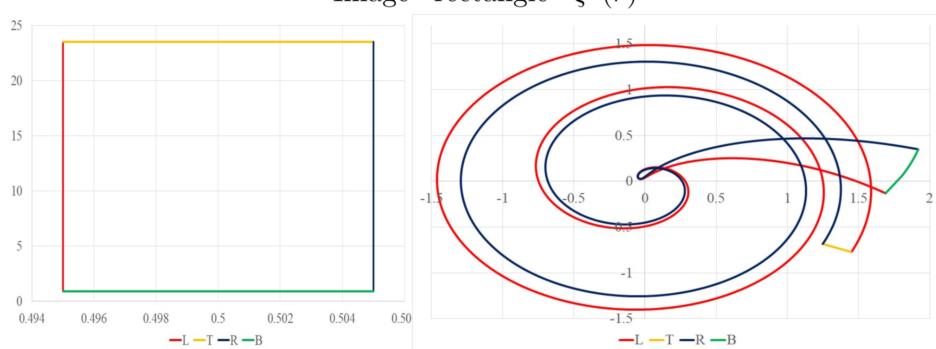


FIGURE 3.
Initial rectangle r delimiting
$\sigma = [0.495, 0.505]$, $t = [0.9, 23.5]$
Image "rectangle" $\zeta''(r)$

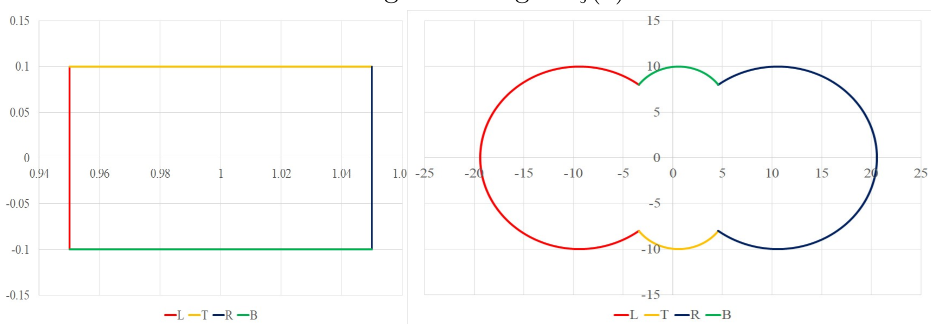


Proof. The Zeta function is a meromorphic [8] function and holomorphic at every point except (1,0). Locally at ρ, its identified n^{th} multiple zero, it can be written as $\zeta(s) = \sum_{m \geq n} a_m (s - \rho)^m$ where a_m are constant complex coefficients. By the previous lemma however, for any finite n, the coefficients a_m remain dependant of s and therefore there is no possible escape to that dependency, thus a contradiction to the property of holomorphy. The Riemann hypothesis is therefore true. $\square$

3. More conformal mappings

The case around the pole (see figure 4) could have been an exception to the general rule as the value of the image diverges in the center. But adding the correspondence $(1, 0) \to \infty$ preserves the bijection. When oriented,

FIGURE 4.
Initial rectangle r delimiting
$\sigma = [0.95, 1.05]$, $t = [-0.1, 0.1]$
Image "rectangle" $\zeta(r)$



the trajectory around the pole of the image is reversed compared to the trajectory of the initial rectangle.

Around the trivial zeros (see figures 5 and 6), there is no special phenomena.

FIGURE 5.
Initial rectangle r delimiting
$\sigma = [-2.05, -1.95]$, $t = [-0.1, 0.1]$
Image "rectangle" $\zeta(r)$

