Number Theory / *Théorie des nombres*

Convexity in the lower half of the critical band and Riemann hypothesis proof.

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Abstract The study of a peculiar second partial derivative extracted from Dirichlet's Eta function enables us to confirm the Riemann hypothesis.

Convexité dans la demi-bande critique inférieure et preuve de l'hypothèse de Riemann

Résumé L'étude d'une dérivée partielle seconde particulière constituée à partir de la fonction Eta de Dirichlet permet de confirmer l'hypothèse de Riemann.

Statute Preprint.
Date V1 : 08/04/2021
V2 : 17/01/2022

Summary

1.Introduction.

“Mathematics consists in proving the most obvious thing in the least obvious way.” George Pólya.

Indeed, the mathematical literature abounds in clues and evidence in favour of Riemann's hypothesis [1]. One of these is the strict adherence to the hypothesis of billions of zeroes obtained by numerical evaluation. Limiting yourself to the accounting of the first zeros, regardless of their number, however, does not give any general property which enables to deduct for sure a rule for those coming next. Therefore, we are expanding the study by looking at all the points of the critical strip and in particular the bottom half of that band. The results brought to light in this way being general apply to the zeros themselves.

Our investigation is based on one among the analytical extensions of Riemann's series, the Dirichlet Eta function. It establishes the existence of a lower boundary for an indicative function of (positive) convexity deduced from it, resulting in the impossibility of symmetrical Riemann zeros on either side of the critical line.

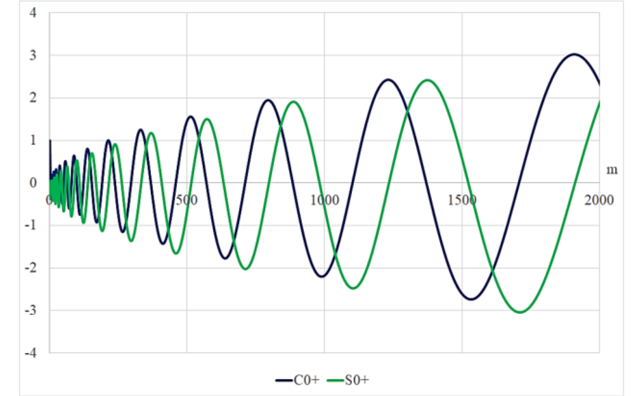
Addressing a wide audience, many graphic illustrations are given here to make the thread of ideas as accessible and clear as possible. Despite all these additions, the article remains relatively short. Can its content then be worth a million of some other one ?

2.Analytic continuations.

Let us have s = a+i.b some complex number.
The parameters a, b and s are taken in the same context throughout this presentation.
The Riemann Zeta function is defined for Re(s) > 1 by the entire function :

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} \qquad (1)$$

The function diverges roughly in the form of an exponentially growing sinusoid for Re(s) < 1 for a given value of the parameter a, the real part of s, and the zeros of this function, called here (non-trivial) zeros of Riemann, correspond to numbers s such as the middle axis of this sinusoid aligns asymptotically with the axis y = 0.



Graphic 1
ζ(s) = C0+i.S0,
s = 1/2+i.14.347251

Note that it is impossible to find precisely the zeros of this function by exploiting only this remark.

Riemann's Zeta function, however, admits, for Re(s) > 0, an analytical continuation based on Dirichlet's entire function Eta η(s).

$$\eta(s) = (1-2^{1-s}).\zeta(s) \qquad (2)$$

This equality shows that the zeros of Dirichlet's Eta function are the union of zeros of $1-2^{1-s}$ and zeros of Riemann's Zeta function. We call the first nominees, the Dirichlet’s zeros.
So, we have the solutions sets :

{Eta function’s zeroes} = {Dirichlet zeroes} U {Riemann zeroes} (3)

The Dirichlet zeros are equal to

$$s = 1+i.2\pi.k/Ln(2) \qquad (4)$$

where k describes the relative integers Z.
These zeros, with constant real value (a = 1), are genuine Siamese brothers of Riemann zeros as we showed in another article (see reference [[6]). The formers are inseparable from the latter and allow us to anticipate the behaviour of Riemann's zeros. They are the trivial image of the veracity of Riemann's hypothesis. Unsurprisingly, we will find them again the upcoming numerical illustrations and elsewhere in this text.

Let us now introduce the functional equation (see reference [2]) :

$$\zeta(s) = 2^s.\pi^{s-1}.\sin(\pi.s/2).\Gamma(1-s).\zeta(1-s) \qquad (5)$$

This further analytical continuation introduces, due to the sinus, additional zeros -2n, called trivial zeros, for any natural (thus positive) integer that are absent in previous functions. This last continuation is essential to our argument because we can state the following theorem :

Theorem 1

The non-trivial Riemann zeros are symmetrical to the axis s = 1/2 in the critical band.

Proof

Let us have $\xi(s) = (1/2).s.(1-s).\pi^{-s/2}.\Gamma(s/2).\zeta(s)$. We get (see reference [3]) immediately $\xi(s) = \xi(1-s)$.
Hence the theorem.

Theorem 2

If the set of all Riemann's zeroes such as $0 < a < 1/2$ is empty, then Riemann's zeros are all on the 1/2 axis.

Proof

This is a trivial consequence of theorem 1.

In 1896, Hadamard and La Vallée-Poussin3] independently proved that no zero could be on the Re(s) = 1 line, and therefore that all non-trivial zeros should be in the interior of the critical band $0 < Re(s) < 1$. For this reason, we have chosen previously to write $0 < a < 1/2$ instead of $0 \leq a < 1/2$, although this second way doesn’t in any way hinder us here, quite the contrary, since it allows us to confirm the work of the authors cited simply by examining case a = 0 (which is actually done in this article).

3.Explicit equations of the Dirichlet Eta function and more expressions.

Let us have Ln(x) the Napierian logarithm of x.
The Eta function writes as :

$$\eta(s) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^s} \qquad (6)$$

We thus get :

$$\eta(s) = \sum_{m=1}^{\infty} (-1)^{m-1}.m^{-a}.\cos(b.Ln(m))+i.\sum_{m=1}^{\infty} (-1)^{m-1}.m^{-a}.\sin(b.Ln(m)) \qquad (7)$$

The search for $\eta(s)$ zeros is therefore tantamount to solving the two equations :

$$\sum_{m=1}^{\infty} (-1)^{m-1}.m^{-a}.\cos(b.Ln(m)) = 0 \qquad (8)$$

and

$$\sum_{m=1}^{\infty} (-1)^{m-1}.m^{-a}.\sin(b.Ln(m)) = 0 \qquad (9)$$

Let us have

$$C0(a,b) = \sum_{m=1}^{\infty} (-1)^{m-1}.m^{-a}.\cos(b.Ln(m)) \qquad (10)$$

and

$$S0(a,b) = \sum_{m=1}^{\infty} (-1)^{m-1}.m^{-a}.\sin(b.Ln(m)) \qquad (11)$$

Then the cancellation of η(s) is equivalent to the following cancellation :

$$(C0(a,b))^2+(S0(a,b))^2 = 0 \qquad (12)$$

Let us have

$$D0(a,b) = (1/2).(C0(a,b))^2+(S0(a,b))^2 \qquad (13)$$

Theorem 3

If the partial second derivative of D0(a,b), versus parameter a, is strictly positive for $0 \leq a \leq 1/2$, then Riemann's hypothesis is true.

Proof

Let us have some given b. Let us place ourselves at $a_0 = 1/2$. By our hypothesis, the second partial derivative versus a is strictly positive in a_0-ε, ε > 0. The D0(a,b) function, positive or null as a sum of squares, is then convex (and therefore the first partial derivative is of constant sign). It necessarily increases on the constant b line when a decreased from $a_0 = -1/2$ to a = 0. The expression D0(a,b) can then be null only for $a_0 = 1/2$.

We note the expressions of successive partial derivatives of C0(a,b) and S0(a,b) versus a as follows:

$$Ck(a,b) = \sum_{m=1}^{\infty} (-1)^{m-1+k}.(Ln(m))^k.m^{-a}.\cos(b.Ln(m)) \qquad (14)$$

and

$$Sk(a,b) = \sum_{m=1}^{\infty} (-1)^{m-1+k}.(Ln(m))^k.m^{-a}.\sin(b.Ln(m)) \qquad (15)$$

This allows us to write successive partial derivatives, versus a, of D0(a,b) as follows:

$$D1(a,b) = C0(a,b).C1(a,b)+S0(a,b).S1(a,b) \qquad (16)$$

and

$$D2(a,b) = C0(a,b).C2(a,b)+S0(a,b).S2(a,b)+(C1(a,b))^2+(S1(a,b))^2 \qquad (17)$$

Our objective is to prove that D2(a,b) > 0 for $0 \leq a \leq 1/2$.

There is a trivially positive part to D2(a,b) that is $P2(a,b) = (C1(a,b))^2+(S1(a,b))^2$. It ought to be compared to the complementary part Q(a,b) = C0(a,b).C2(a,b)+S0(a,b).S2(a,b). As long as Q(a,b) is positive, everything is fine. If Q(a,b) is negative and we have |Q (a,b)| < P(a,b), then the D2(a,b) expression remains positive and Riemann's hypothesis stems from it. It is therefore wise to examine the evolution within the lower critical band of the ratio :

$$R2(a,b) = \frac{C0(a,b).C2(a,b)+S0(a,b).S2(a,b)}{(C1(a,b))^2+(S1(a,b))^2} \qquad (18)$$

From this argument results the following theorem equivalent to theorem 3 :

Theorem 4

If R2(a,b) > -1 for $0 \leq a \leq 1/2$, b any given real number, then Riemann hypothesis is true.

Note

This is a sufficient (and not necessary) condition : A contradictory b (giving R2(a,b) ≤ -1) only excludes the desired result in that value b and its immediate vicinity. We will see below that, indeed, there are b values such as the expression R2(a,b) is less than the -1 value, for $0 \leq a \leq 1/2$, near the origin to abscissas smaller than that of the first Riemann zero (and the first Dirichlet zero).

It should also be noted that because of the symmetry of the functional equation, we only look at the $b \geq 0$ values, the arguments being in any way identical to $b \leq 0$ case.

4.Numerical illustrations.

The reader will refer to Appendix 1 for the conditions to ensure the consistency and accuracy of numerical assessments. All illustrations are given with an interval between points equal to $\Delta b = 1/10$ when not otherwise specified.

The study concerns the critical half-band $0 \leq a \leq 1/2$. However, as Riemann's zeros, at least for those known, are all on the critical line $a = 1/2$, the highlighted expressions are necessarily at their climax and therefore the most prominent on this critical line. Thus, the reader will not be surprised if some calculations focus solely on this line. On a parallel line < 1/2 of the critical band, the situation is similar but rapidly with a (very) smaller magnitude.

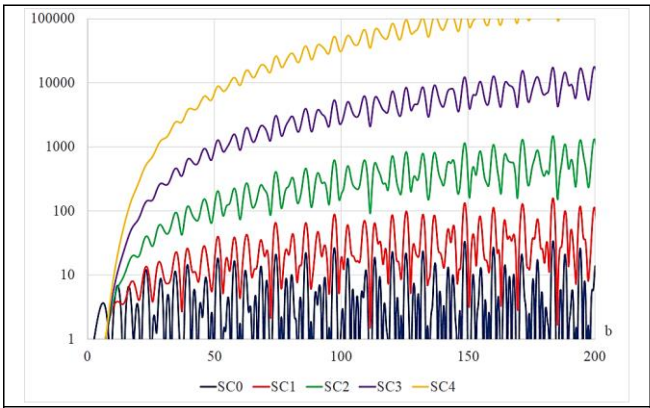
The content of this paragraph 4, called as "illustrations", as well as Appendix 8, contains a set of relationships that are sufficient to show the evidence of the hypothesis. This evidence is overwhelming, the zeros and the neighbourhoods of these zeroes offering only a paroxysm. However, the proof of these relations, for the current paragraph except for the indispensable part 4.1, would surely be a real "tour de force" and is not undertaken here. Hence the obvious need for paragraph 5 that will follow it.

4.1 Staggering of the sums of squares SCk(a,b).

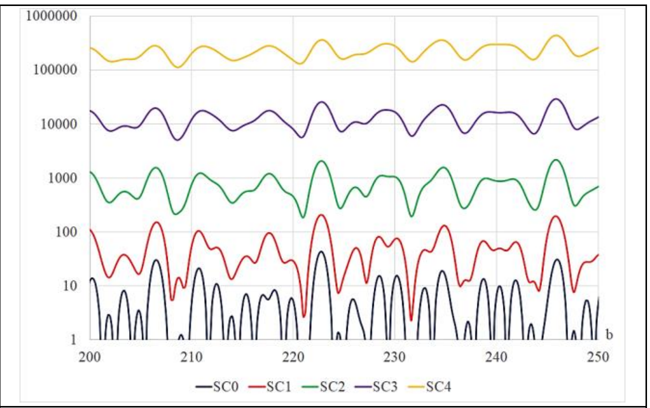
We compare the functions based on sums of Ck(a,b) and Sk(a,b) squares, the latter being defined from relationships (14) and (15) :

$$SCk(a,b) = (Ck(a,b))^2+(Sk(a,b))^2 \qquad (19)$$

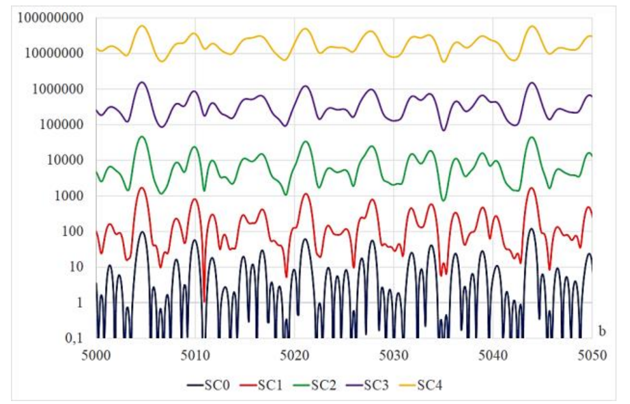
The graphics representations clearly show that, above $b \approx 20$, the $SC_{k+1}(a,b)$ curves are above the $SC_k(a,b)$ curves in relatively nesting positions.



Graphic 2
a = 1/2



Graphic 3
a = 1/2



Graphic 4
a = 1/2

We get immediately :

Theorem 5

We have almost everywhere :

$$SC_{k+1}(a,b) > SC_k(a,b) \qquad (20)$$

Note 1:

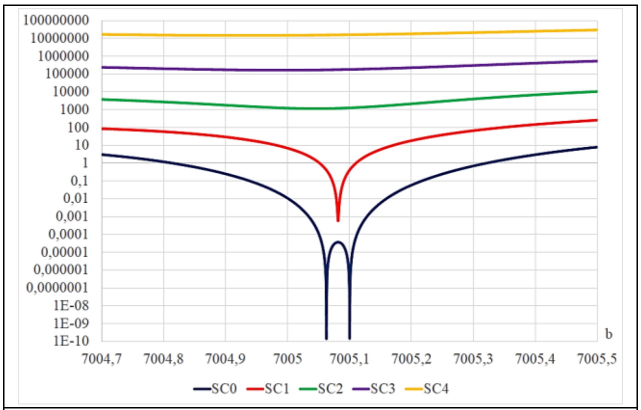
The curves are given with our standard step Δb = 1/10. The downwards peaks are not necessarily fully formed here. Nevertheless, one can see clearly the peak of SC1(a,b) at the level of Riemann's zero corresponding to b ≈ 5010,9331981. The nesting does not prevent in any way to have, close to the Riemann zeros, very small values for $SC_k(a,b)$, whatever k > 0, and this especially for $SC1(a,b) = (C1(a,b))^2+(S1(a,b))^2$. This effectively allows us to find increasingly larger R2(a,b) values here or there, since SC1(a,b) is the denominator of that expression.

Note 2:

The terminology "almost everywhere" is not that of the probability theory. The term only means very often, thus without a strict notion of density, as the study is not completed at this stage.

To improve accuracy, two parameters are to be taken into account:
- The number of terms of the truncation
- The step Δb

In the graphs below, at the peaks' levels, Δb is taken here equal to 1/10000. For SC0, at Riemann's zeros level, both peaks take on lower and lower values (since the theoretical limit here is 0). For SC1, the peak progresses to lower values between truncation with 10,000 terms up to 50,000 terms. This progression then stops. The minimum value statements are 0.00097147 for 10000 terms, 0.00053732 for 50000 terms, 0.00058222 for 150000 terms, nothing in fact prohibiting a higher value in the final instance when accuracy increases.



Graphic 5

Truncation to 150000, a = 1/2

We see above the attraction that constitutes two narrow peaks for $SC_k(a=1/2,b)$ on the expression $SC_{k+1}(a=1/2,b)$ in the hereby k = 0 case. As two peaks create a peak above them, the phenomenon may occur frequently only up to the SC1(1/2,b) level. An imposing peak for SC2(1/2,b) is certainly rare, requiring 3 very close Riemann zeros. A significant spike is undoubtedly exceptional when k > 2.

Theorem 6

We have

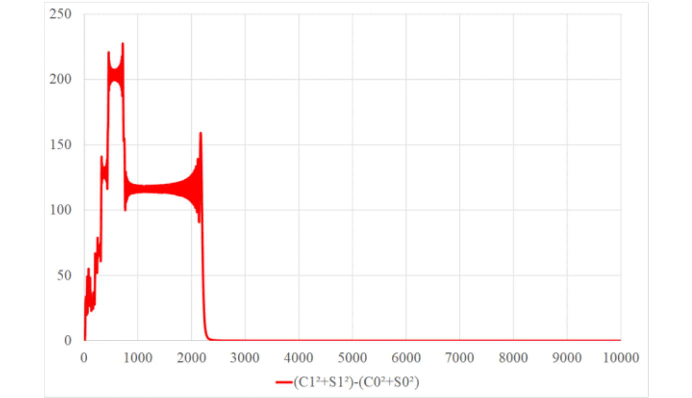
$$SC_{k+1}(a,b) > SC_k(a,b) \qquad (21)$$

Proof

The numerical study clearly shows that the inequality is true except possibly near the peaks' positions and moreover the critical case to examine is that of the relative position of SC0(a,b) and SC1(a,b), higher k cases being even more obvious. So, let us place ourselves at a peak for $SC1(a,b_{peak})$. The expression $SC0(a,b_{peak})$ presents at this abscissa a partial derivative, versus b, close to 0. It can be written, with the notations of paragraph 5.2, $\partial_b((C0(a,b))^2+(S0(a,b))^2) = 2(C0(a,b).S1(a,b)-S0(a,b).C1(a,b))$ with value to take at $b = b_{peak}$. Hence the approximate equality $C0(a,b_{peak}).S1(a,b_{peak}) \approx S0(a,b_{peak}).C1(a,b_{peak})$. Let us simplify the entries by failing to repeat the coordinates (a,b_{peak}). The C0 and S0 functions are non-zero since they are not placed at a Riemann zero. We then have at the peak of SC1, the ordinate difference between SC1 and SC0 equal to $C1^2+S1^2-(C0^2+S0^2) \approx (C0.S1/S0)^2+S1^2-(C0^2+S0^2) = ((S1/S0)^2-1).(C0^2+S0^2)$, and in the same way,

$C1^2+S1^2-(C0^2+S0^2) \approx (C1^2+(C1.S0/C0)^2-(C0^2+S0^2) = ((C1/C0)^2-1).(C0^2+S0^2)$. These expressions, as sums of continuous functions, are continuous. Thus, for the difference $C1^2+S1^2-(C0^2+S0^2)$ to become negative, it must first be able to go to zero. The coordinate point (a,b_{peak}) being intermediate between two Riemann zeros, we have $C0^2+S0^2 \neq 0$. This means that the nullity of $C1^2+S1^2-(C0^2+S0^2)$ results in the joint nullity of $(C1/C0)^2-1$ and$(S1/S0)^2-1$, or simultaneously $C1 \to C0$ and $S1 \to S0$ near the abscissa of the peak. However, C0 is by no means C1, nor S0 compares to S1 and the ratio of their numerical values does not converge due to small, near-random values of C0, S0, C1 and S1. Small values of C0 and S0 are directly linked to small value of the difference $C1^2+S1^2-(C0^2+S0^2)$, creating an increasing oscillation (and therefore a divergence). At the limit $SC1 \to SC0$, the oscillation tends towards infinity making equality impossible.

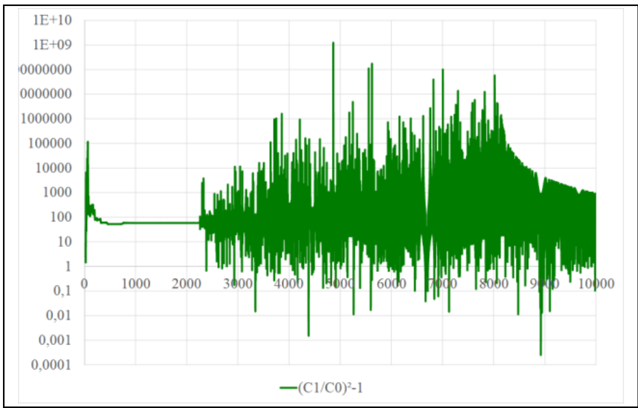
We give the example for the peak near to $b_{peak} \approx 7005.08168$, the phenomenon of oscillations being reproducible with any other example. We do have $C1^2+S1^2-(C0^2+S0^2) \to 0$ (see graphic 6). But $(C1/C0)^2-1 \approx (S1/S0)^2-1 \approx 58$ (which is a first handicap) as long as we take a truncation between 800 and 2300 terms, case where $C1^2+S1^2-(C0^2+S0^2)$ does not yet converge towards 0. When this convergence finally begins with the sufficient number of terms (here above 2500), the ratios C1/C0 and S1/S0 enter an unstable phase due to the low values of C0, S0, C1 and S1 (graphics 7 and 8) oscillating around the previous value. This oscillation remains regardless of the number of terms, and therefore to infinity, that is up to the effective value of $C1^2+S1^2-(C0^2+S0^2)$.



Graphic 6

Truncation to 10000, a = 1/2, bpeak ≈ 7005.08168

Absence of oscillations after m = 2500 for $C1^2+S1^2-(C0^2+S0^2)$

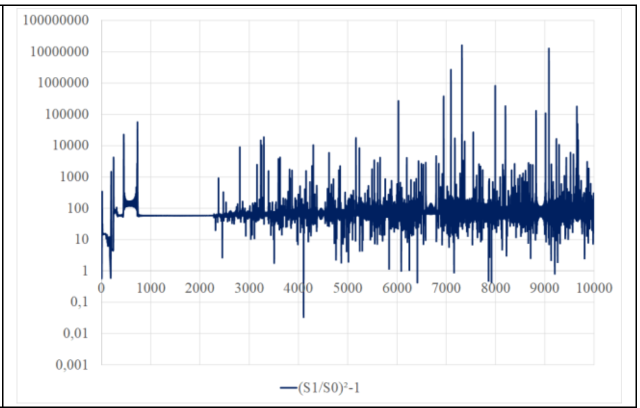


Graphic 7

Truncation to 10000,

a = 1/2, bpeak ≈ 7005.08168

Presence of oscillations after m = 2500 of C1/C0



Graphic 8

Truncation to 10000,

a = 1/2, bpeak ≈ 7005.08168

Presence of oscillations after m = 2500 of S1/S0

Theorem 7

We have for $0 \leq a \leq 1/2$ and $k > 0$:

$$SC_k(a,b) > 0 \qquad (22)$$

Proof

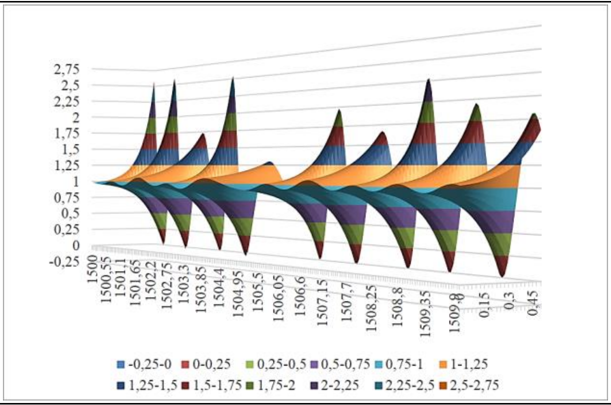
This is immediately due to theorem 6.

Note :

The SC0 ordinate at the intermediate abscissa b_{peak} is, a priori, statistically lower as two Riemann zeros are closer. We return to this point in paragraph 4.5.

4.2 Current values of R2(a,b).

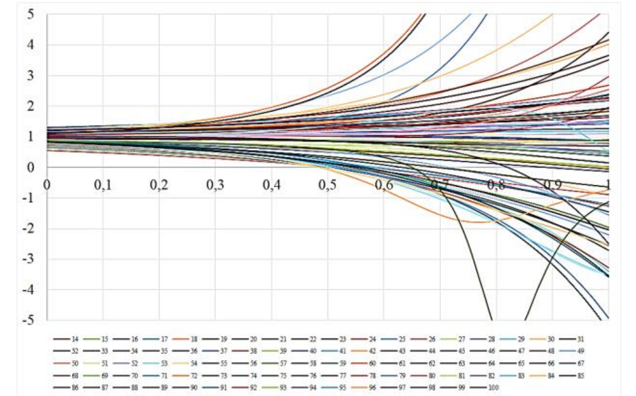
We represent the R2(a,b) ratio as a function of parameter a for fixed b. The view below shows the evolution starting from a = 1/2 (background of the image) with alternating of downwards peaks and upwards peaks. At a = 1/2, the ordinate is null for Riemann and Dirichlet zeroes. The condition of cancellation of C0(a,b).C2(a,b)+S0(a,b).S2(a,b) is more general and may occur without the presence of these two types of zeros. The fifth peak upwards is less marked. It corresponds to the vicinity of the 166th Dirichlet zero ($166*2\pi/Ln(2) \approx 1504.743567$).



Graphic 9

a = [0, 1/2], b ∈ [1500, 1510]

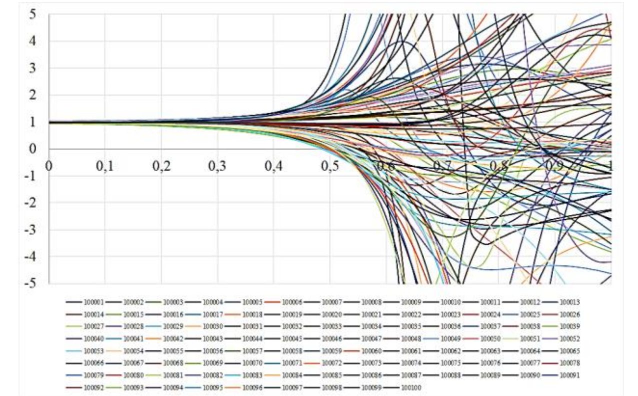
In the graphics that follow, the b values are taken in ℕ for 100 consecutive values. Some b values may be close to the imaginary values of Riemann or Dirichlet zeroes.
The range of b values is given in the legend. The min and max values shown are those of R2(a,b) at a = 1/2. These are of course neither the absolute minimums nor the absolute maximums if b ∈ ℜ.



Graphic 10

b = 14 to 100

min ≈ -0.065923, max ≈ 2.582495

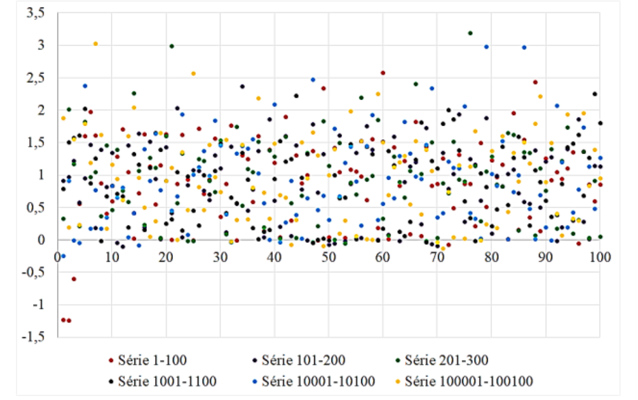


Graphic 11

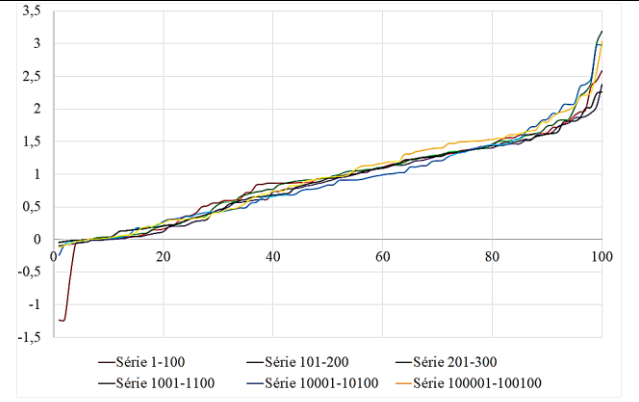
b = 100001 to 100100

min ≈ -0.128374, max ≈ 3.027709

There is some chaos in the variations of R2(a,b) when a > 1/2, but the trend towards the asymptotic value R2(a → -∞,b) → 1 is quickly activated on the side a ≤ 1/2, hence the obvious interest in choosing this side of the critical band.
The graphics below give the values of R2(a = 1/2, b) of the previous series of b numbers. The first graphic is for b in ascending order. The second graphic are the same values where R2(a,b) are sorted in ascending order (the x axis becoming arbitrary).



Graphic 12
a = 1/2



Graphic 13
a = 1/2

The latest graphic shows that the increase in b has no real impact on the order of the distributions of the R2(1/2, b) values and in particular on its minimum value. The last curve of the legend, where b is in the 100001 to 100100 value range, is simply intermediate to the other distributions. We go back to the shape of the latter graphic with increased precision in the following paragraph.

4.3 Random surveys of R2(a,b).

We start with surveys for a = 1/2.
We are talking about random surveys although b is taken with constant spacing 1. Indeed, the value that R2(a,b) versus b, for given parameter a, is not predictable: Riemann's zeroes have b_r imaginary abscissas that we can call random and the value of R2(a,b) for constant b spacing will be at a random distance of the nearby b_r therefore having seemingly random R2(a,b). The same is true of minimums and maximums of R2(a,b) or any other choice. Taking b with random or constant distances thus amounts to the same if we want to statistically analyse the distribution of R2(a,b) given some parameter a.

We then have the following incomplete table :

Table 1

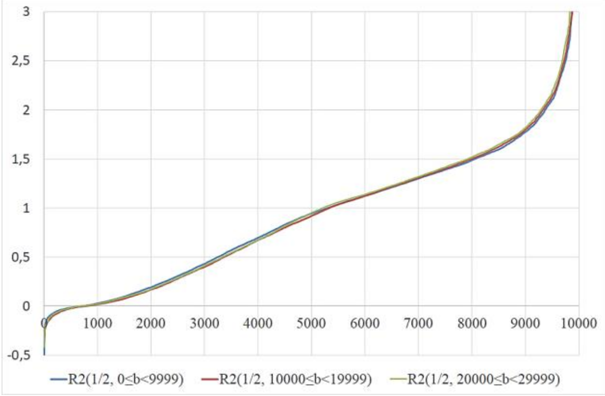
b-k.10000	R2(1/2, 0≤b<9999) k = 0	R2(1/2, 10000≤b<19999) k = 1	R2(1/2, 20000≤b<29999) k = 2
0	-1.08934023	-0.05235938	0.53967998
1	-1.22982782	-0.24315523	0.37783264
2	-1.23743739	0.92023095	1.18166428
3	-0.59258188	-0.00324345	0.43683645
…	…	…	…
9997	1.10427165	0.91778738	0.51817126
9998	-0.02703119	-0.11687438	0.83313405
9999	0.26809106	1.49868353	2.16119046

These values are then sorted in ascending order, with the reference axis becoming a mere index.

Table 2

i	R2(1/2, 0≤b<9999)	R2(1/2, 10000≤b<19999)	R2(1/2, 20000≤b<29999)
0	-1.237437394	-0.367149358	-0.415532661
1	-1.22982782	-0.336819688	-0.389586665
2	-1.089340232	-0.319879054	-0.33263088
3	-0.592581881	-0.314646353	-0.312157279
…	…	…	…
9997	7.164445983	11.20934744	13.13029216
9998	10.87793755	12.42906401	13.19809132
9999	11.3992441	21.28605665	16.7790967

The curves that are representative of the distributions of values obtained in this way are :

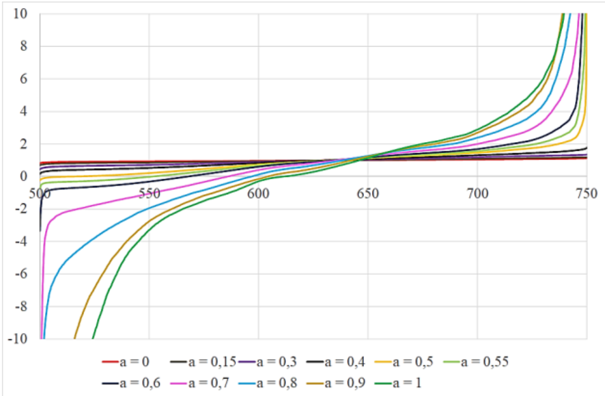


Graphic 14

$x_{graphic} = i$, a = 1/2, b ∈ [0, 29999]

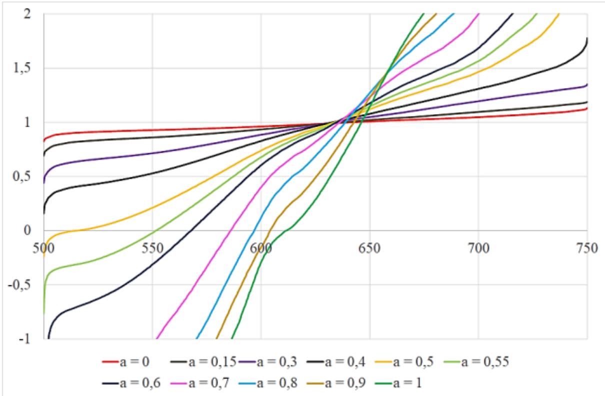
The shape curves in the central part are the same for the three sorted data choices.
It would be the same for any other random choice of abscissas provided that the selected sample has sufficient elements. This random choice may be a Δb spacing different from 1.

Below are the R2(a,b) results for a smaller range of values (b ∈ [500, 750]) and a smaller spacing Δb (Δb = 1/100). The curve's shape for a = 1/2 (in yellow) is the same. Again, the x-axis is not really b since the values of R2(a,b) have been taken up in ascending order.



Graphic 15

a = 0 to 1, x ∈ [500, 750]

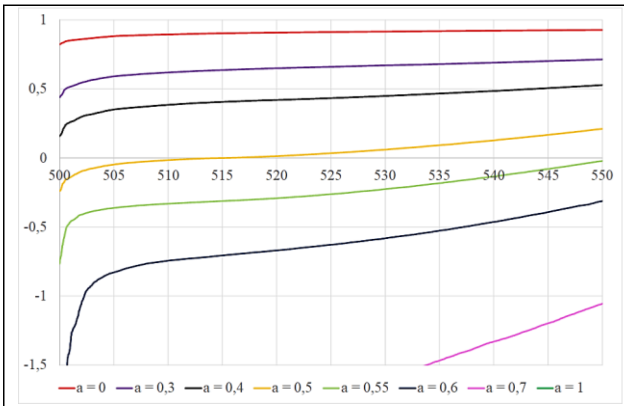


Graphic 16

a = 0 to 1, x ∈ [500, 750]

As the parameter a decreased progressively getting closer to 0, the set of R2(a,b) values get closer to the horizontal axis of ordinate 1.

The graphic below provides a close-up view for the part we are particularly interested in, that is when R2(a,b) < 0. The curves show up in descending order of a values as one would expect. Here, for a = 0.55, we still have R2(a,b) > -1, but this is no longer the case for a = 0.6 (which is without any prejudice for our study for which the proof is necessary only in a ∈ [0, 1/2]).



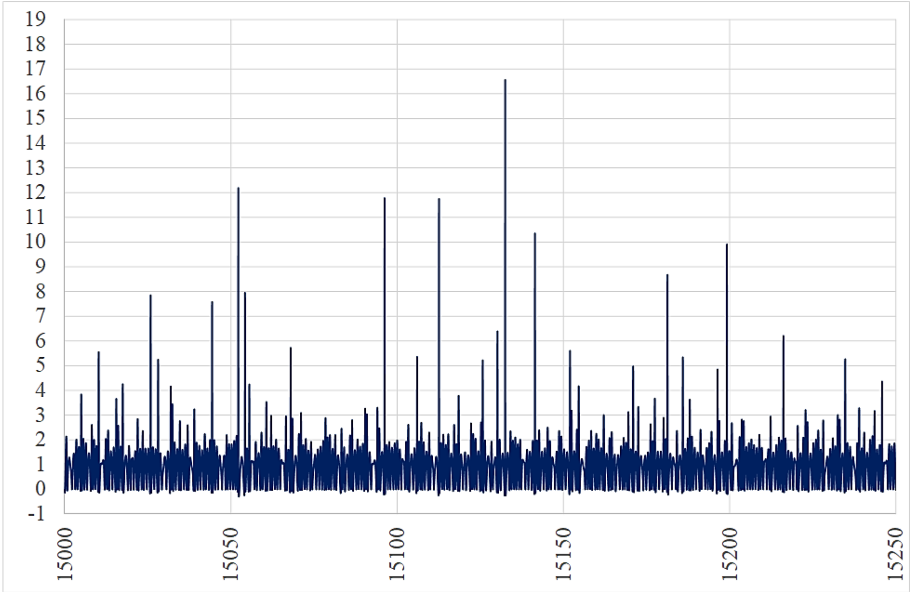
Graphic 17

a = 0 to 1, x ∈ [500, 750]

4.4 Relations between local minimum and maximum of R2(a,b).

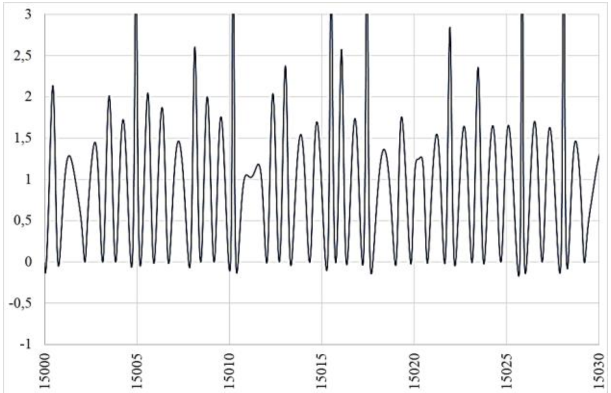
The R(a = 1/2, b) function changes from local minimum to local maximum when b increases. Here we are looking for some relationships between a maximum and the two minima that frame it.

The graphic below gives a sample of the values taken by R2(1/2, b) for b ∈ [15000,15250]. The savvy reader may note, although this is not very visible, that a (positive) peak also corresponds to negative value spikes on either side of this peak.

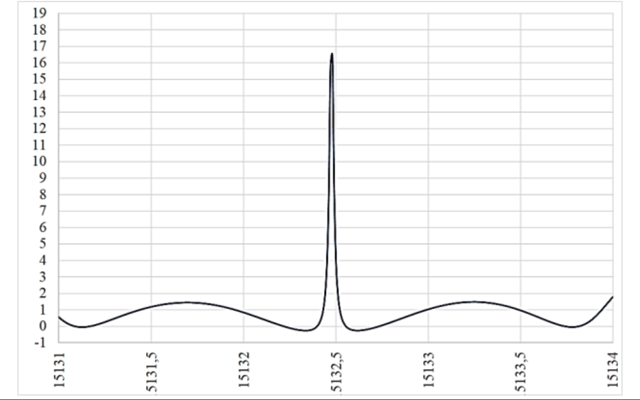


Graphic 18
a = 1/2, b ∈ [15000,15250]

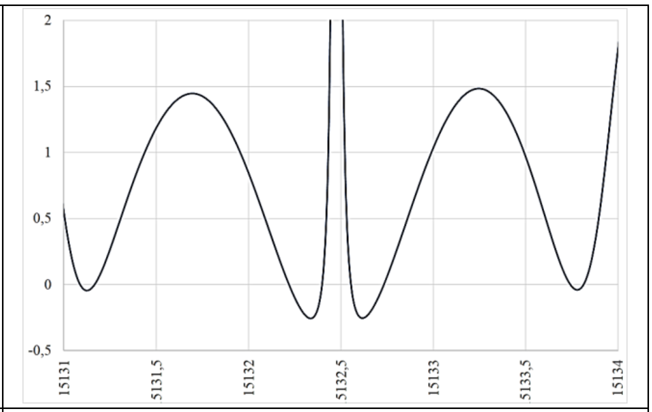
This is more visible by making some magnifications :



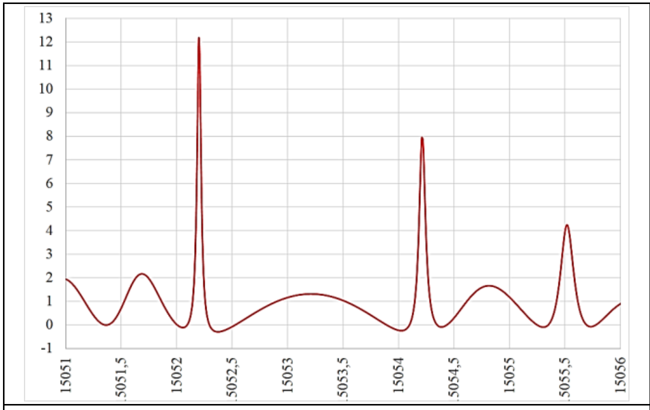
Graphic 19
a = 1/2, b ∈ [15000,15030]



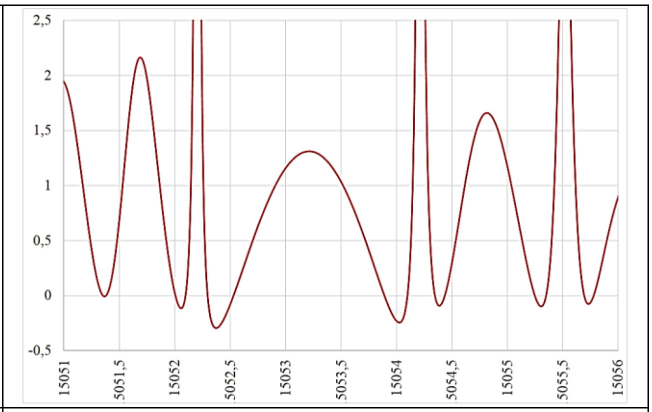
Graphic 20
a = 1/2, b ∈ [15131,15134]



Graphic 21
a = 1/2, b ∈ [15131,15134]



Graphic 22
a = 1/2, b ∈ [15051,15056]

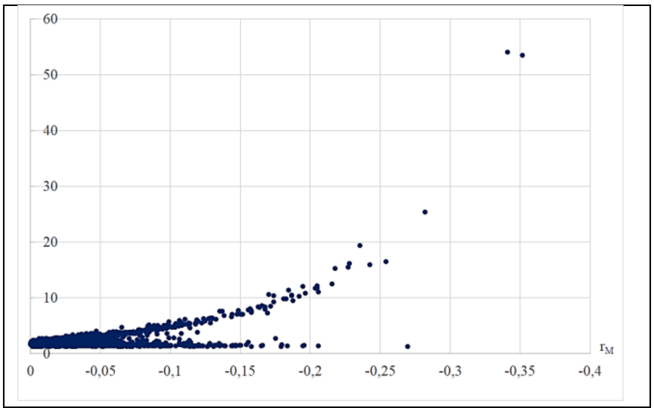


Graphic 23
a = 1/2, b ∈ [15051,15056]

It is as if the rise towards the high values r_{peak} of R2(a,b) requires a spring force acting from under ordinate 0. Indeed, the higher a peak is, the higher the negative values r_{low} surrounding it.

Graphic23 however shows "high" negative values on both sides that do not necessarily cause a high (positive) spike when their forces are already affected in nearby peaks.
Thus, if we represent r_{peak} as a function of r_M, where r_{peak} is the value of a given peak, r_M the average between the two lower values on either side, we necessarily get a "parasitic" branch.
This is what is shown in the graphics (24) below.

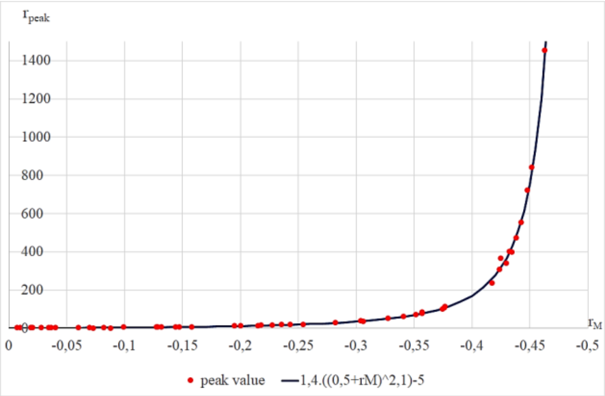
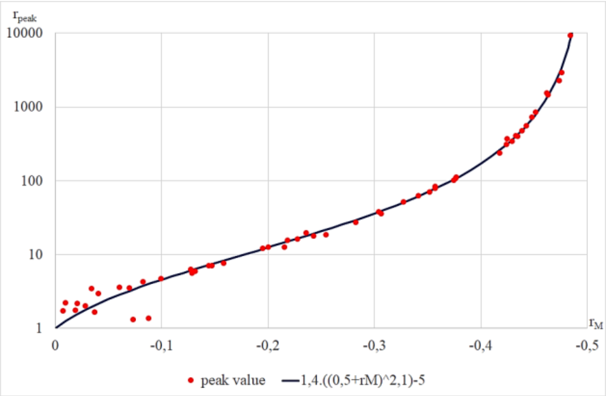
The reader will note that these two graphics (which are the same data except a logarithmic scale for the y-axis in the second graphic) were made by aggregating the data provided in intervals b ∈ [3000, 3250], [6000, 6250], [9000, 9250], [12000, 12500], [15000,15250] with a Δb = 1/100 step.



Graphic 24
r_{peak} function of r_M

The "parasitic" branch is the one extending horizontally. It does not provide any additional useful information to that provided by the ascending branch since it is linked to it. The downside is that it can give the illusion that the abscissa rM is not bounded when what determines it is actually the evolution of the ascending branch. In paragraph 4.5, we specify the criterion after which the mix of horizontal and ascending branches take an overriding character.

The illustration, based on graphic 24, leads also to an important remark. The $\Delta b = 1/100$ step remains too wide to get a good accuracy of the actual values of the peaks r_{peak}. It is imperative to do a point-by-point study. We thus provide in Appendix 3 Table 7 the complete data of the graphics below (which are again the same data with simply a logarithmic scale for the y-axis in the second chart).

	
Graphic 25 r_{peak} function of r_M Main branch	***Graphic 26*** r_{peak} function of r_M Main branch ($r_M < -0,1$)

The points on the first graphic show an increasingly rapid divergence beyond abscissa rM ≈ -0.35. The second graphic shows that this increase is over-exponential. The data near the origin are more erratic because of the combination of the ascending and horizontal branches within graphic (24).

The interpolation function used here is :

$$r_{peak} \approx \frac{1.4}{(0.5+r_M)^{2,1}} -5 \qquad (23)$$

The adjustment parameters are very approximate (except 0.5 which is certainly near effective value). We incline for an exponent with denominator equal to 2, but our data to date indicates the adjustment proposed here. For lack of better, we let it that way.

The important point is that this approximation function diverges at $r_M = -0.5$ which means a bumper value impossible to exceed (as soon as -0.5 instead of -1) because of the continuity of R2(a,b) demonstrated in paragraph 5.1. This then confirms theorem 4.

Note: The term r_M is an average of 2 terms. Nothing prevents one of them from being smaller than -0.5. Cependant les deux coordonnées r_{bas} autour d'un pic étant à l'extérieur de part en d'autre des deux zéros de Riemann, et donc telles que $r_{bas} < 0$, la limite $r_M > -0,5$ signifie qu'aucun des deux r_{bas} ne peut être inférieur à -1. However, since the two r_{low} coordinates around a peak are outside on either side of the two Riemann zeros, and therefore such that $r_{low} < 0$, the limit $r_M > -0.5$ means that neither of the two r_{low} can be less than -1.

4.5 Relationship between Riemann zeros spacings and R2(a,b).

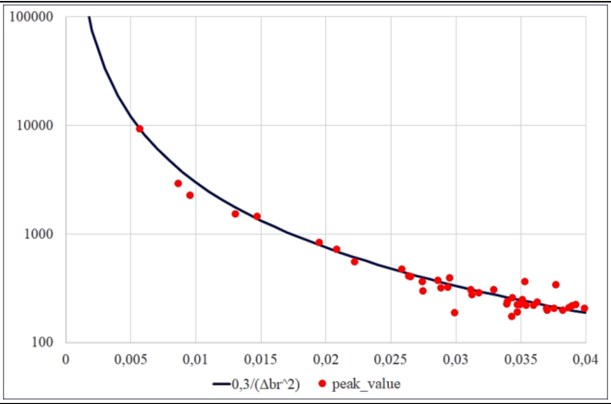
A random search for high-amplitude peaks of R2(a,b) would require enormous computational resources without the existence of a sufficiently simple tracking. Fortunately, there is a link between the gap of two consecutive Riemann zeros and the height of the intermediate peak, a link that then makes the search quite easy thanks to the database referenced in [5].