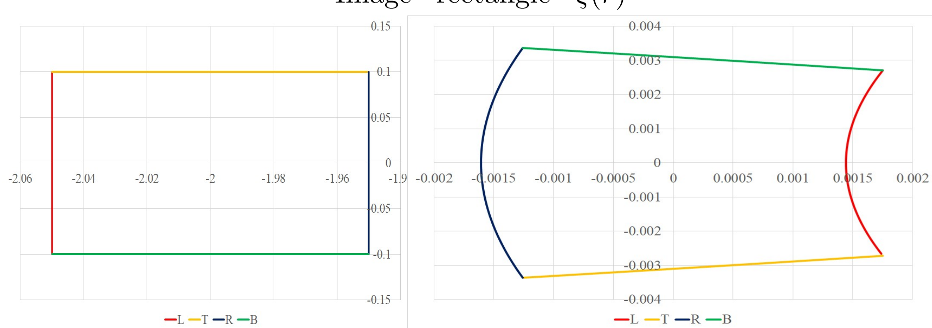
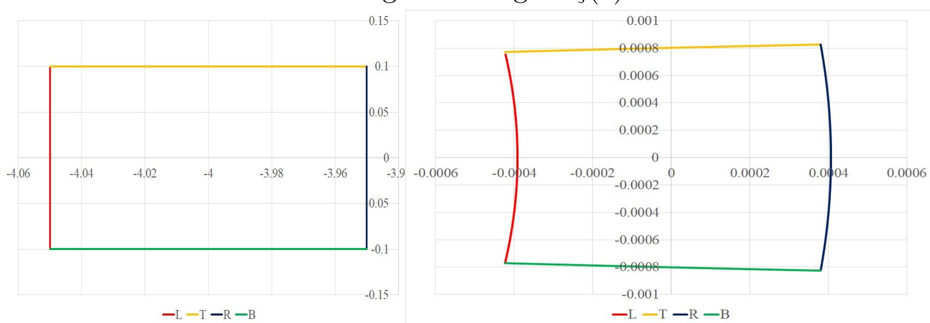
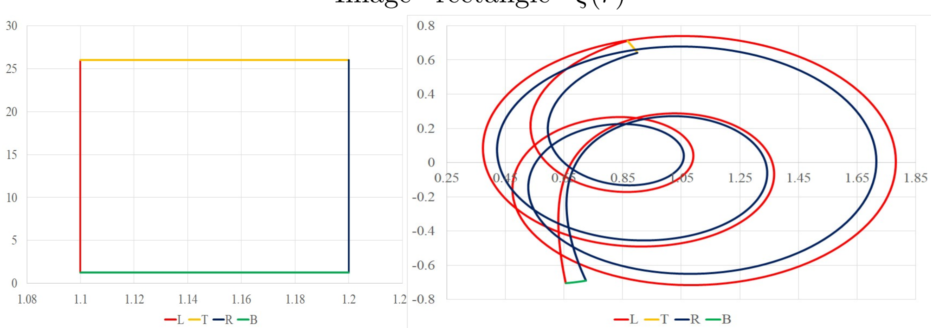


FIGURE 6.
Initial rectangle r delimiting
$\sigma = [-4.05, -3.95]$, $t = [-0.1, 0.1]$
Image "rectangle" $\zeta(r)$



For a typical case (see figure 7), over a broader interval, including zeros, the surfaces overlap and like Riemann surfaces would unfold and spiral around some middle axis in a 3D representation. It seems that except for the

FIGURE 7.
Initial rectangle r delimiting
$\sigma = [1.1, 1.2]$, $t = [1.25, 26]$
Image "rectangle" $\zeta(r)$

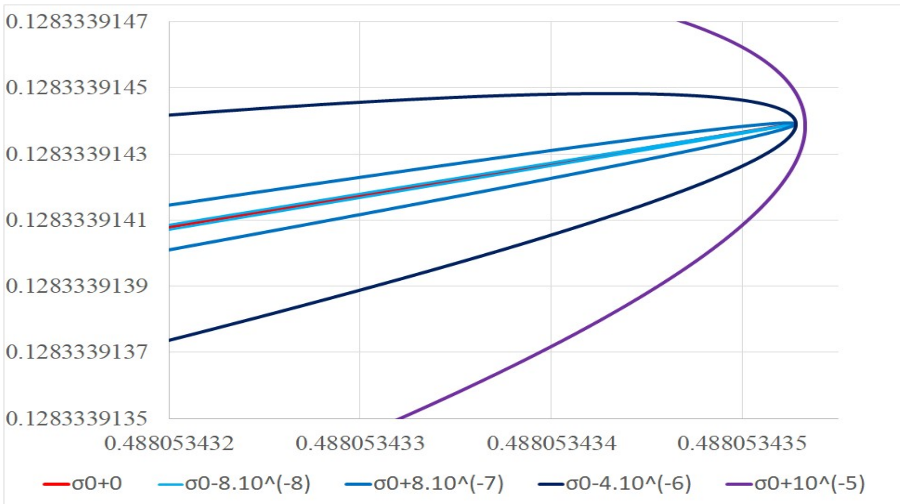


pole's contour, all the other contours are oriented like the initial rectangle.

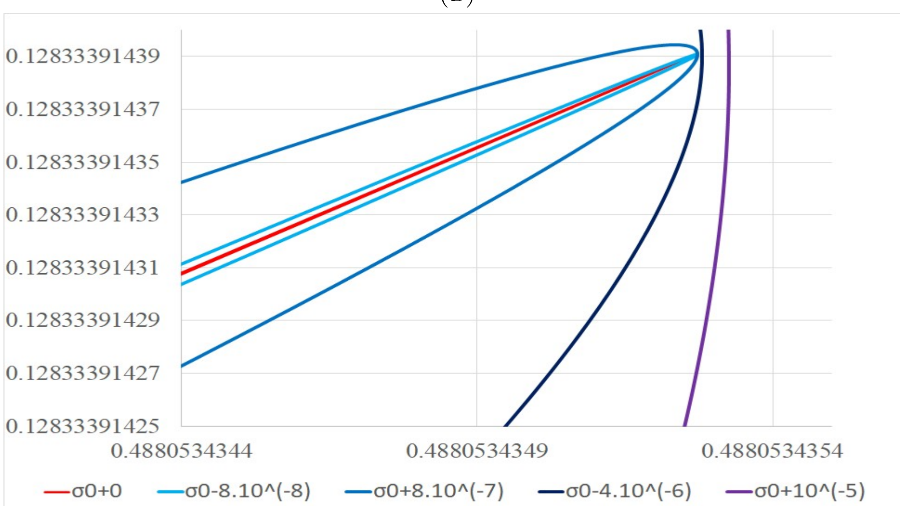
The event where $\zeta'(s_0) = \zeta'(\sigma_0 + i.t_0) = 0$ appears as a limit case of the general feature. The curve $\zeta(\sigma_0 + i.t)$, $t \approx t_0$ is almost a straight line inwards and a straight line outwards (red curve). The surrounding curves are at arbitrary close distances but without ever meeting the limit curve. There is no crossing locally, every curve staying on its respective side of the other. Figure 8 a (and its close-up b) is a typical case. Here the derivative cancels for $s_0 \approx 0.84873532 + i.60.14084577857$. In this example, we alternate the sign of $\Delta\sigma$ while we took increasing absolute values. At very large close-up, a rectangular domain will still give an almost rectangular image.