As it turns on, a peak is generally all the more ample as the gap between two consecutive Riemann zeros (at abscissas noted zero_R- and zero_R+) is smaller. We have the following approximate relationship, where b_r is the peak abscissa, Δb_r the gap between two Riemann's zeroes. The reader will find the numerical elements in Appendix 5 :

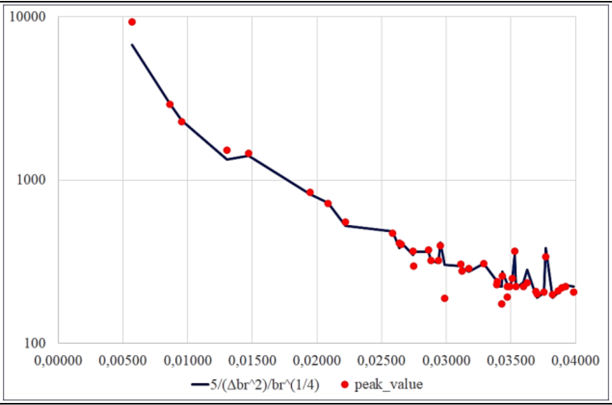
$$r_{peak} \approx 1 + \frac{5}{\Delta b_r^2 . b_r^{1/4}} \qquad (24)$$

According to this relationship, the amplitude of the peak tends towards infinity when the gap tends towards zero. These cases are necessarily more and more common for very high-value abscissas since the average difference between zeros is asymptotically in 2π/Ln(abs_zeroR). The presence of the logarithm, however, makes it difficult to find many cases with very high values here. In particular there is no $r_{peak} > 10000$ for the first 500,000 Riemann zeroes.

The first graph below represents the numerical results and evaluation by an interpolation formula without taking into account the abscissa of the peak br ($b_r^{1/4}$ term at denominator obliterated). In the second, this additional factor is introduced.



Graphic 27
r_{peak} function of Δb_r
(cf. Appendix 5, Table 8)



Graphic 28
r_{peak} function of br and Δb_r
(cf. Appendix 5, Table 8)

The reader will find in appendix 8 a more comprehensive study of the r_{peak} approximation's functions enabling a better picture of the actual value of these extremums.

For large peaks, we can neglect the constants -5 and 1 in the relations 23and 24 In order to compensate for the power 2.1 that we reduce to 2, we increase somewhat the constant in front of the fraction to obtain $r_{peak} \approx 1.8/(0.5+r_M)^2 \approx 5/\Delta b_r^2.b_r^{1/4}$. This results in :

$$r_M \approx -(1/2).(1-1,2.\Delta b_r.b_r^{1/8}) \qquad (25)$$

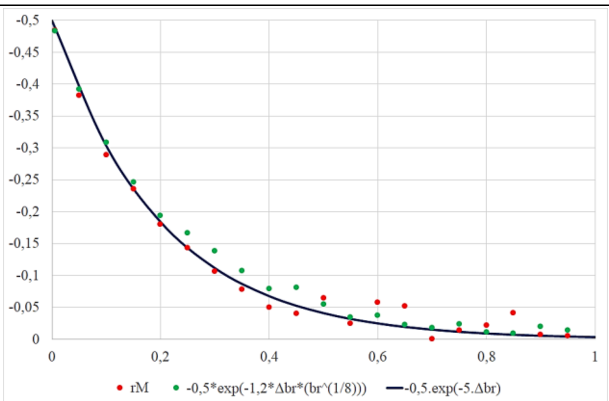
Since $\Delta b_r.b_r^{1/8} > 0$, the term r_M is greater than -0.5.
For values close to 0, exp(-x) ≈ 1-x, and thus :

$$r_M \approx -(1/2).\exp(-1,2.\Delta b_r.b_r^{1/8}) \qquad (26)$$

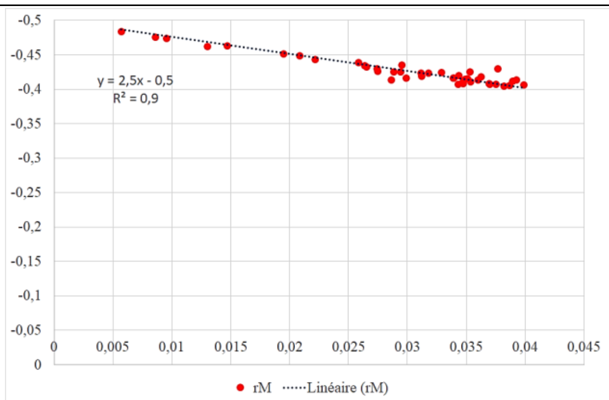
In the range of numerical values examined, we also have the simpler alternative formula :

$$r_M \approx -(1/2).\exp(-5\Delta b_r) \qquad (27)$$

which leads to the following graphs:



Graphic 29
r_M function of Δb_r
(cf. Appendix 5, Table 9)



Graphic 30
r_M function of Δb_r
(cf. Appendix 5, Table 8)

The alignment of the points clearly stalls for a gap between Riemann zeroes larger than $\Delta b_r = 1/2$ (and therefore apparently regardless of the abscissas of these zeroes). Based on this approximate critical value, as the data in Appendix 5 seems to show, the order of abscissas abs_r_{low} (abscissa b of r_M before the peak), abs_zero_R- (abscissa of Riemann's zero before the peak), abs_peak (abscissa of the peak), abs_zero_R+ (abscissa of the Riemann zero after the peak), abs_r_{high} (abscissa of r_M after the peak), is no longer respected (which is perfectly possible since the cancellation of R2(a,b) does not correspond to the cancellation of $(C0(a,b))^2+(S0(a,b))^2$.

Below the critical value of the gap, the points align perfectly here, the only selection criterion having been to take the gap Δb_r among the first 100,000 zeroes such as Δb_r is closest by higher value of 0.05, 0.10, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40 and 0.45, (for the construction of graphic 29), a selection that gives only an almost uniform spacing in abscissa but in no way any predisposition on the value of the ordinate. We have also added to the chart the lower Δb_r gap solution that exists among Riemann's first 500000 zeroes.

It should be noted that the value of the abscissa abs_peak does not intervene in the proposed formula (unlike the relationship involving r_{peak}). Close to the origin ($\Delta b_r < 0.05$), the relationship 27 is quasi-linear by developing exp(x) to the first order and clearly tends (cf. graphic 30) towards the limit r_M =-1/2.

4.6 Geodesics of R2(a,b).

We adopt the word "geodesic" out of sheer convenience. These are more specifically the local extrema of the R2(a,b) function.

Theorem 8

The local maximum value of R2(a,b) is related to the minimums' paths in the vicinity of this peak.

Proof

The extremums of R2(a,b) are determined by the cancellation of the two partial derivatives $\partial_a R2$ and $\partial_b R2$.
This means using relations (30) and (31) that we will establish later on :

$$(C1^2+S1^2+2C0.C2+2S0.S2).(C1.C2+S1.S2)+(C1^2+S1^2).(C0.C3+S0.S3) = 0$$
$$(C1^2+S1^2+2C0.C2+2S0.S2).(C1.S2-S1.C2)+(C1^2+S1^2).(S0.C3-C0.S3) = 0$$

Thus

$$(C1.C2+S1.S2).(S0.C3-C0.S3) = (C1.S2-S1.C2).(C0.C3+S0.S3) \qquad (28)$$

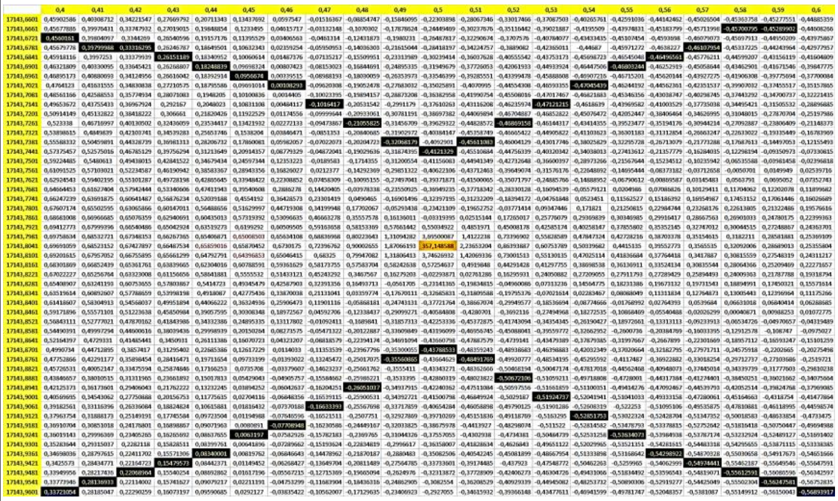
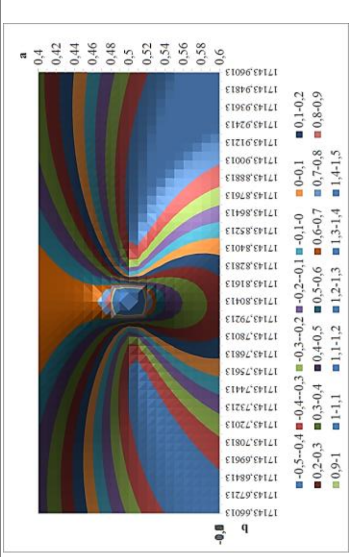
This equation is common to local minimums and maximums, hence the link.

Note 1.

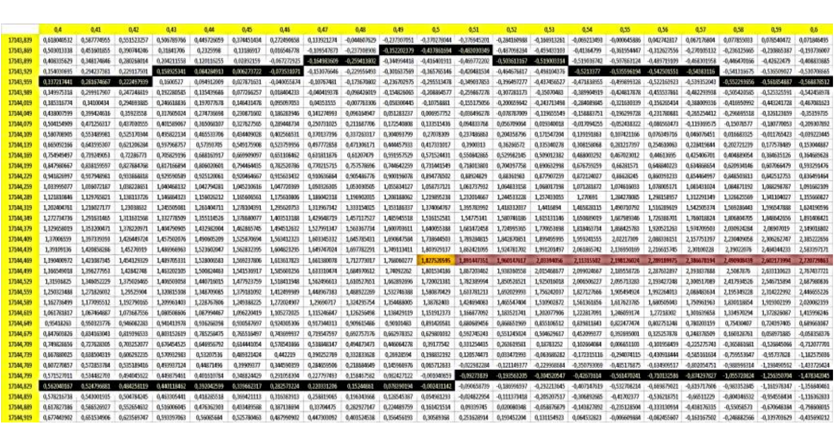
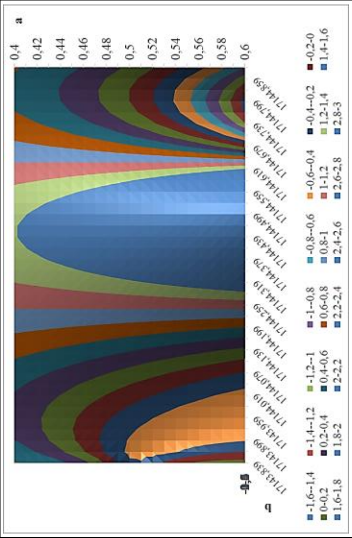
The common equation explains the link between a peak of R2(a,b) and the two minima on either side of that peak observed in the illustrations in the previous paragraph. In fact, what produces the value of the peak is not only the two values on either side, where the parameter a is set in advance, but the entire minimum geodesic "surrounding" that peak. However, the average of the two values examined above is already, when the peak has a significant value above 1, a good representation of the said neighbourhood and thus allows to anticipate the peak value.

Note 2.

The minimums on both sides must have comparable values for the approximation equation to be useful. When this is the case, for a high-value peak, the configuration of the minima (in dark blue) is cross-shaped as in graphic 31, while the altitude isopleths tend vertically as shown by the example of graphic 32.

	
Graphic 31 a = 0.4 to 0.6 b ∈ [17143.66, 17143.96] Δb = 0.006	***Graphic 32***

When the minimums values are dissimilar, the two wings of the minimums are instead oriented in the opposite direction on the side of the extremum with value close to -0.5 and generally horizontal on the side of the other minimum. The altitude isopleths tend horizontally. The graphics that illustrate this point are numbered 33 and 34.
This explains the "parasitic" branch.
In addition, the peak is only its initial draft at a = 0.5 and continues to increase on the side of a > 0.5 (pink line below).

	
Graphic 33 a = 0,4 to 0,6 b ∈ [17143.839, 17144.919] Δb = 0.03	***Graphic 34***

<u>Note 3.</u>

The uncompromising reader will object that an equation such as the relationship 28 can be found not only for geodesics but for any value one wishes to affix to R2 (a,b) and therefore proves nothing. We are not saying the contrary for the first part of this argument. Indeed, no point escapes the equations of the whole set of points. What we are saying here is that there is enough information in the minimums to determine the rest and the argument raised is sufficient as the basis for the evidence.

With all the indicators on green, it is time to get back to something else.

5.Back to the theorems and proofs.

5.1 Continuity of R2(a,b).

<u>Theorem 9</u>

The R2(a,b) function is continuous in interval $0 \leq a \leq 1/2$.

<u>Proof</u>

It suffices to prove that the R2(a,b), i.e. its denominator $SC1(a,b) = (C1(a,b))^2+(S1(a,b))^2$ does not cancel. This is theorem 7.

Let us note that the function is also continuous outside the indicated interval.

5.2 Calculation of the partial derivative linked to R2(a,b).

<u>Writing convention.</u>

In this text, the functions are generally dependent on two variables a and b. The handling of the objects is simplified by writing F instead of F(a,b). The partial derivative of F, versus parameter a, $\partial/\partial a(F(a,b))$ is simplified in $\partial_a F$. The same goes for b. In the text body, we defined the functions Ck(a,b) and $S_k(a,b)$. The non-recalled entries of parameters a and b are equivalent as well as the indexing k : $Ck(a,b) = C_k(a,b) = Ck = C_k$ and $Sk(a,b) = S_k(a,b) = Sk = S_k$.

<u>Evaluation of the partial derivatives of Ck(a,b) and Sk(a,b).</u>

From relations (14) and (15), we get :

$$Ck = \sum_{m=1}^{\infty} (-1)^{m-1+k}.(Ln(m))^k.m^{-a}.\cos(b.Ln(m))$$

and

$$Sk = \sum_{m=1}^{\infty} (-1)^{m-1+k}.(Ln(m))^k.m^{-a}.\sin(b.Ln(m))$$

We deduct immediately

$$\partial_a Ck = \sum_{m=1}^{\infty} (-1)^{m+k}.(Ln(m))^{k+1}.m^{-a}.\cos(b.Ln(m))$$

$$\partial_a Sk = \sum_{m=1}^{\infty} (-1)^{m+k}.(Ln(m))^{k+1}.m^{-a}.\sin(b.Ln(m))$$

$$\partial_b Ck = \sum_{m=1}^{\infty} (-1)^{m+k}.(Ln(m))^{k+1}.m^{-a}.\sin(b.Ln(m))$$

and

$$\partial_b Sk = \sum_{m=1}^{\infty} (-1)^{m-1+k}.(Ln(m))^{k+1}.m^{-a}.\cos(b.Ln(m))$$

In other words :

$$\begin{aligned} \partial_a Ck &= C_{k+1} \\ \partial_a Sk &= S_{k+1} \\ \partial_b Ck &= S_{k+1} \\ \partial_b Sk &= (-1).C_{k+1} \end{aligned} \qquad (29)$$

All of this functions, as sums (finite or infinite) of continuous functions are continuous.

Evaluation of the partial derivatives of R2(a,b).

From relation (18), we get :

$$R2 = \frac{C0.C2+S0.S2}{C1^2+S1^2}$$

It follows using identity (u/v)’ = (u’.v-u.v’)/v^2 :

$$\partial_a R2 = \frac{(C1^2+S1^2+2C0.C2+2S0.S2).(C1.C2+S1.S2)+(C1^2+S1^2).(C0.C3+S0.S3)}{(C1^2+S1^2)^2} \qquad (30)$$

$$\partial_b R2 = \frac{(C1^2+S1^2+2C0.C2+2S0.S2).(C1.S2-S1.C2)+(C1^2+S1^2).(S0.C3-C0.S3)}{(C1^2+S1^2)^2} \qquad (31)$$

Note :

The two previous partial derivatives are continuous due to the fact that $(C1(a,b))^2+(S1(a,b))^2$ doesn’t cancel (see again theorems 7 and 9).

5.3 The impossibility of R2(a,b) = -1.

When we talk about the impossibility of R2 = -1, we mean at the same time the impossibility of R2 < -1 since R2(a,b) is continuous according to both coordinates a and b. In addition, we place ourselves in the conditions a $\in$ [0, 1/2] and b $\in$ [3, +∞[.

Framework of the proof

From relation (18), we get by definition R2 = $(C0.C2+S0.S2)/(C1^2+S1^2)$, so that also (C0.C2+S0.S2) = R2.$(C1^2+S1^2)$. Relation (31) becomes then :

$$\partial_b R2 = \frac{(1+2R2).(C1.S2-S1.C2)+(S0.C3-C0.S3)}{(C1^2+S1^2)} \qquad (32)$$

We seek the values for which the R2 expression is minimal when b varies, thus those such that $\partial_b R2 = 0$, meaning also R2 = (1/2).((S0.C3-C0.S3)/(C1.S2-S1.C2)-1). The solutions are hence those for which we have simultaneously :

$$R2x = \frac{C0.C2+S0.S2}{C1^2+S1^2} \qquad (33)$$

and

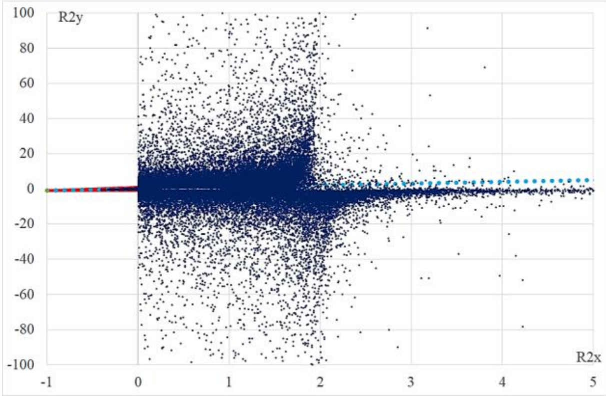
$$R2y = (1/2).\left(\frac{C0.S3-C3.S0}{C1.S2-C2.S1} - 1\right) \qquad (34)$$

and

$$R2 = R2x = R2y \qquad (35)$$

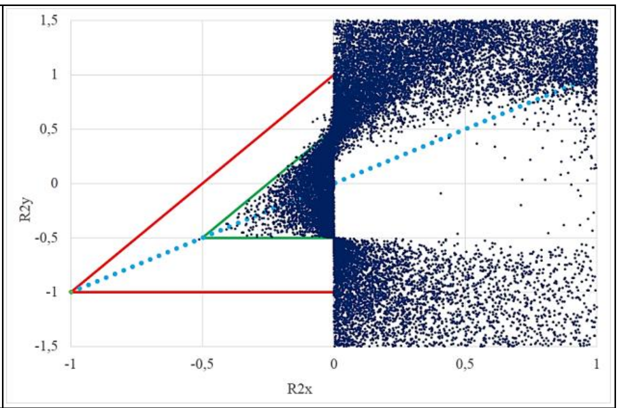
The two graphics below are the same, the second being a close-up view of a particular area. They contain all the (R2x(a,b), R2y(a,b)) points obtained for a 1/2, b = 0 to 20000 and Δb = 1/4, the actual solutions joining these points by continuity. The point (R2x, R2y) ≈ (-0.5122, -0.5121) for (a, b) = (1/2, 78974.87502) corresponding to the only example found where R2(a,b) < -1/2 is also reported on the graphic. The only solutions to retain are on the R2x = R2y axis of this graphic (the light blue dotted line), but the usefulness of spotting all the dots (R2x, R2y) is obvious. This makes it possible to visualize in an obvious or even garish way, the minimum abscissa when simultaneous equalities are obtained. We see that the dots are concentrated, for the part below the abscissa R2x = 0, in a triangle R2y = -1/2, R2x = 0, R2y = 1/2+2.R2x (green frame). The low point of this triangle is -1/2. A few points are slightly outside this triangle, but this does not affect our conclusion. One finds them mostly above the triangle near the 0^- abscissa.

Such a figure with very sharp boundary lines, although slightly permeable, show the absurdity of points extending far beyond R2x < -1/2, namely to a hypothetical R2x = -1, for a = 1/2, complementary information being given for a $\in$ [0,1] in Appendix 10



Graphic 35

a = 1/2, b ∈]3, 20000] and Δb = 0.25

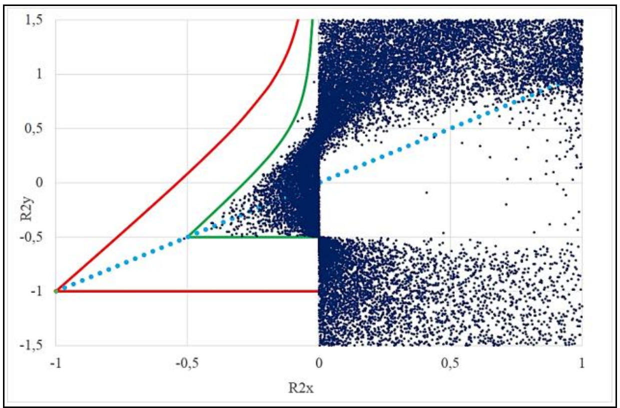


Graphic 36

a = 1/2, b ∈]3, 20000] and Δb = 0.25

The first graph above shows a concentration of coordinates (Rx,Ry) along and below the axis Ry = -1/2 when Rx tends towards infinity. However, the effective solutions are those placed on the axis Rx = Ry. This means the scarcity of high peak values both because of the scarcity of points along the axis Ry = -1/2 and the distance of the line Rx = Ry from the said line. Therefore a double penalty in some way...

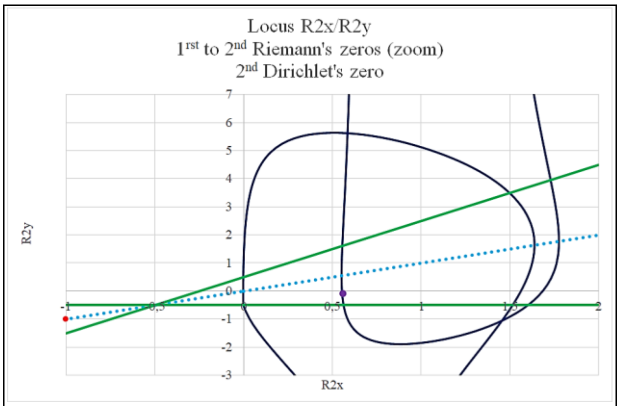
A framework that is a little closer to reality can be given by reparametrizing the initial line R2y = 1/2+2R2x (upper line in green underneath) into a new form R2y = 1/2+2R2x+(1/12).tan(π.(R2x+1/2)), which does not change in any way the hereby argument.



Graphic 37

a = 1/2, b ∈]3, 20000] and Δb = 0.25

The slope of the curve giving R2y as a function of R2x is ∞ on the dotted blue line R2y = R2x as shown in the example below and those of appendix 11. This follows from the construction of this graph which initially uses the relationship ∂_bR2 = 0 (see page 18) and therefore results in ∂bR2x = 0 precisely on the line R2x = R2y.



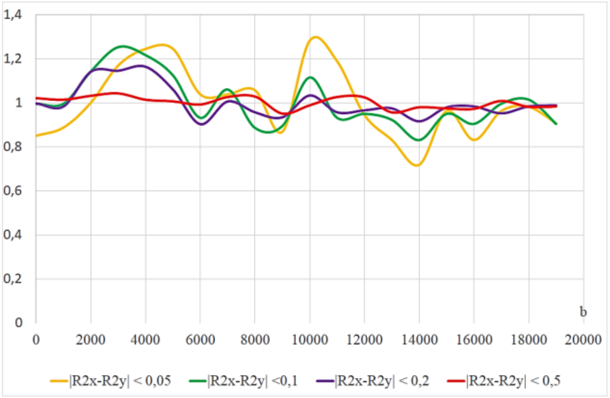
Graphic 38

This slope is also infinite for the divergences of R2y, that is to say when C1.S2-C2.S1 = 0 but then there is of course no corresponding point on the curve itself (see again appendix 11).

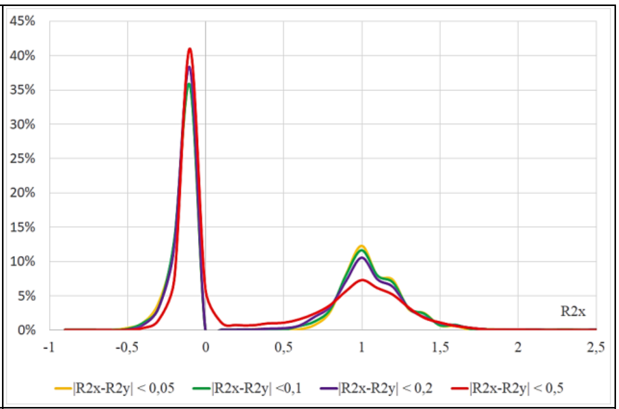
The previous graphics thus show that points cannot reach the -1 abscissa (remaining the fact that overruns of -1/2 are possible).

The population density of points such that |R2x-R2y| < s, s some given threshold, is relatively constant regardless of intervals b such as [10000k, 10000.(k+1)[, k = 0 to 19, chosen from our data collecting. This is the subject of the graphic (39).

Graphic40 shows the population density relative to the R2x parameter for intervals like [0.1k, 0.1(k+1)[, k -10 to 25, crossed with the criterion |R2x-R2y| < s. Points' population peaks are centred on R2x ∈ [-0.1, 0] and R2x ∈ [1, 1.1] intervals and collapse very rapidly on each side of the first peak instance.



Graphic 39
a = 0.5 and b ∈ [10000k, 10000(k+1)[, k = 0 to 19



Graphic 40
R2x ∈ [k/10, (k+1)/10[, k = -10 to 25

Theorem 10

The minimal value of R2(a,b) is -1 excluded.

Proof

From relations (18) and (32), because $\partial_b R2 = 0$, we get the two following equations to be solved :

$$(C2).C0+(S2).S0-R2.(C1^2+S1^2) = 0$$
$$(S3).C0-(C3).S0-(1+2R2).(C1.S2-S1.C2) = 0$$

So that :

$$\begin{pmatrix} C0 \\ S0 \end{pmatrix} = \begin{pmatrix} C2 & S2 \\ S3 & -C3 \end{pmatrix}^{-1} \begin{pmatrix} R2.(C1^2+S1^2) \\ (1+2R2).(C1.S2-S1.C2) \end{pmatrix}$$

Then

$$\begin{pmatrix} C0 \\ S0 \end{pmatrix} = \left(\frac{1}{C2.C3+S2.S3}\right).\begin{pmatrix} C3 & S2 \\ S3 & -C2 \end{pmatrix} \begin{pmatrix} R2.(C1^2+S1^2) \\ (1+2R2).(C1.S2-S1.C2) \end{pmatrix}$$

Let us write

$$\alpha = R2.(C1^2+S1^2)$$
$$\beta = (1+2R2).(C1.S2-S1.C2)$$

Then :

$$C0^2+S0^2 = \frac{\alpha^2.(C3^2+S3^2)+\beta^2.(C2^2+S2^2)+2\alpha.\beta.(C3.S2-C2.S3)}{(C2.C3+S2.S3)^2}$$

Let us write the ratio :

$$DSC(\ldots) = \frac{\alpha^2.(C3^2+S3^2)+\beta^2.(C2^2+S2^2)+2\alpha.\beta.(C3.S2-C2.S3)}{(C2.C3+S2.S3)^2.(C0^2+S0^2)} \quad (36)$$

and

$$DL(\ldots) = Ln(DSC(\ldots))$$

A good understanding of the argument requires, as before, the distinction between the two cases R2(a,b) = -1/2 and R2(a,b) = -1, knowing that continuity gives perfectly accessible intermediate values, if necessary, between these two cases.

If R2 = -1/2, we write DSC0 = DSC(-1/2), expression DSC being defined above, so that also :

$$DSC0 = \frac{(C1^2+S1^2)^2.(C3^2+S3^2)}{4.(C0^2+S0^2).(C2.C3+S2.S3)^2}$$

and

$$DL0 = Ln(DSC0)$$

The R(a,b) = -1/2 is reached when

$$DSC0 = 1$$

or as well

$$DL0 = 0$$

If R(a,b) = -1, we write DSC1 = DSC(-1), so that :

$$DSC1 = \frac{(C2^2+S2^2).(C1.S2-C2.S1)^2+(C1^2+S1^2).((C1^2+S1^2).(C3^2+S3^2)+2.(C1.S2-C2.S1)(C3.S2-C2.S3))}{(C0^2+S0^2).(C2.C3+S2.S3)^2}$$

and

$$DL1 = Ln(DSC1)$$

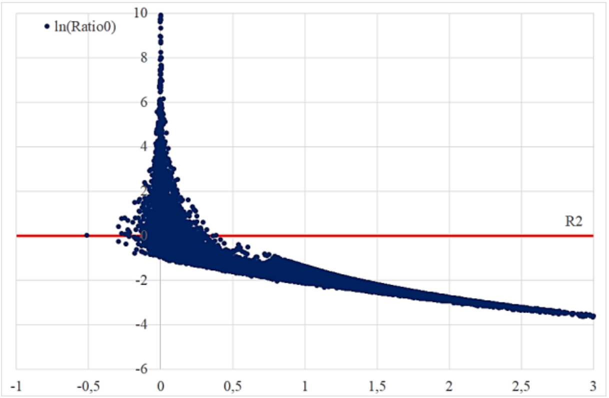
The R(a,b) = -1 case is reached when

$$DSC1 = 1$$

or else

$$DL1 = 0$$

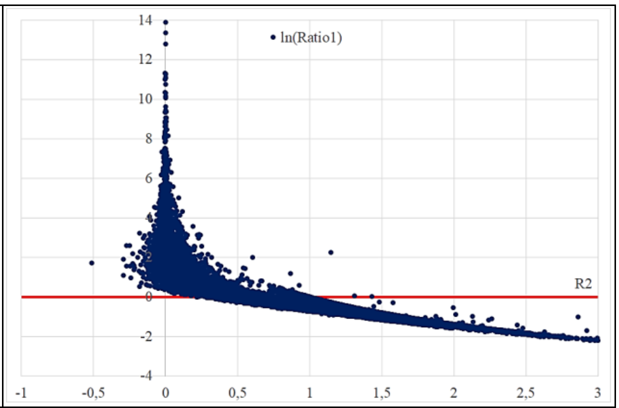
Graphics of DSC0 and DSC1 below are, in fact, point clouds' variants undergoing a continuous distortion of the graphics obtained in the first part of this demonstration. They show the same thing in a slightly different form.



Graphic 41

a = 0.5 and b ∈]50, 5050]

Ratio0 = DSC0

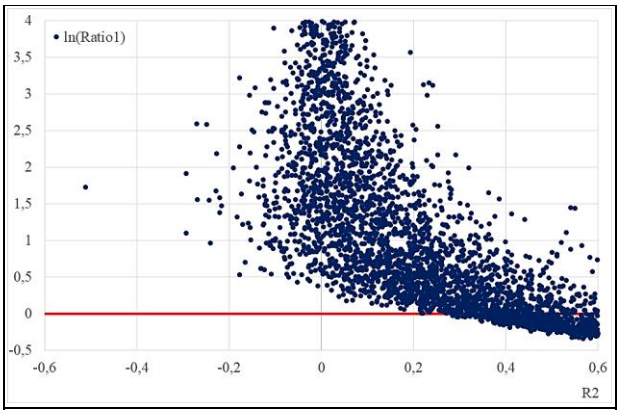


Graphic 42

a = 0.5 and b ∈]50, 5050]

Ratio1 = DSC1

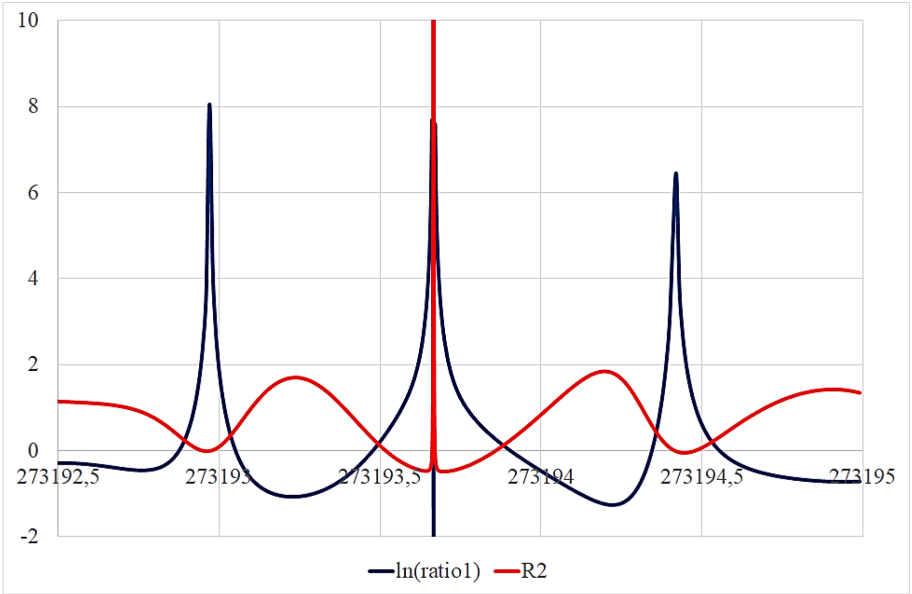
The first graphic (graphic41) shows the outgrowth of the minimums, on the negative R2 side, aligned on the line Ln(DSC0) = 0. For the second graphic (graphic 42), the outgrowth is deflected upwards showing the impossibility of reaching R2 = -1 values. On line Ln(DSC1) = 0, where this event R2 = -1 must be effective to reject Riemann's hypothesis, the intersection is not only above -1/2, but much further beyond 0^+.



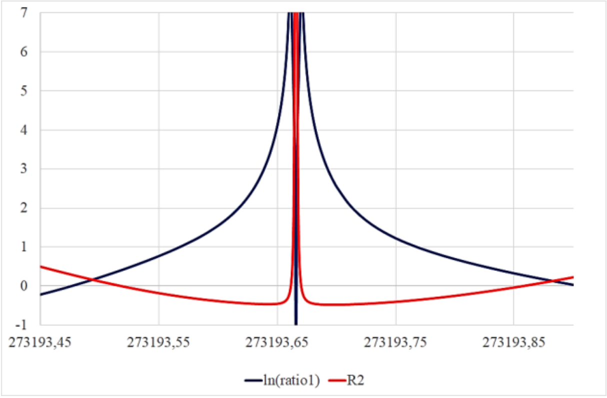
Graphic 43
a = 0.5 and b ∈]50, 5050]
Ratio1 = DSC1

Finally, let us look at the precise reason why R2(a,b) = -1 events are not achieved by local punctual examples. For this, we choose the case for which we detected the smallest difference between Riemann's zeroes among the first 500000 of them and a few others.

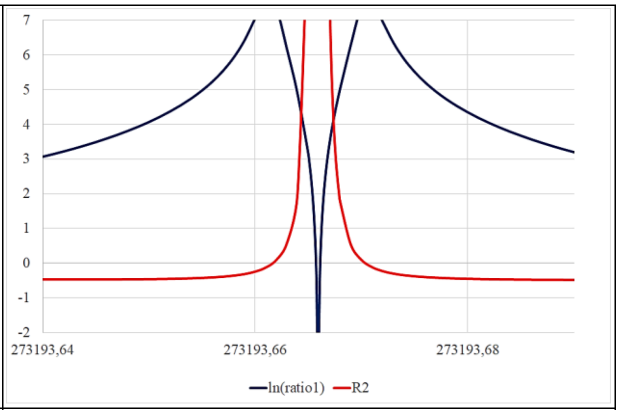
The graphic below shows simultaneously, one on one, the evolution of R2 and Ln(DSCl). We recall that Ln(DSC1) = 0 (or DSC1 = 1) is the target value right above the R2 minimums.



Graphic 44
Ratio1 = DSC1, a = 0.5 and b ∈ [273192.5, 273195]

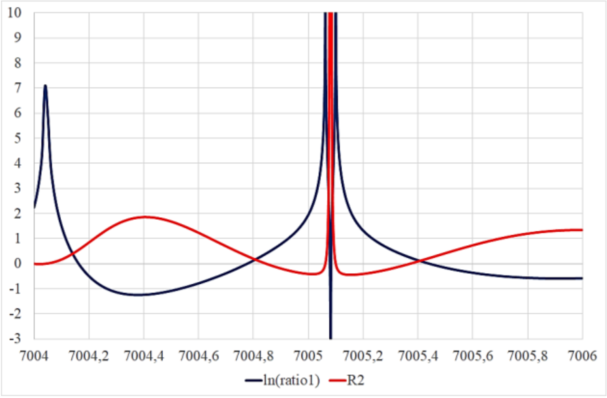


Graphic 45
Zoom 1 : Ratio1 = DSC1
Outer crossings : ordinates ≈ 0.68 and ≈ 0.695

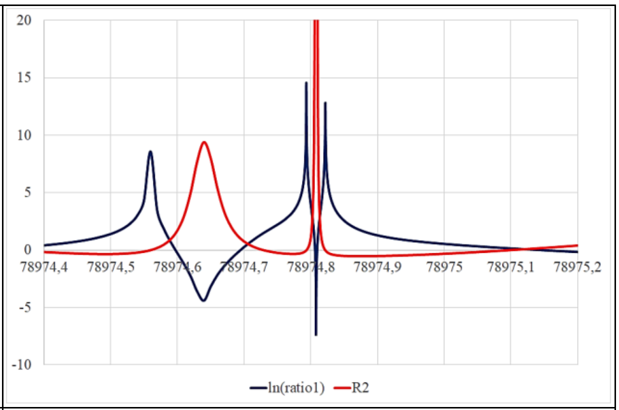


Graphic 46
Zoom 2 : Ratio1 = DSC1
Inner crossings : ordinates ≈ 4.3 and ≈ 4.15

The initiation of an R2 descent induces the initiation of a Ln(DSC1) ascent and vice versa as shows the examples below. This behaviour is perfectly reproducible, as shown by the two graphics that follow.



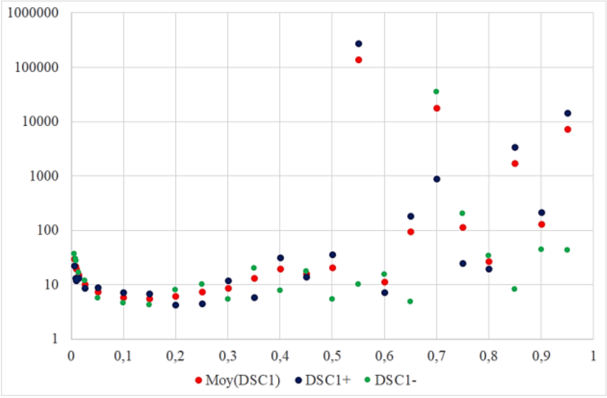
Graphic 47
a = 0.5 et b ∈ [7004, 7006]
Outer crossings : ordinates ≈ 0.16 et ≈ 0.1
Inner crossings : ordinates ≈ 2.8 et ≈ 2.8



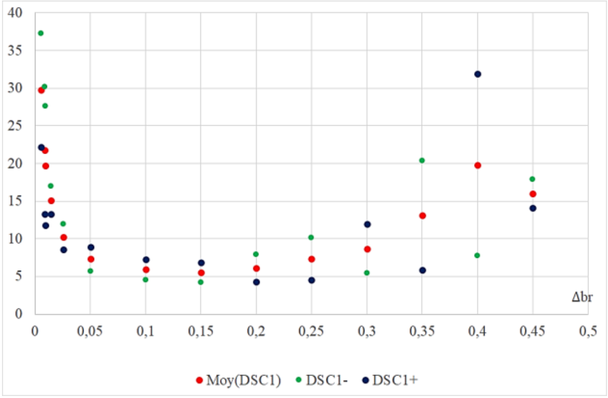
Graphic 48
a = 0.5 et b ∈ [78974.4, 78975.2]
Outer crossings : ordinates ≈ 0.55 et ≈ 0.1
Inner crossings : ordinates ≈ 2.8 et ≈ 2.85

The crossing of R2 and Ln(DSC1) curves at the approach of a low-R2 zone is at the level of ordinate 0 and Ln(DSC1) then quickly increases.
Let us consider the intersections of the R2 and Ln(DSC1) curves. We call inner crossings those whose abscissas are between two Riemann's zeros and outer crossings the other two to the right and left (of graphic 45). The term Ln(DSC1) necessarily diverges according to the relationship 36 since $C0^2+S0^2 = 0$ for any Riemann (and Dirichlet) zero. So, the inner crossings are trivially above ordinate 0. The outer crossings are also, a point that however seems difficult to establish. The easiest way is to assess the value of DSC1 at the points that matter to us, that is where R2 is minimum.

For this, we pick the data used to establish graphic 29 with the same selection criterion chosen at that time. The corresponding table is in Appendix 6 Table 10. Doing so, we get the graphics below (the second graph being a zoom of the first on the area of low values of Δbr the gaps between Riemann zeros) :



Graphic 49
DSC1
a = 1/2



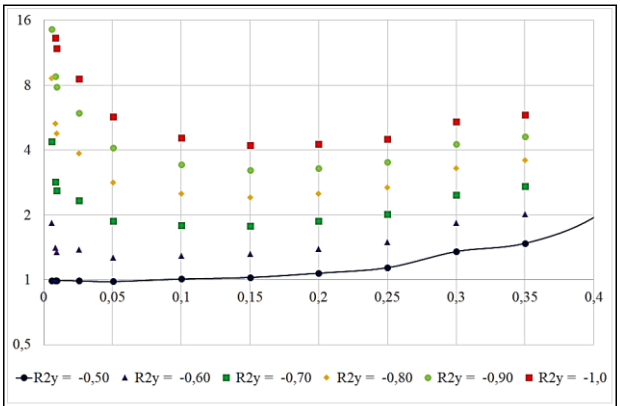
Graphic 50
DSC1
a = 1/2

We recorded DSC1- the value of DSC1 for the r_M- abscissa before the peak and DSC1+ the value for the r_M+ abscissa after the peak. We have also reported in the graphics the average DSC1 value of these two values, knowing that what really is important here is rather the minimum value of the two values DSC1- and DSC1+.

The alignment of the points fails (which then offers no useful information) for a gap between zeroes of Riemann higher than approximatively $\Delta b_r = 1/2$ in exactly the same way we had found in the case of graphic 29.

However, the points interesting us, i.e. cases where the R2 ratio is likely to be close to the -1 value, are necessarily points sticking to the origin of the abscissas (extremely small Δbr). This area corresponds to the uprise of DSC1 well beyond the critical value DSC1 = 1. This upswing is due to the simple fact that the closer two Riemann zeros are, the more pronounced the corresponding peak is and the steeper the flanks of the peak, including until the abscissas of the minimums of R2. Thus, the abscissa of a minimum (rM- or rM+) of R2 is close to that of its corresponding zero, in other words, when Δbr → 0, then $C0^2+S0^2 \rightarrow 0$ at the abscissas rM- and rM+ also. But $C0^2+S0^2 \rightarrow 0$ is at the DSC1 denominator and no term in C0 or S0 is within the numerator for compensation. The term $C1^2+S1^2$, and even more $C2^2+S2^2$, will tend to 0 with many decades of delay as shown by the typical example of graphic 5, the number of decades increasing rapidly with the lowering of Δbr. The other terms do not tend in any way towards 0. The compensation remains effective in DSC0 because of the square of $C1^2+S1^2$ in the DSC0 numerator, but it would take a power of at least 4 effected to $C2^2+S2^2$ in DSC1 (plus 3 very close zeros at least) to obtain the said compensation. Thus, DSC1 necessarily diverges when Δbr → 0 and so in a very steep manner.

We extended the study to the intermediate value of R2y = -0.5 to -1 by 1/10 steps. Appendix 6 Table 11 gives the values corresponding to the underneath graphic. For R2y = -0.5, we collected points below DSC(X) = 1 which is of course expected and thus authorizes the existence of R2(a,b) < -0.5, of which we found a unique example (see appendix 5). The calibration thanks to the graphic below shows, at the same time, that the limit value is close to it. The existence of points such as R2(a,b) < -0.6 without being totally unthinkable (because the points represented here are only the image of a larger dot cloud if one uses more data) is certainly a quite rare event.



Graphic 51
DSC(X)
a = 1/2

The rise of the points obtained for the DSC1 expression near the origin ($\Delta b_r < \approx 0.15$) and their alignment besides that ($\Delta b_r < \approx 0.35$) is independent of the abscissas of the Riemann's zeroes (is dependent only on the gap between the said zeroes) completing this proof.

Note :

The reader will note that all the examples of this proof are built with a = 1/2. Indeed, the peak values are (for the part of the critical band $a \leq 1/2$) on the critical line. The presentation of the numerical results is interesting only for this critical line, as the values of the extremums decline otherwise very quickly (for a < 1/2) not allowing to find additional solutions that can contradict our presentation in a relevant way.

Theorem 11

The asymptotic value of the R2(a,b) minimums is -1/2.

Proof

By the term "asymptotic," we mean the minimums of R2 (a,b) when b tends towards infinity (and the parameter a is fixed). In this case, Riemann's zeros are, on average, at a distance of about $2\pi/\text{Ln}(br)$, meaning nearer and nearer. According to the relation (31), the numerator of $\partial_b R2$ is equal to $(C1^2+S1^2+2C0.C2+2S0.S2).(C1.S2-S1.C2)+(C1^2+S1^2).(S0.C3-C0.S3)$. The cancellation of $\partial_b R2$ occurs for $(C0.C2+S0.S2)/ (C1^2+S1^2) = (1/2).((C0.S3-S0.C3)/(C1.S2-S1.C2)-1)$, in other words when :

$$R2 = \frac{-1}{2} + \frac{C0.S3-S0.C3}{2.(C1.S2-S1.C2)} \qquad (37)$$

Asymptotically, as we saw in the last part of the proof of the impossibility of R2 = -1, the terms C0 and S0 tend towards 0 much faster than all the terms of the Ck and Sk’s type, k > 0. It immediately follows R2 → -1/2.

Note 1 :

This result reminds us that negative overruns of -0.5 are possible. These will become more frequent when b increases. But, asymptotically, these overruns will also be more and more restricted to the immediate vicinity of -0.5 and therefore without the possibility of joining -1, thus confirming again theorem 10.

Note 2 :

At the peak abscissa r_{peak}, the expression C1.S2-S1.C2 necessarily takes values very close to 0, taking away this prerogative from the other two extremums (the minimums).

Numerical examples.

The examples below are realized thanks to the online computer application Pari gp using the computer program given in appendix 9

These are the three cases with smallest gaps between Riemann’s zeros of abscissas less than b = 2000000. We actually systematically obtain values close to -0.5 (between -0.48 and -0.51).

The "theoretical" value of the peak (truncation $+\infty$) is obtained from the formula $r_{peak} \approx 1+5/(\Delta b_r^2.\ b_r^{1/4})$. It is difficult to obtain the actual precis value of these peaks numerically (see again appendix 9). Some values concerning the minimums r_{M^-} and r_{M+} of R2 to the left and right of the Riemann’s zeros surrounding a peak also remain imprecise if the truncation does not include enough terms.
The reader will be able to compare the final truncations used to the numbers of terms $m \approx 1/(\exp(\pi/b)-1)$ corresponding to the last jump of values of the terms $\sum(-1)^{m+k}.(\text{Ln}(m))^{k+1}.m^{-a}.\cos(b.\text{Ln}(m))$ and $\sum(-1)^{m+k}.(\text{Ln}(m))^{k+1}.m^{-a}.\sin(b.\text{Ln}(m))$. For more information, refer to Appendix 1 relationship 38.