Figure 8.
$\sigma_0 = 0.84873532$ $t = [60.140715, 60.140967]$
Image $\zeta(\sigma_0 + \Delta\sigma + t)$

(a)



(b)

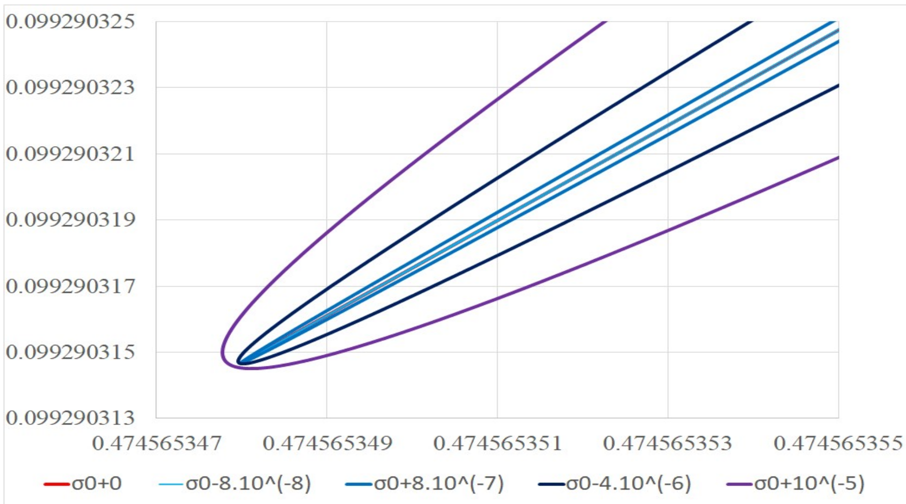


In the case of second derivative's cancelling, it is this time the first derivative of ζ that shows the previous pattern. Figure 9 a (and its close-up b) is a typical case. Here the second derivative cancels for $s_0 \approx 0.9691707 + i.295.16838$.

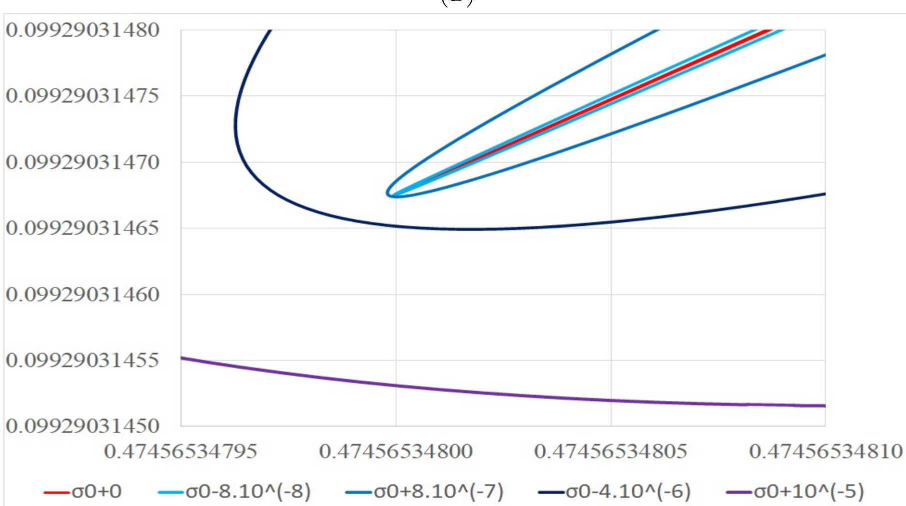
For a curious reader, a relevant remark may be: how to explain the simultaneous property of the $\zeta-$function being infinitely derivable and showing

Figure 9.
$\sigma_0 = 0.9691707$ $t = [295.16825, 295.1685]$
Image $\zeta'(\sigma_0 + \Delta\sigma + t)$

(a)



(b)



in the same time straight lines coming in and going out within the former graphics? What happens in the vicinity of $\sigma_0 + i.t_0$? The answer is a size-diminishing node (crunode), barely visible even at high magnification, therefore avoiding a cusp, and allowing any derivative's value and a smooth continuation through the non-singular point $\sigma_0 + i.t_0$.

Literature and sources

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