Example 1 :
1115578th Riemann zero, b = abs_zeroR- = 663318.508310486
1115579th Riemann zero, b = abs_zeroR+ = 663318.511269140
Gap between zeros = 0.002958654
Number of terms for the last values’ jump : 211200.

abs_r_M-	truncation	r_M-	abs_r_M+	truncation	r_M+	abs_peak	truncation	value_peak
							+∞ (th)	20016.79
663318.493	400000	-0.47470	663318.531	400000	-0.49702	663318.509792	2000000	12083.47
663318.493	500000	-0.48127	663318.531	500000	-0.48719	663318.509792	3000000	12857.79
663318.493	600000	-0.47625	663318.531	600000	-0.49116	663318.509792	4000000	30072.08
663318.493	700000	-0.48800	663318.531	700000	-0.49903	663318.509792	5000000	15770.54
663318.493	800000	-0.48891	663318.531	800000	-0.49867	663318.509792	6000000	21286.54
663318.493	900000	-0.48151	663318.531	900000	-0.49167	663318.509792	7000000	27207.50
663318.492	1000000	-0.48170	663318.530	1000000	-0.49245	663318.509792	8000000	25831.65
663318.493	1000000	-0.48176	663318.531	1000000	-0.49245	663318.509792	9000000	19475.39
663318.494	1000000	-0.48161	663318.532	1000000	-0.49234	663318.509792	10000000	20324.01

Example 2 :
3637897th Riemann zero, b = abs_zeroR- = 1961773.9933561
3637898th Riemann zero, b = abs_zeroR+ = 1961773.9966154
Gap between zeros = 0.003259290
Number of terms for the last values' jump : 634000.

abs_r_M-	truncation	r_M-	abs_r_M+	truncation	r_M+	abs_peak	truncation	value_peak
1961773.979	1000000	-0.49666	1961774.010	1000000	-0.51715		+∞ (th)	12577.56
1961773.979	1500000	-0.50639	1961774.010	1500000	-0.50081	1961773.995	3000000	2476.21
1961773.979	2000000	-0.49447	1961774.010	2000000	-0.48523	1961773.995	4000000	6172.31
1961773.979	4000000	-0.48805	1961774.010	4000000	-0.48598	1961773.995	7000000	3971.08
1961773.979	6000000	-0.49153	1961774.010	6000000	-0.48904	1961773.9949	10000000	20729.31
1961773.979	8000000	-0.49178	1961774.010	8000000	-0.48832	1961773.9949	15000000	15082.05
1961773.978	10000000	-0.49072	1961774.009	10000000	-0.487921	1961773.99497	20000000	4659.89
1961773.979	10000000	-0.49081	1961774.010	10000000	-0.487927	1961773.99498	20000000	13634.89
1961773.980	10000000	-0.49075	1961774.011	10000000	-0.487741	1961773.99499	20000000	11987.72

Example 3 :
3271858th Riemann zero, b = abs_zeroR- = 1779292.80366586
3271859th Riemann zero, b = abs_zeroR+ = 1779292.80782699
Gap between zeros = 0.00416113
Number of terms for the last values' jump : 566400.

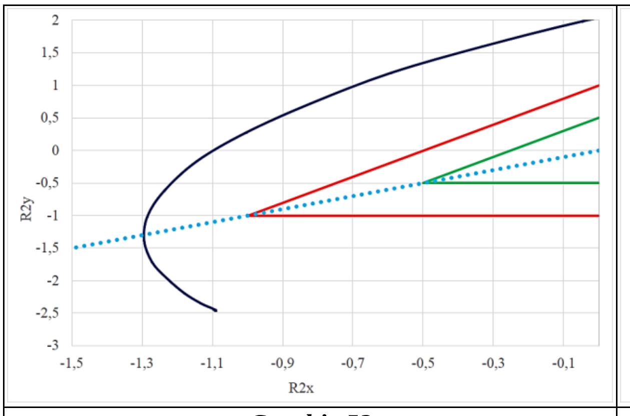
abs_r_M-	truncation	r_M-	abs_r_M+	truncation	r_M+	abs_peak	truncation	value_peak
							+∞ (th)	7907.52
1779292.7940	1000000	-0.41291	1779292.830	1500000	-0.50279	1779292.805757	3000000	28888.16
1779292.7940	1500000	-0.44111	1779292.830	3000000	-0.51002	1779292.805757	4000000	6634.78
1779292.7940	3000000	-0.47259	1779292.835	5000000	-0.50784	1779292.805757	5000000	6030.39
1779292.7920	5000000	-0.46615	1779292.835	7000000	-0.50774	1779292.805756	6000000	7235.79
1779292.7915	7000000	-0.46839	1779292.830	9000000	-0.50774	1779292.805746	7000000	8360.63
1779292.7920	7000000	-0.468493	1779292.835	9000000	-0.50789	1779292.805747	7000000	8361.39
1779292.7925	7000000	-0.468489	1779292.840	9000000	-0.50647	1779292.805748	7000000	8301.65

5.4 The exception to the rule.

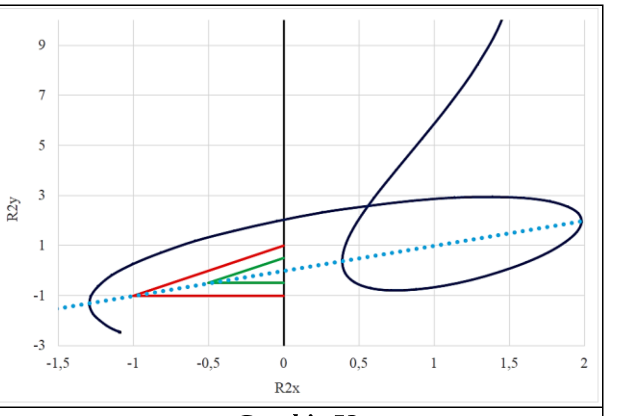
We found an exception to the minimum rule of -1 very early in this text (see note of theorem 4). It is essential to give the reason for it because, although of no practical importance as it is local and therefore of easily verifiable effect, it is nevertheless like a thorn in the foot from the theoretical point of view.

The very particular case b < br1

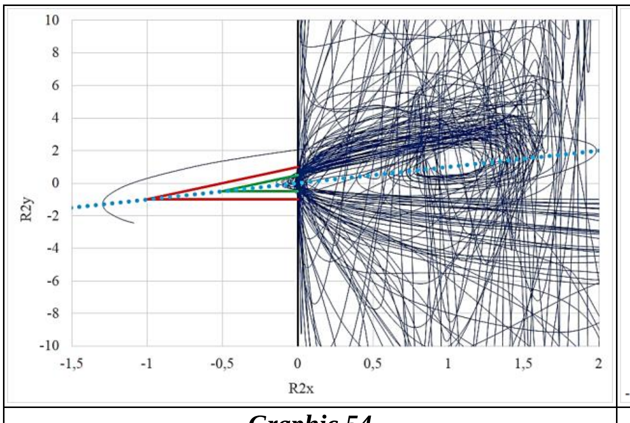
We examine the case where abscissa b is lower than that of the first Riemann zero and its development out from this area. The types of curves and choice of colours are the same as before. In particular, the dark blue curve represents the points (R2x, R2y).



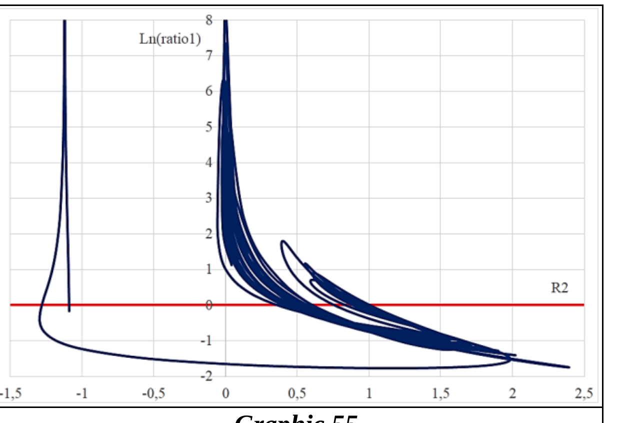
Graphic 52
a = 0.5 and b ∈]0, 3.5]



Graphic 53
a = 0.5 and b ∈]0, 11.5]



Graphic 54
a = 0.5 and b ∈]0, 200]

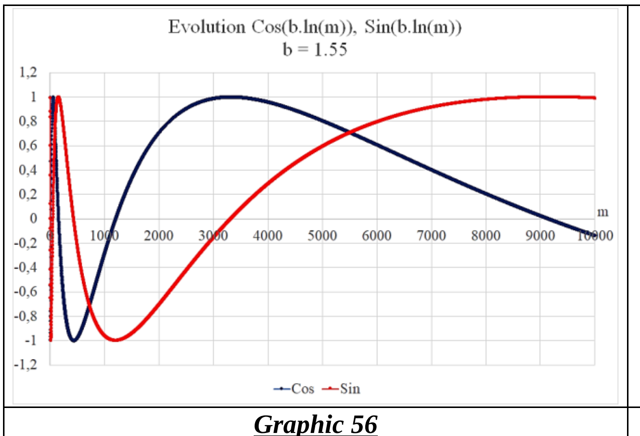


Graphic 55
a = 0.5 and b ∈]0, 50]

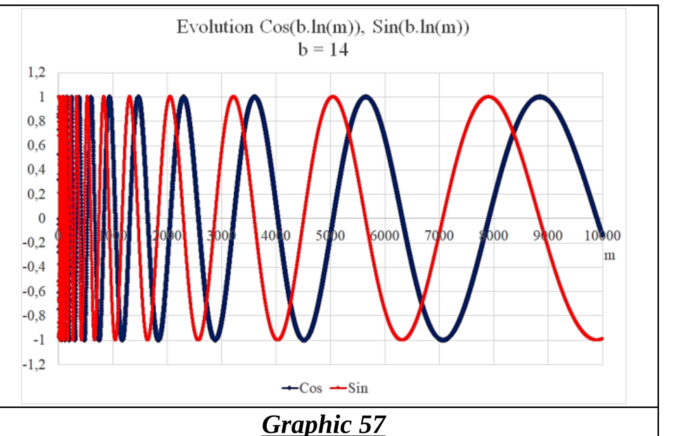
The reader can see that as soon as the blue curve crosses abscissa R2x = 0, it is trapped in the areas described above despite all the restlessness, to say the least, that reigns there.

<u>Why can R2x be less than -1 for small values of b ?</u>

Some rule will apply in a context and only in this case. It is not otherwise here.
Indeed, we note the evolutions of the values of cos(b.ln(m)) and sin(b.ln(m)), in the sums Ck and Sk as a function of m, for truncation m = 1 to 10000 and two values of b, as examples below :

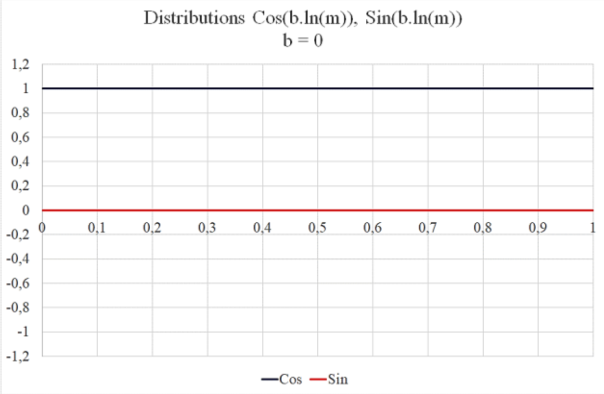


Graphic 56
a = 0.5 and b = 1.55

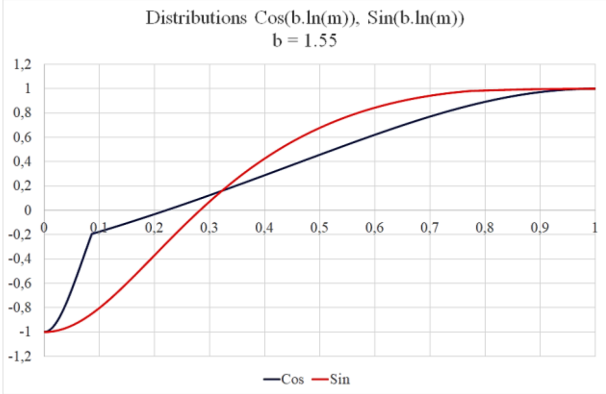


Graphic 57
a = 0.5 and b = 14

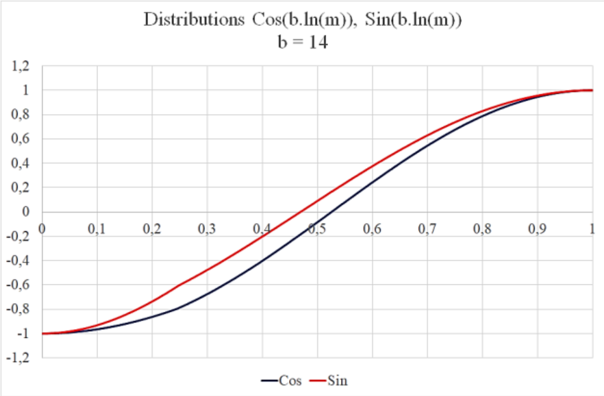
Let us take these values for two additional cases, with further examples provided in Appendix 12. Let us list the values of these two expressions (intimately linked by the sum of their squares). Then let us list them by increasing values. We get the graphs below. We observe that the distribution is not according to a fixed scheme for small values of b. It gradually tends however, as b increases, towards a unique sinusoidal distribution common to the elements of ∑cos(b.ln(m)) and those of ∑sin(b.ln(m)). The minimum -1 rule is necessarily subject to a certain strict framework. We note that this frame is the existence of this sinusoidal distribution. Thus, the deviation from the minimum rule can be acceptable up to the somewhat very approximate value b ≈ br1 (≈ 14). Beyond this region, a type distribution cos(π.(m/m_{max}+1)) settles permanently and will remain unique up to b infinite. Of course, truncation cannot be limited to m_{max} = 10000 terms when b increases.



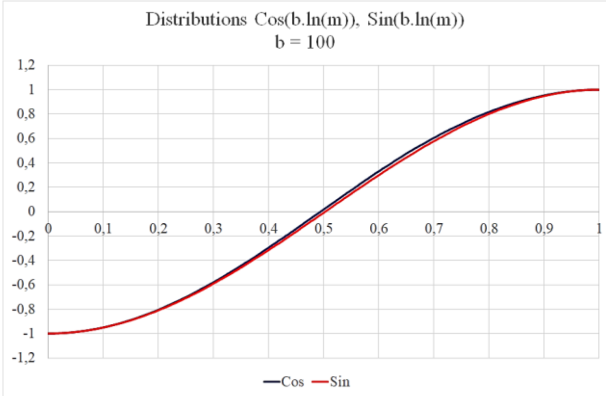
Graphic 58
a = 0.5 et b = 0



Graphic 59
a = 0.5 et b = 1.55



Graphic 60
a = 0.5 et b = 14



Graphic 61
a = 0.5 et b = 100

Of course also, except for b = 0, by taking a truncation with more terms (than 10000), we can find a sinusoidal profile for small values of b. But this takes place while the additional terms have only a negligible effect on the asymptotic value of R2x, the latter being essentially built on the first terms. The profile of the distribution must be "complete" in the useful truncation zone, where it has a real effect on the value of R2x (that is R2), otherwise it is strictly speaking effectively "incomplete".

Is the unique asymptotic distribution sufficient to impose R2x greater than -1 for b > br1 ?

In other words, how many b-values need to be checked before concluding that the minimum value cited is legitimate each time (and that we are in the presence of a theorem) ?

Well, to whom will object that this is only a few calculations on a tiny part of the values that can take b, we recall that the b-parameter is encapsulated in the cosine and sinus functions that can only take values between -1 to 1. The neighbourhoods of all values within this interval are reached thousands of times (for b < 20000 for example) and the functions are continuous. Of course, not all possibilities are covered, but the sample is quite representative of the whole system of equations. In addition, if the examples are necessarily specific, the relationships and thus conclusions are general.

Note: We do not say, however, that giving random values to cos(b.ln(m)) between -1 and 1 (with corresponding values deduced for sin(b.ln(m))) would give results for R2 greater than -1 because this is in fact not the case. It is necessary to have (cos(b.ln(m)), sin(b.ln(m))) and m = 0, 1, 2, etc. in this order in the equations for everything to work according to the expectation.

6.Conclusion.

We studied a convexity condition to confirm Riemann's hypothesis. This condition is a sufficient condition, meaning a violation, apart from that of the one already cited, would not necessarily deny the hypothesis. The way the proof was implemented makes it possible to calibrate the "distance" to a possible denial and shows a much to wide gap to this possibility. Several formulas, such as relationships (23), (24), (27), (36), (44) have been established implying geometric parameters links that impose the existence of the sole critical line for Riemann zeroes without any point departing from it. We used an approximate truncation method for assessing these parameters, with appendix 1 legitimizing it. It would be interesting, however, to find an alternative method similar to that used for the evaluation of Riemann's zeroes to determine the relationships, or points clouds, both much faster and with greater precision (see provision made in Appendix 9). The particular shape of the curves in graphic 14, the set of parameters leading to it and the relationship between them also deserve extra attention.

This done and said, in a thousand years, when another eminent reader, that the one who reads us here, will wake up, his wish will be all the more satisfied. If not, we will tell him : “ “Young man”, in mathematics, you don't understand things, you get used to them”. John Von Neumann.

References

[1] G.F.B. Riemann. Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse. Monatsberichte der Berliner Akademie. Nov 1859.

[2] Luis Báez-Duarte. Fast proof of functional equation for ζ(s). 14 May 2003 (arXiv math/0305191).

[3] http://fr.wikipedia.org/wiki/Hypothèse_de_Riemann.
http://fr.wikipedia.org/wiki/Fonction_zéta_de_Riemann

[4] https://fr.wikipedia.org/wiki/Fonction_zêta_de_Riemann#La_bande_critique_et_l'hypothèse_de_Riemann

[5] Database of L-functions, modular forms, and related objects.
https://www.lmfdb.org/zeros/zeta/

[6] The Siamese brothers of the Riemann zeroes. Hubert Schaetzel.
https://hubertschaetzel.wixsite.com/website. Dirichlet’s Sheet.

Appendix 1 : Truncation. Precision of evaluations.

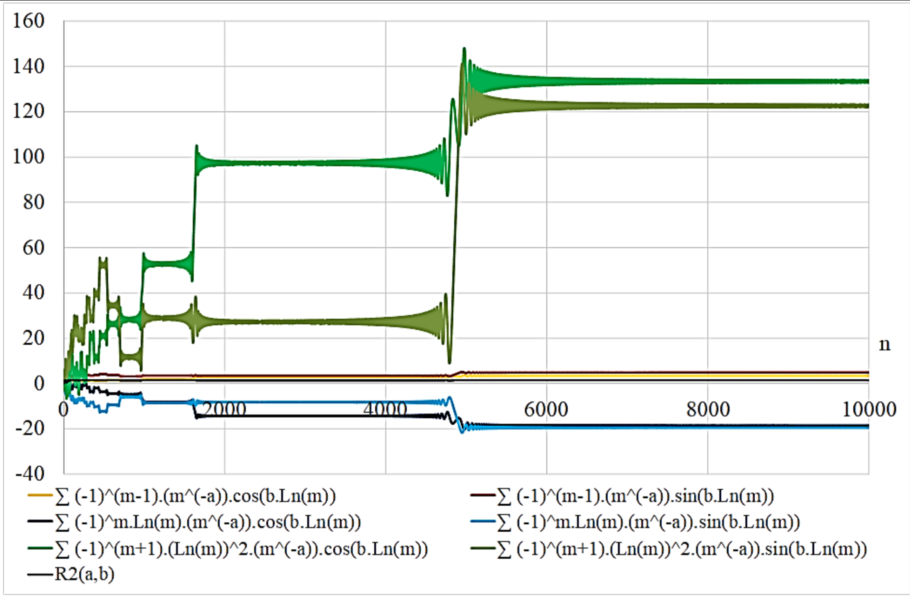
The numerical evaluations within of the body of text are drawn from expressions with infinite numbers of terms. They are based on approximations by truncation at a certain rank n. Expressions are the combinations of

$$Ck(a,b) = \lim_{n \to +\infty} \sum_{m=1}^{n} (-1)^{m-1+k}.(Ln(m))^k.m^{-a}.\cos(b.Ln(m))$$

and

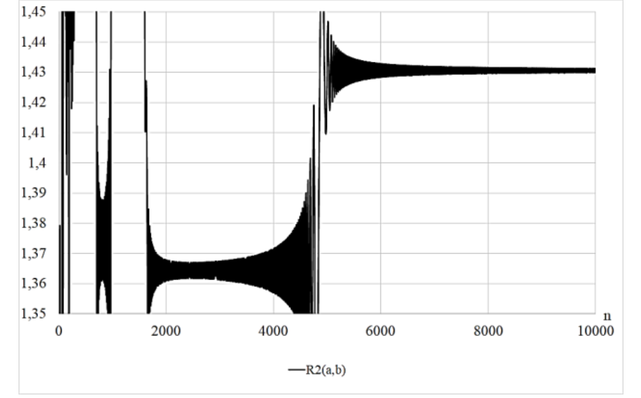
$$Sk(a,b) = \lim_{n \to +\infty} \sum_{m=1}^{n} (-1)^{m-1+k}.(Ln(m))^k.m^{-a}.\sin(b.Ln(m))$$

which shapes as a function of n is typically the followings :

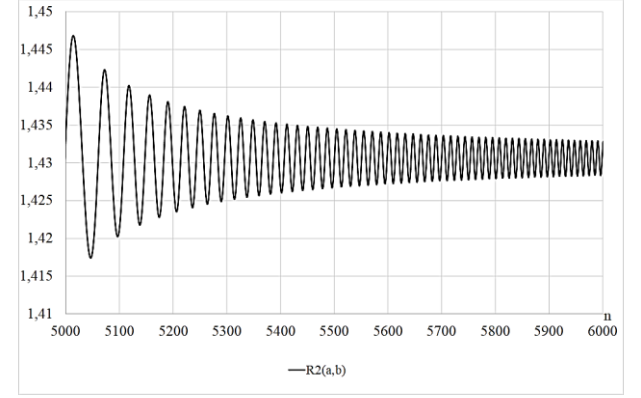


Graphic 62

a = 1/2, b = 15300



Graphic 63



Graphic 64

The particular look of these graphics can give the reader the impression that it is impossible to assess the value of expression to infinity. Indeed, leaps in values appear at abscissas that may seem random. What guarantee do we have here that a new jump will not arise somewhere asymptotically? To find out what is happening, it is necessary to trace the origin of these jumps. The sums we are talking about here are alternating sums. A jump comes from the fact that a given term is followed by a term of the same sign and this "many" times. So let us consider what produces the sign of two terms that follow each other. Within $(-1)^{m-1+k}.(Ln(m))^k.m^{-a}.\cos(b.Ln(m))$, neither Ln(m) in general nor m^{-a} have any effect on the change of sign. It remains therefore $(-1)^m.\cos(b.Ln(m))$, k being a constant term that can be eliminated. For two successive terms to be the same sign, it is sufficient asymptotically that $(-1)^m.\cos(b.Ln(m)) \approx (-1)^{m+1}.\cos(b.Ln(m+1))$ since Ln(m) and Ln(m+1) are of close values. From that, we deduce $\cos(b.Ln(m+1)) \approx -\cos(b.Ln(m))$, or $\cos(b.Ln(m+1)) \approx \cos(\pi+b.Ln(m))$, and then $b.Ln(m+1) \approx (1+2k).\pi+b.Ln(m)$, or finally :

$$b.Ln(1+1/m))/\pi \approx 1+2k$$

where $k \in Z$.
For $b > 0$ and $m > 0$, k is necessarily in N.
When $m \to +\infty$, and b has some given value, the product $b.Ln(1+1/m))/\pi \to 0$, so that the values of m for which $b.Ln(1+1/m))/\pi \approx 1$ are the last ones for which a jump occurs. The initial expression will converge after this last leap which intervenes at abscissa :

$$m \approx 1/(\exp(\pi/b)-1) \qquad (38)$$

In the case of the graphics 62 to 64 examples, $m \approx 1/(\exp(\pi/15300)-1) \approx 4870$.
The other jumps occur around m abscissas such as :

$$m \approx 1/(\exp((1+2k).\pi/b)-1)$$

That is for hereby example

Table 3

k	m
…	…
10	231
9	256
8	286
7	324
6	374
5	442
4	541
3	695
2	974
1	1623
0	4869

This table explains the "chaos" near the origin of the abscissas.

The so found expression also allows to give approximately the rank n sufficient, versus some b, to have a good asymptotic evaluation despite the truncation. Typically, one can chose 2 times the abscissa of the last jump :

$$n \approx 2/(\exp(\pi/b)-1)$$

or approximately when b is large enough in front of π (which is the case in general) :

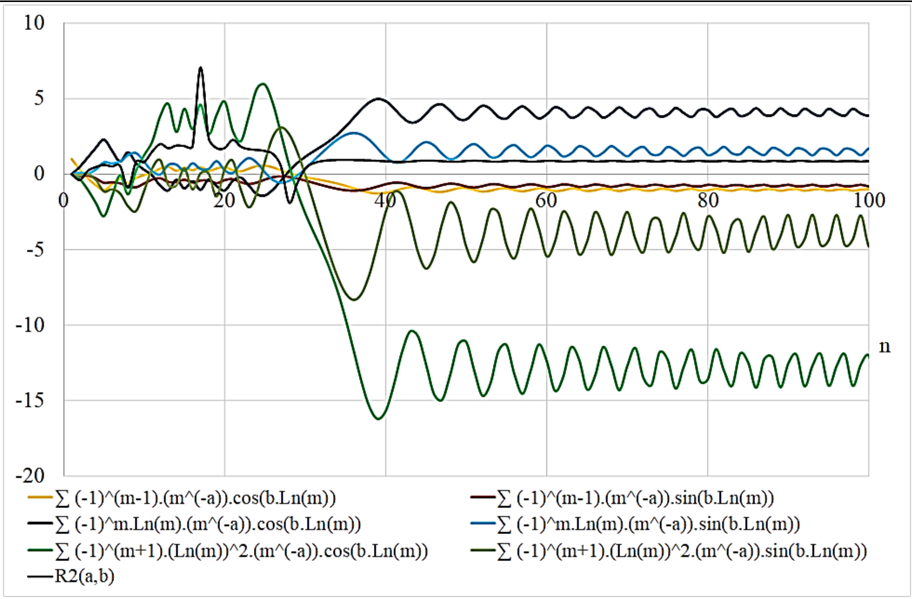
$$n \approx 2b/\pi$$

Table 4

Parameter b	Rank n
100	63
250	158
500	317
1000	636
2500	1591
5000	3182
10000	6365
25000	15914
50000	31830
100000	63661

Roughly speaking, the accuracy of the asymptotic evaluation therefore depends on a linear variation in the number of terms of the truncation with respect to b ($n \approx 0,6366198.b$).

The graphic below gives the example of $b = 100$.
The reader will therefore note, that this simple previous calculation does not apply to "small" b values ($b < 50$) due to the presence of significant oscillations. These particular cases are discussed in Appendix 2.



Graphic 65

a = 1/2, b = 100

For the sake of accuracy, all the calculations were conducted with 10000 terms except in the case of b > 100000 for which we used 100000 terms and even more when specified so.

When, on the contrary, we are interested in areas where the function studied is not subject to a jump but is close to a zero slope, the equation to be solved is $(-1)^m.\cos(b.\mathrm{Ln}(m)) \approx -(-1)^{m+1}.\cos(b.\mathrm{Ln}(m+1))$ and therefore :

$$b.\mathrm{Ln}(1+1/m))/\pi \approx 2k$$

The corresponding abscissas are :

$$m \approx 1/(\exp(2.\pi.k/b)-1)$$

So that for our example

Table 5

k	m
…	…
10	243
9	270
8	304
7	347
6	405
5	487
4	608
3	811
2	1217
1	2435
0	+∞

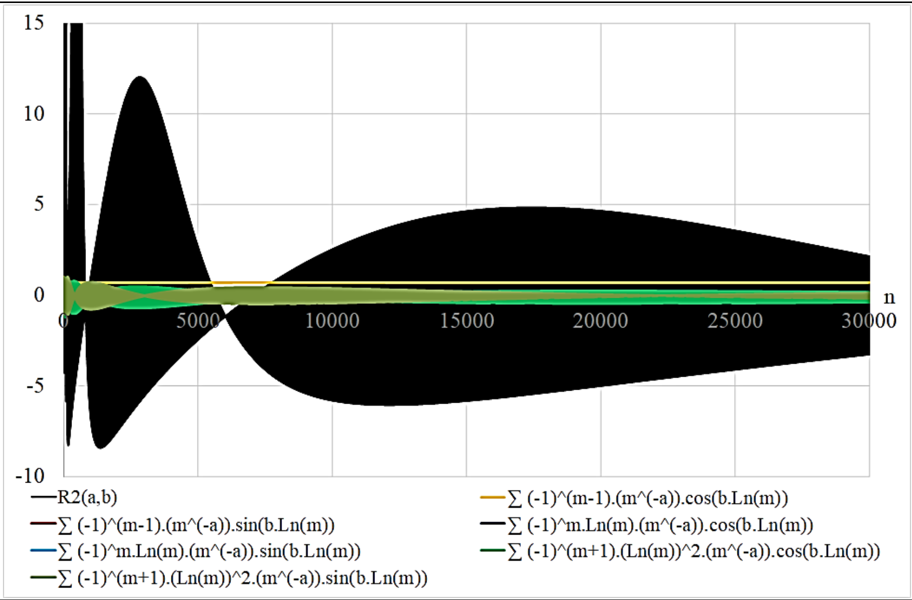
Let us note that for the sinus, the expressions of the sought abscissas result in exactly the same.

Finally, in view of graphic64, and directly related to the fact of having an alternating sum, the accuracy of the evaluation is subject to oscillations. Thus, the sum ∑ is corrected by half of the last term (or equivalently the average of the sum at ranks n-1 and n is made). Eventually, when necessary, the average of several terms in even number is made (up to 100 terms when b > 100000).

Appendix 2 : Case of low value abscissas b.

The range concerned here is before the occurrences of both the first Riemann zero and the first Dirichlet zero. This therefore has no bearing on the conclusions as to the Riemann hypothesis made in this text. However, it is studied for a simple reason : It is an exception to rule R2(a,b) ≥ -1 in a clear way.

We mentioned in the previous appendix the vigorous oscillations of functions studied for low b values. An example for b ≈ 1,569 (a = 1/2), is given below. As the reader can see, it is R2(a,b) that is subject to the greatest amplitudes.
It should also be noted that the example given corresponds to the minimum value of R2(a=1/2,b).



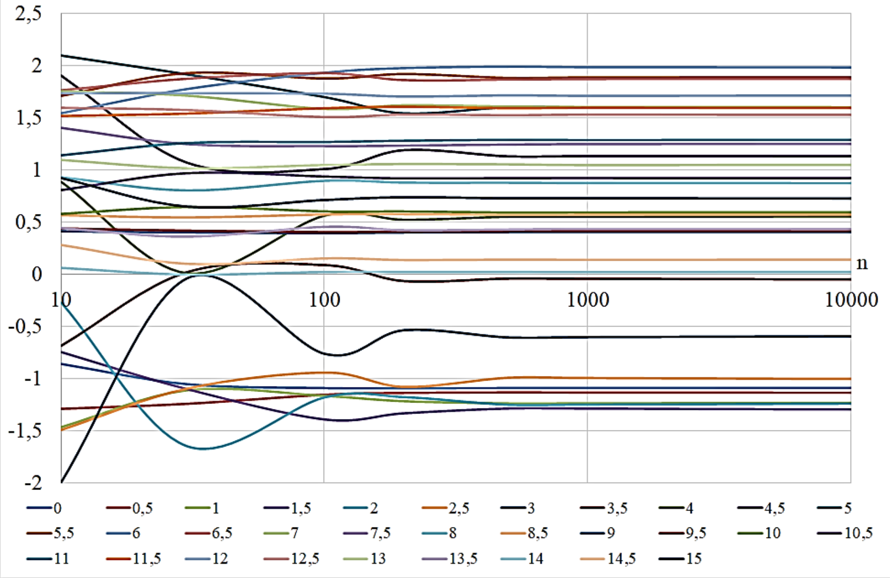
Graphic 66
a = 1/2, b = 1.569

The rule of the rank n ≈ 2/(exp(π/b)-1) for the truncation of the sums is no longer suitable (see also appendix 1). However, although strong oscillations are perpetuated beyond n equal to 100000 or even 1000000 (based on research not replicated here), a good approximation of the asymptotic value can be found using less than 1000 terms as shown in the data below simply by correcting the last term by half its value.
In fact, our usual 10000 terms are more than necessary.

Table 6

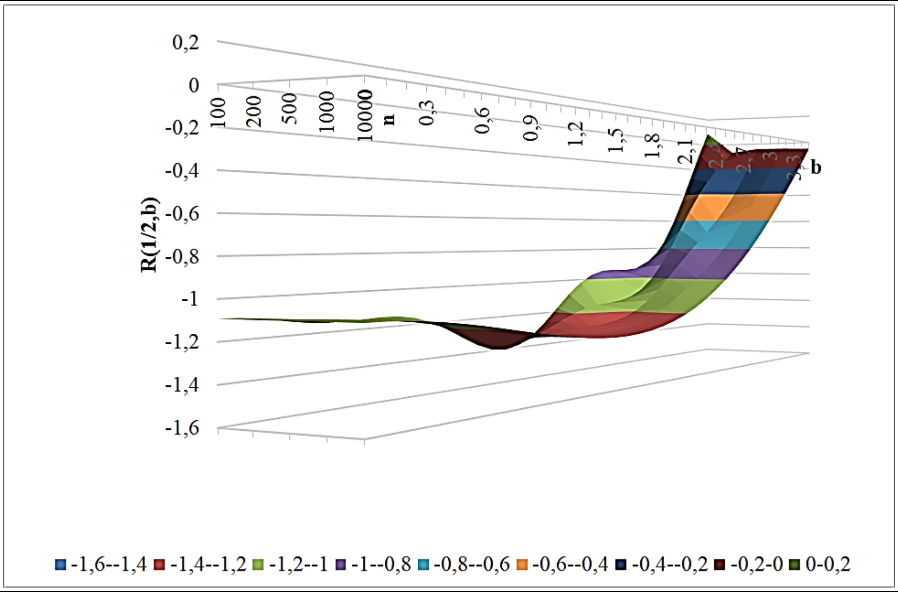
b \ n	10	30	100	200	500	1000	10000	30000
0	-0.85956	-1.05385	-1.09105	-1.09244	-1.09119	-1.09031	-1.08934	-1.08928
0.5	-1.28557	-1.23779	-1.15162	-1.13482	-1.13032	-1.13101	-1.13309	-1.13322
1	-1.46522	-1.10872	-1.16395	-1.21415	-1.23549	-1.23499	-1.22983	-1.22988
1.5	-0.74064	-1.09805	-1.39187	-1.33224	-1.28549	-1.28649	-1.29483	-1.29435
2	-0.27214	-1.64351	-1.17701	-1.17433	-1.24905	-1.24678	-1.23744	-1.23738
2.5	-1.48644	-1.09573	-0.94321	-1.07924	-0.99159	-0.995	-1.00367	-1.00449
3	-1.98609	-0.03799	-0.7605	-0.5338	-0.6039	-0.60031	-0.59258	-0.59259
3.5	-0.68545	0.022304	0.090951	-0.06263	-0.03853	-0.04072	-0.04684	-0.04599
4	0.892506	0.011444	0.568647	0.527649	0.556349	0.556195	0.559474	0.559301
4.5	1.904682	1.073586	1.013366	1.193721	1.133629	1.136235	1.13581	1.135286
5	2.096723	1.917201	1.70033	1.545182	1.605281	1.600985	1.599867	1.600158
5.5	1.712557	1.933361	1.88094	1.9242	1.886159	1.891117	1.892714	1.892854
6	1.547342	1.777149	1.932112	1.976638	1.990187	1.985294	1.983331	1.983066
6.5	1.769115	1.876386	1.927718	1.864439	1.868721	1.873344	1.875756	1.875847
7	1.762121	1.720717	1.59327	1.628834	1.61671	1.612395	1.609922	1.610079
7.5	1.406636	1.249643	1.23055	1.235197	1.245458	1.249354	1.251331	1.251174
8	0.936602	0.806794	0.902373	0.88496	0.88363	0.880682	0.879466	0.87945
8.5	0.570413	0.548674	0.577347	0.579683	0.574039	0.575494	0.575958	0.576033
9	0.412599	0.405027	0.400975	0.404862	0.407771	0.407947	0.408194	0.408146

n / b	10	30	100	200	500	1000	10000	30000
9.5	0.444887	0.421798	0.404804	0.4112	0.417511	0.416348	0.415433	0.415489
10	0.584259	0.648408	0.607762	0.607423	0.597503	0.598429	0.599839	0.599811
10.5	0.806697	0.968425	0.936422	0.918462	0.920861	0.921329	0.919697	0.919642
11	1.145472	1.261794	1.271709	1.283515	1.291946	1.289679	1.291374	1.291465
11.5	1.523251	1.546281	1.597019	1.609937	1.597047	1.600738	1.598947	1.598899
12	1.736473	1.736638	1.731341	1.704849	1.716336	1.711668	1.713658	1.713612
12.5	1.602475	1.580445	1.513442	1.537447	1.530961	1.536661	1.534524	1.534641
13	1.097688	1.023908	1.051837	1.060742	1.05814	1.051752	1.053757	1.053705
13.5	0.444616	0.366165	0.456156	0.425299	0.431911	0.436701	0.435334	0.435271
14	0.067268	0.002344	0.027503	0.026967	0.026939	0.02624	0.026498	0.026525
14.5	0.280177	0.105767	0.155764	0.141745	0.144891	0.143578	0.14431	0.14433
15	0.920883	0.650266	0.713763	0.739415	0.729604	0.728874	0.727859	0.727941



Graphic 67
a = 1/2, b = 0 to 15

The threshold R2(a=1/2,b) > -0.5 comes up around b ≈ 3.093.



Graphic 68
a = 1/2, b = 3.093, n = 100 to 10000

Appendix 3 : Table of data (r_M, r_{peak}).

a = 1/2.

Tableau 7

b_{peak}	r_{peak}	r_{peak} computed		b_{peak}	r_{peak}	r_{peak} computed
b_{low1}	r_{low1}	r_M		b_{low1}	r_{low1}	r_M
b_{low2}	r_{low2}	($r_{peak\ computed}$ - r_{peak})/r_{peak} in %		b_{low2}	r_{low2}	($r_{peak\ computed}$ - r_{peak})/r_{peak} in %
3002.618200	1.359173	4.01781474		6169.467230	27.345310	29.4536211
3001.771300	-0.096865	-0.08811628		6169.334230	-0.256178	-0.28244545
3003.417000	-0.079367	195.6%		6169.610730	-0.308713	7.7%
3006.331500	1.634847	2.0509957		6221.504800	7.551661	8.35691952
3005.786400	-0.004200	-0.03692003		6221.305800	-0.092370	-0.15838718
3006.941200	-0.069640	25.5%		6221.719400	-0.224404	10.7%
3129.626100	7.052130	7.51220619		6247.471100	2.924391	2.17021567
3129.392300	-0.187105	-0.14759275		6247.173400	-0.071029	-0.04060265
3129.850600	-0.108081	6.5%		6247.782000	-0.010177	-25.8%
3210.503600	7.080494	7.27977773		7005.081792	340.003331	363.985697
3210.276300	-0.195948	-0.14443203		7005.016468	-0.415558	-0.4296605
3210.718400	-0.092916	2.8%		7005.154610	-0.443763	7.1%
3230.167000	6.245951	6.14068648		9003.959300	16.217260	16.5248074
3229.932700	-0.084404	-0.12756089		9003.799700	-0.292237	-0.22782277
3230.411000	-0.170718	-1.7%		9004.100100	-0.163408	1.9%
4108.734600	12.558510	12.6205689		9006.100700	1.710784	1.19891661
4108.548300	-0.246659	-0.20060678		9005.625800	-0.010883	-0.00763005
4108.907100	-0.154555	0.5%		9006.554200	-0.004378	-29.9%
4474.251404	35.369779	39.0518577		9059.798700	12.578443	14.6135309
4474.115304	-0.312496	-0.30647162		9059.631600	-0.211381	-0.21550016
4474.386104	-0.300448	10.4%		9059.966900	-0.219620	16.2%
4990.396680	51.082057	51.1197566		11705.671515	82.890826	78.4779095
4990.270000	-0.372382	-0.32754631		11705.569200	-0.407862	-0.35725835
4990.504780	-0.282711	0.1%		11705.754715	-0.306654	-5.3%
6001.830700	2.203104	1.26225366		12000.173100	5.951227	6.4168697
6001.478400	-0.011755	-0.01000773		11999.971300	-0.136701	-0.1318787
6002.169100	-0.008260	-42.7%		12000.374600	-0.127057	7.8%
6014.952800	3.517755	3.23003171		12000.943800	1.308211	3.36710657
6014.662400	0.000667	-0.06979119		12000.374600	-0.127057	-0.07316187
6015.230600	-0.140249	-8.2%		12001.847600	-0.019267	157.4%
6093.237892	69.638348	72.1312585		12002.197000	2.014218	1.78808707
6093.129592	-0.352250	-0.35178109		12001.847600	-0.019267	-0.02846429
6093.345992	-0.351312	3.6%		12002.479600	-0.037661	-11.2%
6139.700600	12.040684	11.9738615		12002.950400	1.757631	1.51715263
6139.524900	-0.206133	-0.19522809		12002.479600	-0.037661	-0.01922911
6139.871900	-0.184324	-0.6%		12003.318800	-0.000797	-13.7%
6161.598900	4.713395	4.57029263		12003.695800	2.175975	1.5710824
6161.351400	-0.194926	-0.09961525		12003.318800	-0.000797	-0.0211121
6161.837200	-0.004304	-3.0%		12004.018900	-0.041427	-27.8%

b_{peak}	r_{peak}	r_{peak} computed		b_{peak}	r_{peak}	r_{peak} computed
b_{low1}	r_{low1}	r_M		b_{low1}	r_{low1}	r_M
b_{low2}	r_{low2}	($r_{peak\ computed}$ - r_{peak})/r_{peak} in %		b_{low2}	r_{low2}	($r_{peak\ computed}$ - r_{peak})/r_{peak} in %
12006.362900	3.555481	2.8667187		17144.469200	1.828259	17.1765337
12006.110900	-0.117381	-0.0604418		17143.863100	-0.453107	-0.23166147
12006.624500	-0.003503	-19.4%		17144.814700	-0.010216	839.5%
12034.390500	4.290657	3.76902467		17366.547404	112.104669	108.875545
12034.155500	-0.077290	-0.0825924		17366.464800	-0.385264	-0.37687899
12034.626900	-0.087895	-12.2%		17366.626804	-0.368494	-2.9%
12080.850300	19.415848	17.8938103		18017.866410	78.328211	78.0836037
12080.718800	-0.216104	-0.23569829		18017.762400	-0.404163	-0.35693617
12080.987400	-0.255292	-7.8%		18017.950110	-0.309709	-0.3%
12139.152200	15.562344	15.0146792		25704.555998	101.538403	104.587834
12139.021700	-0.135245	-0.21822987		25704.472400	-0.378628	-0.37460813
12139.306200	-0.301215	-3.5%		25704.637798	-0.370589	3.0%
12154.092800	5.518834	6.2277013		33179.383619	235.251553	259.985908
12153.877600	-0.193953	-0.12893817		33179.325819	-0.392109	-0.41764894
12154.298700	-0.063923	12.8%		33179.451000	-0.443189	10.5%
12224.698400	62.781801	61.8836669		36510.181139	397.063404	426.909465
12224.589100	-0.378055	-0.34137021		36510.123700	-0.440079	-0.43474168
12224.793800	-0.304686	-1.4%		36510.236050	-0.429404	7.5%
12232.205100	17.898485	19.3031213		50965.883362	308.438872	309.327147
12232.055100	-0.281015	-0.24311088		50965.827610	-0.414817	-0.42408016
12232.342700	-0.205207	7.8%		50965.942980	-0.433343	0.3%
14334.247440	37.891963	37.9563388		57273.674907	473.483908	487.60865
14334.131540	-0.319945	-0.30413684		57273.627600	-0.421399	-0.43870276
14334.357740	-0.288329	0.2%		57273.729747	-0.456007	3.0%
15032.366600	3.432032	1.97486777		63137.222002	721.842318	698.901883
15032.118200	0.001073	-0.03452005		63137.182402	-0.425509	-0.4482838
15032.601000	-0.070113	-42.5%		63137.272052	-0.471059	-3.2%
15132.485400	18.420514	21.7803598		66678.085608	840.428101	805.983424
15132.345400	-0.255724	-0.25471431		66678.041828	-0.454095	-0.45165618
15132.625800	-0.253705	18.2%		66678.127951	-0.449217	-4.1%
15471.584480	107.439301	108.07436		71732.908569	1454.109584	1437.89763
15471.492480	-0.393990	-0.37646434		71732.872599	-0.455031	-0.46325596
15471.669680	-0.358939	0.6%		71732.949009	-0.471481	-1.1%
17143.298900	2.345076	20.9315022		85877.882424	402.544511	401.14131
17143.017000	-0.071725	-0.25092307		85877.833720	-0.429200	-0.43280181
17143.752800	-0.430121	792.6%		85877.932490	-0.436404	-0.3%
17143.804309	366.725948	317.714636		139735.509308	553.753874	564.984215
17143.745900	-0.408860	-0.42502625		139735.456808	-0.474679	-0.44281671
17143.869200	-0.441192	-13.4%		139735.548398	-0.410954	2.0%

Appendix 4 : Functions Rk(a,b).

In this appendix, we deviate somewhat from our goal by the fact that we are not only focusing on R2(a,b), but also on the related Rk = Rk(a,b) functions :

$$R_k(a,b) = \frac{C_{k-2}(a,b).Ck(a,b)+S_{k-2}(a,b).S_k(a,b)}{(C_{k-1}(a,b))^2+(S_{k-1}(a,b))^2} \qquad (39)$$

We thus have :

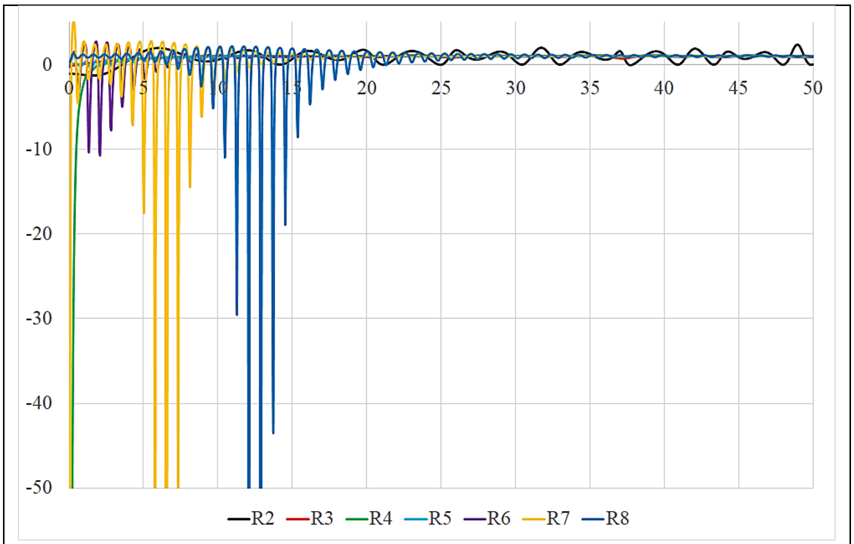
$$R2(a,b) = \frac{C0(a,b).C2(a,b)+S0(a,b).S2(a,b)}{(C1(a,b))^2+(S1(a,b))^2} \qquad (40)$$

$$R3(a,b) = \frac{C1(a,b).C3(a,b)+S1(a,b).S3(a,b)}{(C2(a,b))^2+(S2(a,b))^2} \qquad (41)$$

and so on.

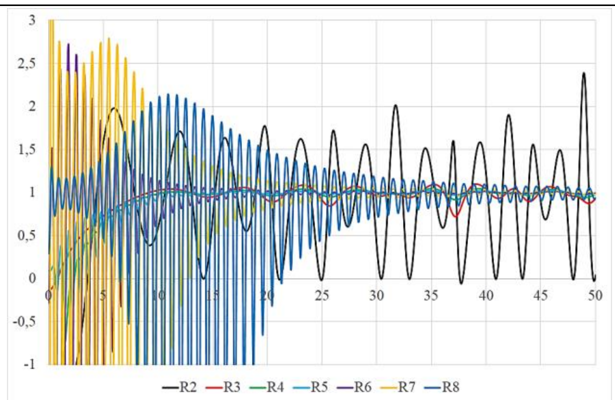
The reader will refer to relationships (14) and (15) for the definition of Ck(a,b) and Sk(a,b).
In fact, the k = 2 case, from previous studies, is constituting a kind of initial boundary case. We observe the effect of the increase in power affecting the Napierian logarithm of that starting expression.

The aim is of course to check whether Rk(a,b) functions, especially its negative values, can instruct us on the R2(a,b) reference function.

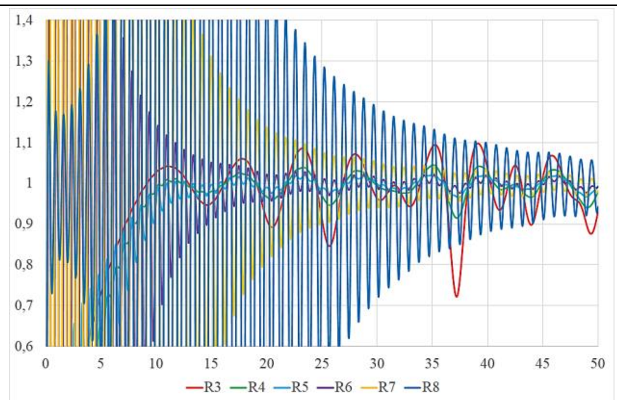


Graphic 69
a = 1/2, b ∈ [0.50]

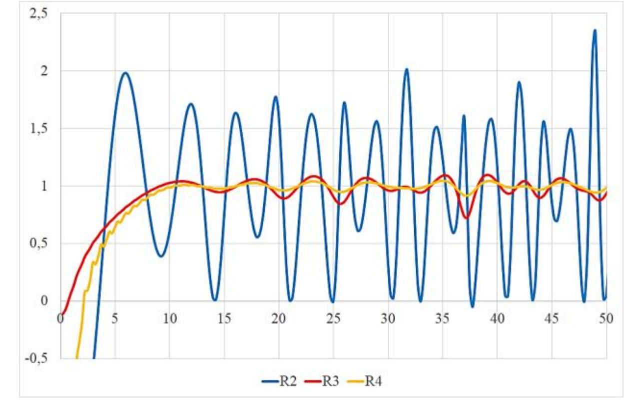
The undulations of the Rk(a,b) functions, particularly on the negative values side, are much broader than those of the initial R2(a,b) function in the b < 20 zone, recalling that this area (near origin and without adverse impact on the study) is also a source of exceptional behaviour for R2(a,b).



Graphic 70

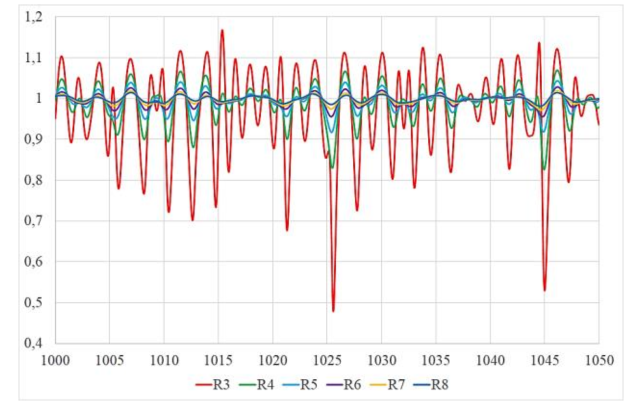


Graphic 71



Graphic 72
a = 1/2, b = 0 to 50

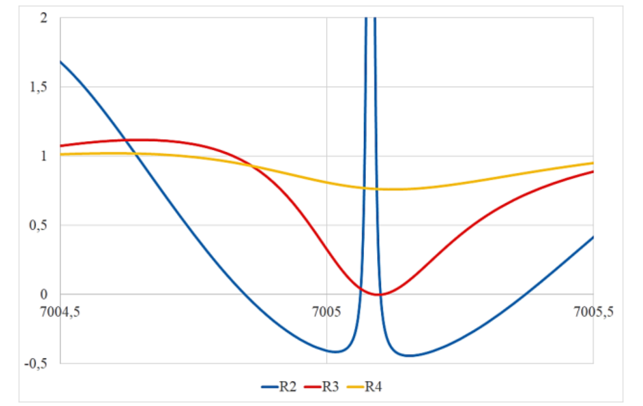
Past the zone of low b values, the Rk(a,b) functions are getting closer and closer to the horizontal y = 1 axis as k increases. For large enough k, it is therefore likely that any negative value of Rk(a,b) does exist beyond abscissa b = 20.



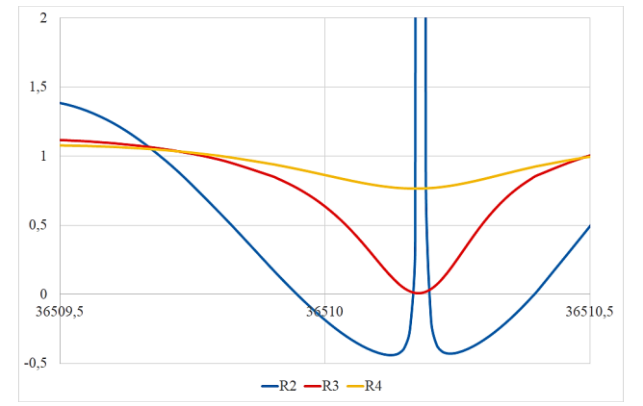
Graphic 73

Graphic 74

Thus, just as R2 has a minimum (in the order of -1/2 when b > 5), the minimum of the R3 function seems to be likely around 0 (excluded in the figures below).



Graphic 75
a = 1/2, b = 7004.5 to 7005.5



Graphic 76
a = 1/2, b = 36509.5 to 36510.5

Appendix 5 : Numeric data for r_M and r_{peak}.

The only case of undershooting -0.5 for R2(1/2,b) is shown in red font in the table below.
The values of the peaks are given as mere indicative information (see Appendix 9).

Table 8
Truncation to 200000 terms.

abs_r_M-	abs_zeroR-	Δpr-	abs_peak	abs_zeroR+	abs_r_M+	Δpr+	r_M-	r_M+	r_M	zeroes gap	value_peak
273193,64713	273193,66314	0,01601	273193,66600	273193,66884	273193,69030	0,02146	-0,47529	-0,49284	-0,48406	0,00570	9333
270071,27520	270071,29406	0,01886	270071,29840	270071,30270	270071,33260	0,02990	-0,45654	-0,49562	-0,47608	0,00864	2910
302700,28161	302700,30084	0,01923	302700,30563	302700,31040	302700,34210	0,03170	-0,45149	-0,49558	-0,47354	0,00956	2274
234016,87060	234016,89498	0,02438	234016,90151	234016,90804	234016,93620	0,02816	-0,45482	-0,46989	-0,46236	0,01305	1528
71732,87260	71732,90121	0,02861	71732,90857	71732,91591	71732,94901	0,03310	-0,45503	-0,47148	-0,46326	0,01470	1454
66678,04183	66678,07586	0,03403	66678,08561	66678,09534	66678,12795	0,03261	-0,45409	-0,44922	-0,45166	0,01948	840
63137,18240	63137,21153	0,02913	63137,22200	63137,23238	63137,27205	0,03967	-0,42551	-0,47106	-0,44828	0,02085	722
139735,45681	139735,49829	0,04148	139735,50931	139735,52048	139735,54840	0,02792	-0,47468	-0,41095	-0,44282	0,02219	554
57273,62760	57273,66193	0,03433	57273,67491	57273,68777	57273,72975	0,04198	-0,42140	-0,45601	-0,43870	0,02584	473
123474,56761	123474,60963	0,04202	123474,62274	123474,63601	123474,66770	0,03169	-0,45895	-0,40894	-0,43394	0,02638	410
85877,83372	85877,86915	0,03543	85877,88242	85877,89568	85877,93249	0,03681	-0,42920	-0,43640	-0,43280	0,02653	403
135079,60050	135079,63490	0,03440	135079,64864	135079,66231	135079,70037	0,03806	-0,42005	-0,43887	-0,42946	0,02741	367
109565,91390	109565,94581	0,03191	109565,95961	109565,97330	109566,01218	0,03888	-0,40796	-0,44458	-0,42627	0,02749	298
78974,77310	78974,79335	0,02025	78974,80805	78974,82196	78974,87502	0,05306	-0,31586	**-0,51209**	**-0,41397**	0,02861	382
123377,76550	123377,80717	0,04167	123377,82151	123377,83602	123377,86850	0,03248	-0,44950	-0,40157	-0,42554	0,02884	321
122031,68080	122031,72052	0,03972	122031,73518	122031,74992	122031,78477	0,03485	-0,43828	-0,41194	-0,42511	0,02940	322
36510,12370	36510,16639	0,04269	36510,18114	36510,19592	36510,23605	0,04013	-0,44008	-0,42940	-0,43474	0,02954	397
116527,20102	116527,23373	0,03271	116527,24873	116527,26361	116527,30070	0,03709	-0,40259	-0,42944	-0,41601	0,02988	188
91686,05548	91686,09871	0,04324	91686,11421	91686,12986	91686,16475	0,03489	-0,44486	-0,40254	-0,42370	0,03115	306
99658,87400	99658,91664	0,04264	99658,93213	99658,94787	99658,98050	0,03263	-0,44706	-0,39077	-0,41892	0,03123	278
107457,27870	107457,31325	0,03455	107457,32925	107457,34502	107457,39000	0,04498	-0,39677	-0,45003	-0,42340	0,03177	286
50965,82761	50965,86687	0,03926	50965,88336	50965,89978	50965,94298	0,04320	-0,41482	-0,43334	-0,42408	0,03291	308
105639,02740	105639,06546	0,03806	105639,08244	105639,09935	105639,13990	0,04055	-0,40893	-0,42340	-0,41616	0,03389	228
139456,17730	139456,20826	0,03096	139456,22548	139456,24221	139456,29230	0,05009	-0,36418	-0,46784	-0,41601	0,03395	239
130640,15568	130640,19831	0,04263	130640,21530	130640,23262	130640,26430	0,03168	-0,44057	-0,37428	-0,40742	0,03431	174
56646,87565	56646,91430	0,03865	56646,93153	56646,94865	56646,99243	0,04378	-0,40679	-0,43228	-0,41954	0,03435	258
104605,47909	104605,52221	0,04312	104605,53945	104605,55693	104605,59160	0,03467	-0,43549	-0,38764	-0,41157	0,03472	223
106665,22879	106665,27119	0,04240	106665,28845	106665,30592	106665,33890	0,03298	-0,43689	-0,37882	-0,40785	0,03473	192
118523,18580	118523,21930	0,03350	118523,23695	118523,25420	118523,30210	0,04790	-0,37555	-0,45410	-0,41483	0,03490	223
72677,17939	72677,22772	0,04833	72677,24509	72677,26284	72677,29697	0,03413	-0,45277	-0,37826	-0,41552	0,03512	249
17143,74590	17143,78654	0,04064	17143,80431	17143,82184	17143,86920	0,04736	-0,40886	-0,44119	-0,42503	0,03531	367
113224,63253	113224,66735	0,03482	113224,68517	113224,70273	113224,74673	0,04400	-0,38537	-0,43546	-0,41042	0,03539	222
70902,92790	70902,96757	0,03967	70902,98561	70903,00356	70903,04550	0,04193	-0,40738	-0,42031	-0,41385	0,03599	222
33179,32582	33179,36529	0,03948	33179,38362	33179,40158	33179,45100	0,04942	-0,39211	-0,44319	-0,41765	0,03628	235
118935,46690	118935,51144	0,04454	118935,52978	118935,54839	118935,58380	0,03541	-0,43305	-0,38251	-0,40778	0,03695	207
134985,54371	134985,58931	0,04560	134985,60765	134985,62632	134985,66090	0,03458	-0,43848	-0,37598	-0,40723	0,03701	200
91166,84014	91166,87802	0,03788	91166,89686	91166,91555	91166,95880	0,04325	-0,39255	-0,42196	-0,40726	0,03754	206
7005,01647	7005,06287	0,04640	7005,08179	7005,10056	7005,15461	0,05405	-0,41556	-0,44376	-0,42966	0,03770	340
104230,44079	104230,47771	0,03692	104230,49692	104230,51592	104230,55943	0,04351	-0,38567	-0,42341	-0,40454	0,03821	199
73146,92828	73146,96705	0,03877	73146,98646	73147,00573	73147,04856	0,04283	-0,39373	-0,41702	-0,40537	0,03868	209
52126,13432	52126,18544	0,05112	52126,20472	52126,22440	52126,26151	0,03711	-0,44741	-0,37682	-0,41212	0,03896	219
42525,75198	42525,79594	0,04395	42525,81557	42525,83517	42525,87994	0,04477	-0,41109	-0,41558	-0,41333	0,03923	223
40094,91230	40094,94904	0,03674	40094,96917	40094,98891	40095,03817	0,04927	-0,37307	-0,43906	-0,40606	0,03987	206

Table 9

Truncation to 200000 terms

abs_r_M-	abs_zeroR-	Δpr-	abs_peak	abs_zeroR+	abs_r_M+	Δpr+	r_M-	r_M+	r_M	zeroes gap	value_peak
273193.64713	273193.66314	0.01601	273193.66600	273193.66884	273193.69030	0.02146	-0.47529	-0.49284	-0.48406	0.00570	9333
68398.90460	68398.95599	0.05139	68398.98100	68399.00662	68399.04519	0.03857	-0.41900	-0.34532	-0.38216	0.05063	122
69035.15059	69035.20719	0.05660	69035.25610	69035.30735	69035.34845	0.04110	-0.33788	-0.24115	-0.28952	0.10015	30.88
56666.98370	56667.04491	0.06121	56667.11754	56667.19497	56667.23917	0.04420	-0.28543	-0.18490	-0.23517	0.15006	15.60
59404.67366	59404.71336	0.03970	59404.81787	59404.91338	59404.97320	0.05982	-0.12590	-0.23442	-0.18016	0.20002	9.243
32417.35933	32417.39493	0.03560	32417.52684	32417.64496	32417.70526	0.06030	-0.08628	-0.19976	-0.14302	0.25003	6.884
26041.07526	26041.12896	0.05370	26041.27057	26041.42897	26041.46147	0.03250	-0.15003	-0.06325	-0.10664	0.30001	5.237
32907.10105	32907.12525	0.02420	32907.31255	32907.47525	32907.52535	0.05010	-0.03367	-0.12197	-0.07782	0.35000	4.123
46217.24820	46217.28890	0.04070	46217.47501	46217.68893	46217.70752	0.01859	-0.08138	-0.01929	-0.05034	0.40003	3.311
16183.81710	16183.84543	0.02833	16184.07443	16184.29543	16184.32783	0.03240	-0.03448	-0.04556	-0.04002	0.45000	3.293
35015.36860	35015.41941	0.05081	35015.61128	35015.91942	35015.90230	-0.01712	-0.11123	-0.01760	-0.06442	0.50001	2.682
71084.67160	71084.70208	0.03048	71084.92200	71085.25209	71085.25223	0.00014	-0.04840	-0.00004	-0.02422	0.55000	1.942
29126.11340	29126.08620	-0.02720	29126.46100	29126.68621	29126.72951	0.04330	-0.04117	-0.07512	-0.05815	0.60000	2.294
57478.81015	57478.85675	0.04660	57479.04785	57479.50675	57479.50135	-0.00540	-0.10261	-0.00166	-0.05214	0.65000	1.696
61432.66359	61432.66409	0.00050	61433.01420	61433.36411	61433.36081	-0.00330	-0.00004	-0.00054	-0.00029	0.70002	1.758
17205.58085	17205.58795	0.00710	17205.93718	17206.33796	17206.31526	-0.02270	-0.00202	-0.02515	-0.01359	0.75000	1.848
67219.89461	67219.87721	-0.01740	67220.26660	67220.67721	67220.65411	-0.02310	-0.01546	-0.02771	-0.02158	0.80000	1.692
61148.29712	61148.25912	-0.03800	61148.74032	61149.10913	61149.10746	-0.00167	-0.08215	-0.00014	-0.04114	0.85000	1.620
6612.02748	6612.00798	-0.01950	6612.49020	6612.90798	6612.91678	0.00880	-0.01191	-0.00204	-0.00697	0.90000	1.835
8875.18991	8875.17281	-0.01710	8875.69210	8876.12284	8876.12184	-0.00100	-0.00959	-0.00003	-0.00481	0.95002	1.649
1512.59584	1512.58976	-0.00607	1513.04176	1513.58977	1513.60217	0.01240	-0.00092	-0.00355	-0.00223	1.00000	1.741

Appendix 6 : Numeric data DSC1 and DSC(X).

Table 10

abs_peak	abs_zeroR-	abs_zeroR+	abs_r_M-	DSC1-	abs_r_M+	DSC1+	Δb_r	Average(DSC1)
273193.66600	273193.66314	273193.66884	273193.64683	37.297	273193.69084	22.220	0.0057	29.758
270071.29840	270071.29406	270071.30270	270071.27527	30.188	270071.33280	13.288	0.00864	21.738
302700.30563	302700.30084	302700.31040	302700.28170	27.649	302700.34180	11.831	0.00956	19.740
71732.90857	71732.90121	71732.91591	71732.87260	16.951	71732.94901	13.245	0.0147	15.098
57273.67491	57273.66193	57273.68777	57273.62760	11.921	57273.72975	8.601	0.02584	10.261
68398.98100	68398.95599	68399.00662	68398.90460	5.708	68399.04519	8.940	0.05063	7.324
69035.25610	69035.20719	69035.30735	69035.15059	4.572	69035.34845	7.286	0.10016	5.929
56667.11754	56667.04491	56667.19497	56666.98370	4.216	56667.23917	6.860	0.15006	5.538
59404.81787	59404.71336	59404.91338	59404.67366	7.938	59404.97320	4.263	0.20002	6.101
32417.52684	32417.39493	32417.64496	32417.35933	10.184	32417.70526	4.498	0.25003	7.341
26041.27057	26041.12896	26041.42897	26041.07526	5.417	26041.46147	11.975	0.30001	8.696
32907.31255	32907.12525	32907.47525	32907.10105	20.331	32907.52535	5.853	0.35	13.092
46217.47501	46217.28890	46217.68893	46217.24820	7.749	46217.70752	31.905	0.40003	19.827
16184.07443	16183.84543	16184.29543	16183.81710	17.891	16184.32783	14.070	0.45	15.980
35015.61128	35015.41941	35015.91942	35015.36860	5.452	35015.90230	35.480	0.50001	20.466
71084.92200	71084.70208	71085.25209	71084.67160	10.179	71085.25223	276451.850	0.55001	138231.015
29126.46100	29126.08620	29126.68621	29126.11340	15.585	29126.72951	7.270	0.60001	11.427
57479.04785	57478.85675	57479.50675	57478.81015	4.813	57479.50135	184.527	0.65	94.670
61433.01420	61432.66409	61433.36411	61432.66359	34979.548	61433.36081	881.129	0.70002	17930.339
17205.93718	17205.58795	17206.33796	17205.58085	203.069	17206.31526	24.982	0.75001	114.025
67220.26660	67219.87721	67220.67721	67219.89461	33.764	67220.65411	19.667	0.8	26.715
61148.74032	61148.25912	61149.10913	61148.29712	8.171	61149.10746	3385.101	0.85001	1696.636
6612.49020	6612.00798	6612.90798	6612.02748	44.805	6612.91678	214.978	0.9	129.892
8875.69210	8875.17281	8876.12284	8875.18991	43.426	8876.12184	14604.454	0.95003	7323.940
1513.04176	1512.58976	1513.58977	1512.59584	586.860	1513.60217	70.703	1.00001	328.782

Table 11

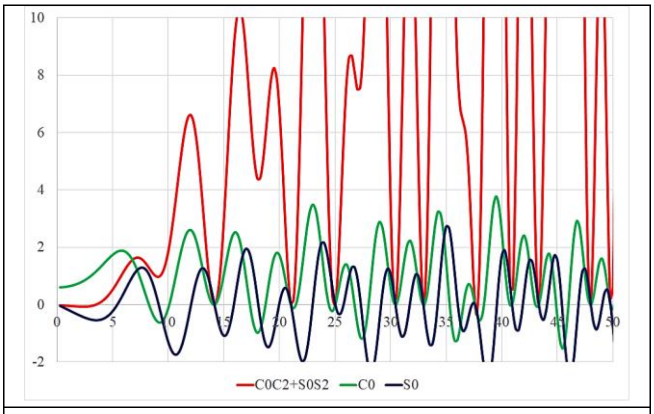
DSC(R2y)

rM	rM-	rM+	rM-	rM+	rM-	rM+	rM-	rM+	rM-	rM+	rM-	rM+
R2y	-0.5	-0.5	-0.6	-0.6	-0.7	-0.7	-0.8	-0.8	-0.9	-0.9	-1	-1
Δb_r	DSC(…)											
0.0057	1.053	0.995	2.937	1.842	7.503	4.388	14.752	8.633	24.683	14.577	37.297	22.220
0.00864	1.131	0.994	2.887	1.411	6.670	2.849	12.481	5.308	20.320	8.788	30.188	13.288
0.00956	1.148	0.995	2.805	1.344	6.284	2.602	11.584	4.769	18.706	7.845	27.649	11.831
0.02584	1.106	0.991	1.869	1.379	3.332	2.335	5.494	3.857	8.357	5.945	11.921	8.601
0.05063	0.984	1.274	1.266	1.979	1.879	3.098	2.824	4.631	4.100	6.578	5.708	8.940
0.10016	1.011	1.469	1.297	2.115	1.796	3.019	2.508	4.183	3.434	5.605	4.572	7.286
0.15006	1.028	1.551	1.316	2.178	1.779	3.022	2.417	4.084	3.229	5.363	4.216	6.860
0.20002	1.895	1.076	2.660	1.396	3.647	1.875	4.856	2.512	6.286	3.308	7.938	4.263
0.25003	2.481	1.142	3.504	1.501	4.786	2.016	6.327	2.687	8.126	3.515	10.184	4.498
0.30001	1.357	2.966	1.835	4.190	2.480	5.704	3.292	7.506	4.271	9.596	5.417	11.975
0.35	5.054	1.481	7.204	2.015	9.807	2.720	12.863	3.594	16.370	4.639	20.331	5.853
0.40003	1.952	7.957	2.706	11.383	3.663	15.491	4.822	20.281	6.184	25.752	7.749	31.905
0.45	4.462	3.504	6.344	4.962	8.627	6.748	11.313	8.862	14.401	11.302	17.891	14.070
0.50001	1.404	8.806	1.896	12.627	2.547	17.205	3.356	22.540	4.325	28.632	5.452	35.480
0.55001	2.541	69129.5	3.571	99538.5	4.849	135475.2	6.376	176939.6	8.153	223931.9	10.179	276451.9

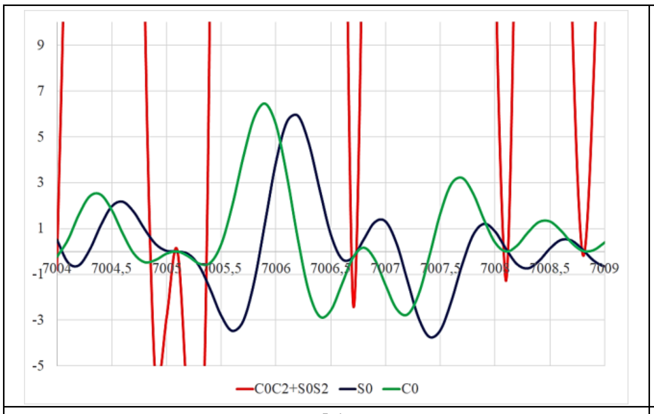
rM	rM-	rM+	rM-	rM+	rM-	rM+	rM-	rM+	rM-	rM+	rM-	rM+
R2y	-0.5	-0.5	-0.6	-0.6	-0.7	-0.7	-0.8	-0.8	-0.9	-0.9	-1	-1
Δb_r	DSC(…)											
0.60001	3.857	1.848	5.483	2.550	7.469	3.443	9.815	4.527	12.520	5.803	15.585	7.270
0.65	1.233	46.381	1.656	66.588	2.225	90.507	2.941	118.136	3.804	149.476	4.813	184.527
0.70002	8746.556	220.365	12594.147	317.202	17141.242	431.7	22387.840	563.850	28333.9	713.661	34979.5	881.129
0.75001	51.062	6.245	73.308	8.908	99.632	12.113	130.033	15.860	164.512	20.150	203.069	24.982
0.8	8.446	4.921	12.076	6.998	16.422	9.511	21.486	12.460	27.267	15.846	33.764	19.667
0.85001	2.048	846.081	2.848	1218.370	3.861	1658.4	5.085	2166.2	6.522	2741.8	8.171	3385.1
0.9	11.198	53.760	16.044	77.327	21.828	105.233	28.549	137.476	36.208	174.058	44.805	214.978
0.95003	10.757	3651.7	15.455	5258.0	21.071	7156.5	27.604	9347.0	35.056	11829.7	43.426	14604.5
1.00001	147.008	18.151	211.473	25.829	287.690	34.923	375.661	45.433	475.384	57.360	586.860	70.703

Appendix 7 : Cancellation of R2(a,b).

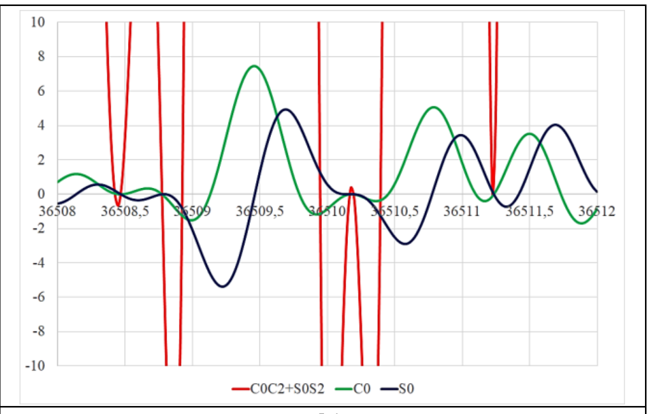
Expression R2(a,b) is null at Riemann and Dirichlet zeroes. However, these are not the only cancellation solutions since the condition to fill is not C0²+S0² = 0, but C0.C2+S0.S2. The purpose of the graphics below is simply to illustrate the fact that cancellations at critical abscissas of this article generally go in pairs, meaning two abscissas "close" to each other, one abscissa a zero and the other one not.



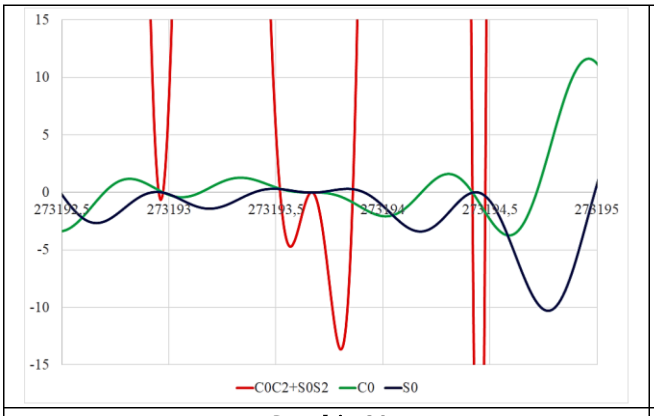
Graphic 77



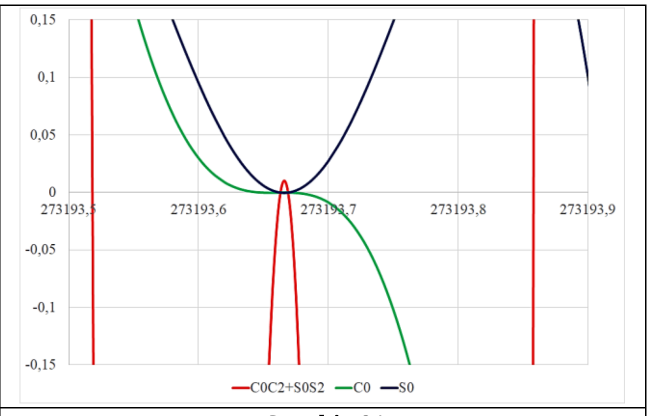
Graphic 78



Graphic 79



Graphic 80



Graphic 81

Appendix 8 : Functions to approximate r_{peak}.

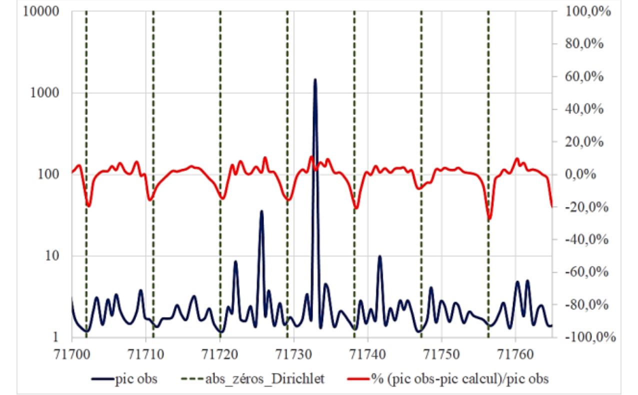
A first approximation function relative to the height of the peaks of R2(1/2,b) values, leading to differences in percentage between the observed values and said measurement function as noted in graphic82, obeys to the relationship :

$$r_{peak} \approx 1 + \frac{5}{(b_{peak})^{1/4}.\Delta b_r^2} \qquad (42)$$

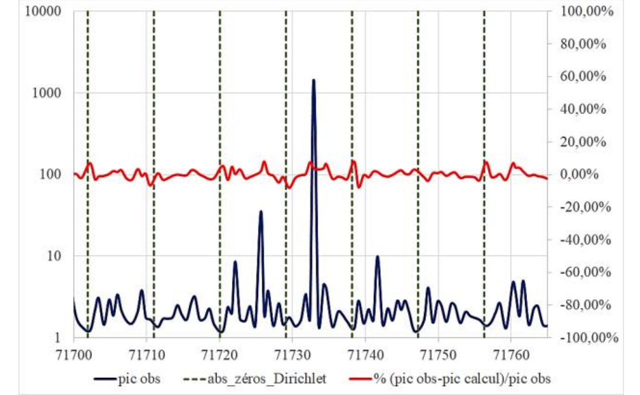
The term Δb_r is the gap between two Riemann's consecutive zeroes surrounding the peak r_{peak} of R2(1/2,b).
The height of the peak thus reacts in a certain way as the inverse of a radiation temperature in relation to the abscissa of the peak (Stefan-Boltzmann law type $b_{peak} \approx \alpha.(1/r_{peak})^4$) and as an energy for the atomic level Δb_r (law type $r_{peak} \approx \beta/\Delta b_r^2$).

Graphic82, where Dirichlet's zero abscissas are represented by vertical lines, clearly shows that the error is amplified by the fact that the existence of these zeros is not taken into account at the moment.
Modification below introduced by a second approximation function, and corresponding to graphic83, takes into account this point :

$$r_{peak} \approx 1 + \frac{1}{(b_{peak})^{1/4}}\left(\frac{1.238}{(b_{peak} - abs\_zeroR-).(abs\_zeroR+ - b_{peak})} + \frac{-18.1295}{(b_{peak} - abs\_zeroD-).(abs\_zeroD+ - b_{peak})}\right) \qquad (43)$$



Graphic 82
Before correction



Graphic 83
After correction

Close to the origin of b's, the relationship (43) still has errors in the order of 30% or more. The following relationship, a simple variant of the previous one, improves this point:

$$r_{peak} \approx 1 + \frac{3}{Ln(b_{peak})} + \frac{1}{(b_{peak})^{1/4}}\left(\frac{1.238}{(b_{peak} - abs\_zeroR-).(abs\_zeroR+ - b_{peak})} + \frac{-18.1295}{(b_{peak} - abs\_zeroD-).(abs\_zeroD+ - b_{peak})}\right) \qquad (44)$$

Graphic83still presents errors between the actual values of the peaks and the approximation. We searched for a better version in the form

$$r_{peak} \approx 1 + \frac{\gamma}{Ln(b_{peak})} + \frac{1}{(b_{peak})^{1/4}}\sum\left(\frac{\alpha_i}{(b_{peak} - abs\_zeroRi-).(abs\_zeroRi+ - b_{peak})} + \frac{\beta_i}{(b_{peak} - abs\_zeroDi-).(abs\_zeroDi+ - b_{peak})}\right) \qquad (45)$$

taking into account the Riemann and Dirichlet zeros further from the peak. For our part, this does not seem to significantly lessen errors in a general application.

Besides, some relation like

$$r_{peak} \approx 1 + \frac{1}{(b_{peak})^{1/4}}\sum\left(\frac{\alpha_i}{|b_{peak} - abs\_zeros|^n}\right) \qquad (46)$$

i.e. without associating pairs of zeros, but simply taking into account all the differences between abscissas of the peak and those of the zeros (n being a power to adapt) seems doomed to failure.

Appendix 9 : Errors on the peak values of R2(a,b).

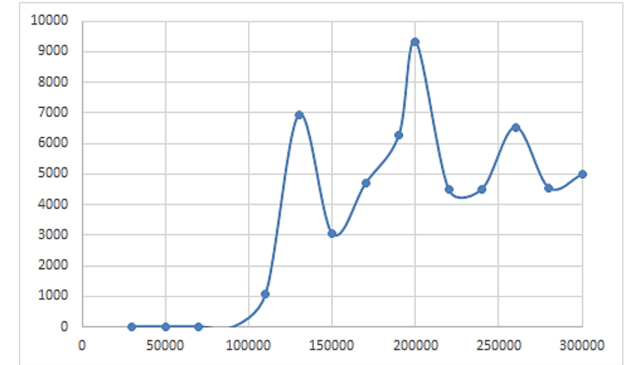
The relative error on the actual value of the peaks R2 = (C0. C2+S0. S2)/($C1^2$+$S1^2$) has not been specified so far. These peaks of values correspond to situations where $C1^2$+$S1^2$ tends towards 0 at the denominator of the ratio R2(a,b). Numerically, the value of this denominator can be largely distorted in the absence of precaution. As this is obtained using a truncation of the Dirichlet Eta function, it is necessary to ensure a sufficient number of terms to obtain a valid result when the abscissa b increases (see Appendix 1 page 31). The repercussion is all the greater as $C1^2$+$S1^2$ is smaller. Calculations must be carried out with a suitable truncation. For the production of the tables below, with the Excel tool, we have extended the investigation up to 300000 terms. However, some results for peaks values remain fluctuating in a very significant way.

The graphics below thus give, as a mere information, some data relating to these valuation uncertainties.

Truncations at / b	30000	50000	70000	90000	110000	130000	150000	170000
273193,666	0,79223436	0,79233452	0,79238882	-23,0571782	1066,88405	6938,16401	3040,68025	4688,02971
270071,298	0,78701346	0,7865257	0,78655872	-93,6405373	4690,2529	2852,85517	2409,05929	3034,69195
302700,306	0,84673934	0,75410234	0,75408475	0,75486075	696,780461	2444,2157	2780,2796	2248,68861
234016,902	0,77122518	0,77121376	0,77159857	3320,92683	1408,7961	1340,67378	1357,42807	1521,02584
71732,9086	807,012008	1260,06896	1543,56257	1659,75052	1463,47145	1477,5109	1328,09192	1311,98807

Truncations at / b	190000	200000	220000	240000	260000	280000	300000
273193,666	6264,18214	9332,50484	4481,24146	4503,3149	6533,48089	4547,31808	4974,56835
270071,298	3590,93864	2910,36473	2863,11206	3895,65609	3261,84607	2955,0297	3120,00748
302700,306	2687,97803	2274,00819	2411,96066	2664,51978	2838,83817	2409,74561	2593,14038
234016,902	1439,41531	1527,88393	1378,01132	1408,08006	1347,32708	1370,19658	1574,97404
71732,9086	1369,65022	1454,10964	1521,62094	1461,61479	1550,77267	1496,18688	1501,541

The case with the smallest gap between Riemann zeros among the first 500000 of them is then shown below and in the first row of the two previous tables. The magnitude of the discrepancies depending on the truncation chosen is obvious. Note that in our numerical reports in the main text, we used for this example the truncation to 200000 terms (which is not the best).



Graphic 84
R2 as a function of the truncation
b = 273193,666

Comparing these data with those obtained with the Pari gp tool shows that the discrepancies are not due to the imprecision of the spreadsheet. They are mainly related to the choice of truncation. In fact, the confidence limit, say at 10%, already manifests itself around a peak of height between 100 and 200. This means that at 10000, we are almost at the limit of any kind of reliable appreciation (if we want to stay in a reasonable calculation time). This does not call into question the nature of the formulas proposed in the text for peaks' values but the reader must remain vigilant on this point (which was recalled in our conclusion in the main text). It should be noted, however, that for the minima of R2(a,b), calculation errors are generally of lesser incidence.

The reader will be able to do some tests thanks to underneath programming in Pari gp. He will be able to appreciate for himself the differences of evaluation with Excel spreadsheet as well as the sometimes staggering effect of the truncation. In particular, it can use for the comparison the data in appendix 5(with large discrepancies here or there).

```
{a = -1/2; b = 273193.6;
n =100000; \\truncation
c0 = sum(i = 1, n, -((-1)^i)*(i^a)*cos(b*log(i)));
c0 = c0-(1/2)*((-1)^(n+1))*(n^a)*cos(b*log(n));
c1 = sum(i = 1, n, ((-1)^i)*(i^a)*log(i)*cos(b*log(i)));
c1 = c1+(1/2)*((-1)^(n+1))*(n^a)*log(n)*cos(b*log(n));
c2 = sum(i = 1, n, -((-1)^i)*(i^a)*log(i)*log(i)*cos(b*log(i)));
c2 = c2-(1/2)*((-1)^(n+1))*(n^a)*log(n)*log(n)*cos(b*log(n));
s0 = sum(i = 1, n, -((-1)^i)*(i^a)*sin(b*log(i)));
s0 = s0-(1/2)*((-1)^(n+1))*(n^a)*sin(b*log(n));
s1 = sum(i = 1, n, ((-1)^i)*(i^a)*log(i)*sin(b*log(i)));
s1 = s1+(1/2)*((-1)^(n+1))*(n^a)*log(n)*sin(b*log(n));
s2 = sum(i = 1, n, -((-1)^i)*(i^a)*log(i)*log(i)*sin(b*log(i)));
s2 = s2-(1/2)*((-1)^(n+1))*(n^a)*log(n)*log(n)*sin(b*log(n));
sc1 = c1*c1+s1*s1;
sc2 = c0*c2+s0*s2;
r2 = sc2/sc1;
print(sc1); print(sc2); print(r2)}
```

When the size of the truncation is insufficient, we may find values of R2 well below -0.50 as shown by the first calculation below.

```
? {a = -1/2; b = 273193.6896; n =100000; print(r2)}
-0.6166038284211750574916648100
? {a = -1/2; b = 273193.6896; n =150000; print(r2)}
-0.4271955965872622470275834475
? {a = -1/2; b = 273193.6896; n =200000; print(r2)}
-0.5261384266503698375318264730
? {a = -1/2; b = 273193.6896; n =300000; print(r2)}
-0.4796643516828380156432379411
? {a = -1/2; b = 273193.6896; n =400000; print(r2)}
-0.5016208422615278776668411402
? {a = -1/2; b = 273193.6896; n =500000; print(r2)}
-0.4858750009077289531691875641
? {a = -1/2; b = 273193.6896; n =600000; print(r2)}
-0.4919079569086871791624603385
? {a = -1/2; b = 273193.6896; n =700000; print(r2)}
-0.4920769664916562265010777527
? {a = -1/2; b = 273193.6896; n =800000; print(r2)}
-0.4929477589891854606600505430
? {a = -1/2; b = 273193.6896; n =900000; print(r2)}
-0.4890131904821112999770340327
? {a = -1/2; b = 273193.6896; n =1000000; print(r2)}
-0.4891392347130997219482404937
? {a = -1/2; b = 273193.6896; n =1100000; print(r2)}
-0.4900429046216253353805337294
? {a = -1/2; b = 273193.6895; n =1200000; print(r2)}
-0.4933191008740246565228320951
? {a = -1/2; b = 273193.6896; n =1200000; print(r2)}
-0.4933198070145980866491230746
? {a = -1/2; b = 273193.6897; n =1200000; print(r2)}
-0.4933195499724307993347376930
```

Some examples of peaks' assessment are given below. The sensitivity to the abscissas is of course high.

```
? {a = -1/2; b = 273193.66604; n =400000; print(r2)}
785.5071969861662090556709993
? {a = -1/2; b = 273193.66605; n =400000; print(r2)}
948.7846204060245073619972671
? {a = -1/2; b = 273193.66606; n =400000; print(r2)}
786.5941885569144741261967395
```

```
? {a = -1/2; b = 273193.6659; n =500000; print(r2)}
451.90765575487468386421997773
? {a = -1/2; b = 273193.6660; n =500000; print(r2)}
1388.88365947718256521622410
? {a = -1/2; b = 273193.6661; n =500000; print(r2)}
568.93498358852821059418856781

? {a = -1/2; b = 273193.66597; n =1000000; print(r2)}
2734.89671306647973391666392
? {a = -1/2; b = 273193.66598; n =1000000; print(r2)}
3014.98026645601698714957131
? {a = -1/2; b = 273193.66599; n =1000000; print(r2)}
2982.13699907617419224030220

? {a = -1/2; b = 273193.665999; n =1500000; print(r2)}
6605.22008926159842463546453
? {a = -1/2; b = 273193.666; n =1500000; print(r2)}
6693.60038655462761507811622
? {a = -1/2; b = 273193.66601; n =1500000; print(r2)}
6387.94768945809348908313737
```

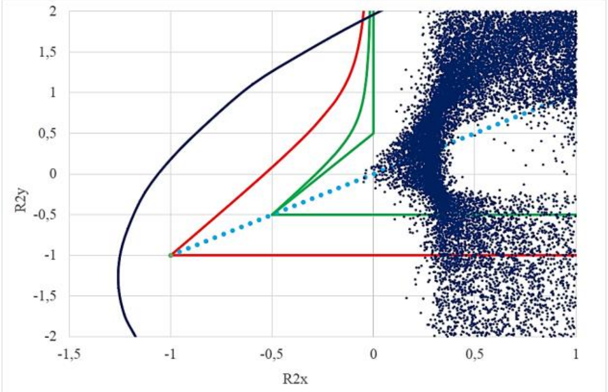
When the numerator itself is small, in addition to the denominator, a sign inversion of the first one can numerically give a totally aberrant results R2 << -1 if the truncation is not suitable. The reader should pay attention to it for his own trials.

```
? {a = -1/2; b = 1961773.995; n =2500000; print(sc1); print(sc2); print(r2)}
3.1102492692639370843185242 E-6
-0.0018391504598070022237670824
-591.319312565018443384267994
```

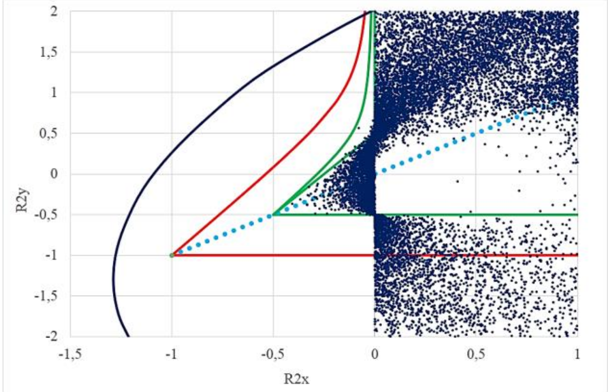
Appendix 10 : Locus of (R2x(a,b), R2y(a,b)).

The graphs below represent R2y as a function of R2x. For each of them, the parameter a is fixed and the represensative points are (R2x(a,b), R2y(a,b)), b describing the 40000 positions between 0 excluded and 10000 with 1/4 spacing.
The goal is to show the evolution according to the choice of a which is a global shift to the left when a increases.

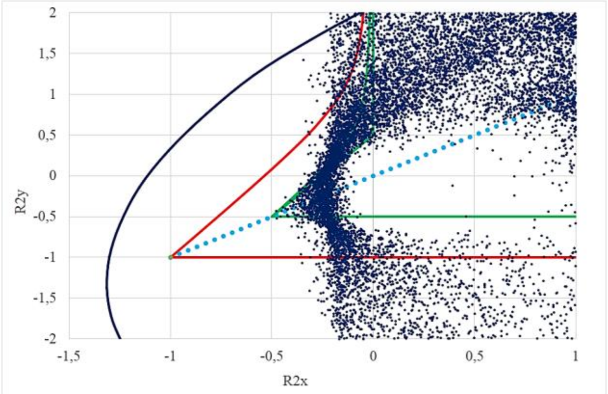
This result is not surprising, the interesting point being here to show the special case of a = 1/2 whose demarcation lines R2x = 0 and R2y = -1/2, have no equivalent for the other values of a, other values giving increasingly blurred boundaries.



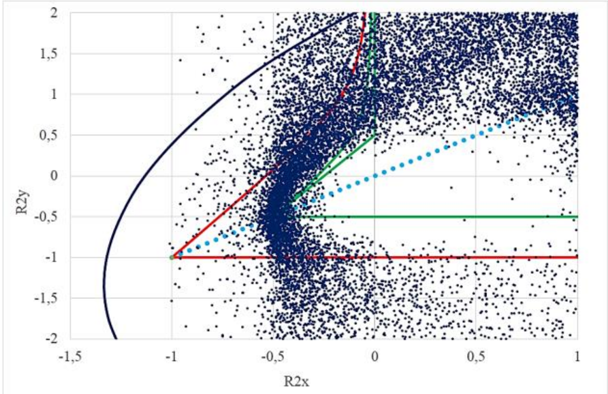
Graphic 85
a = 0.45, b ∈]0, 10000] et Δb = 0.25



Graphic 86
a = 0.50, b ∈]0, 10000] et Δb = 0.25

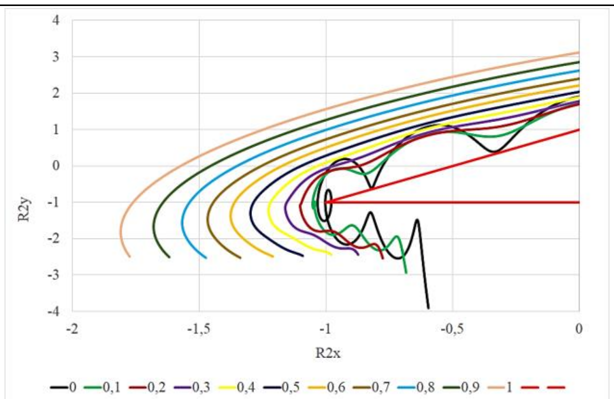


Graphic 87
a = 0.525, b ∈]0, 10000] et Δb = 0.25

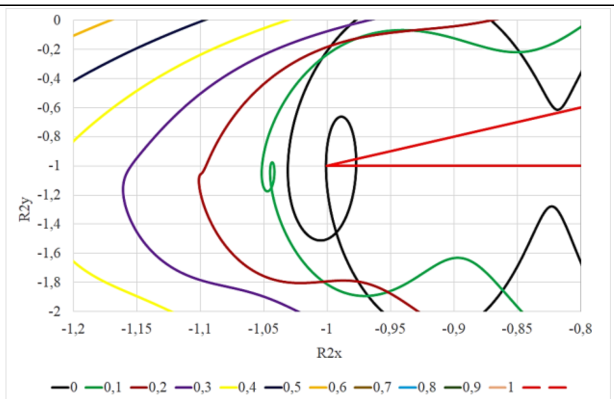


Graphic 88
a = 0.55, b ∈]0, 10000] et Δb = 0.25

We can notice that the points are not necessarily always deported to the left for the right part of the critical line (a < 1/2) especially in the area where the Riemann zeros do not yet appear (b < 4 < 14). The mix of equations being based on sinusoids it is not surprising to find undulating figures for some of the values of a (here the one close to 0) and position's inversions.



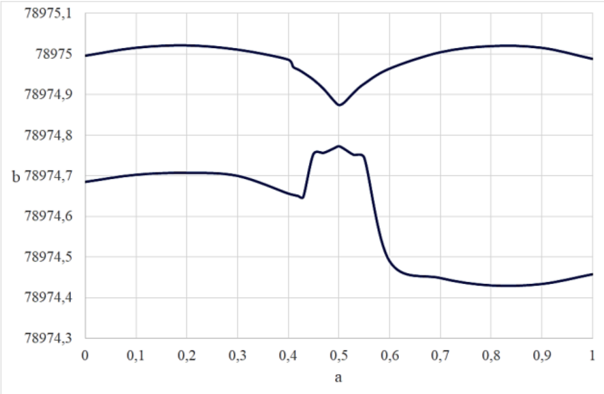
Graphique 89
a = 0 à 1, Δa = 0.1, b ∈]0, 4], Δb = 0.005

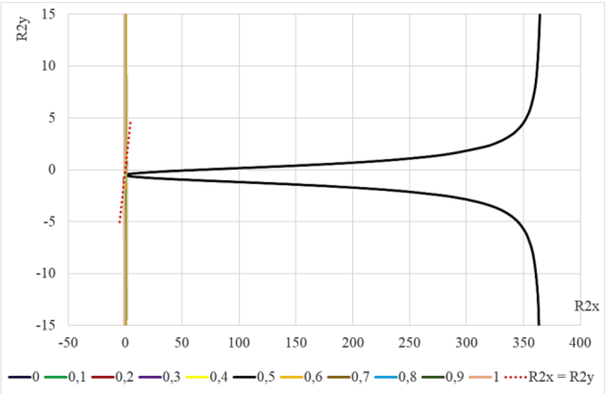


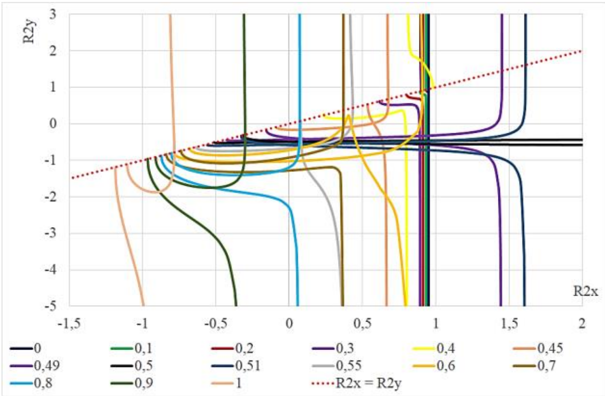
Graphique 90
a = 0 à 1, Δa = 0.1, b ∈]0, 4], Δb = 0.005

The graphs below show the evolution between the minimas around the peak located between the 106073th and the 106074th Riemann zeros (b ≈ 78974.79335 and b ≈ 78974.82196). Note before, that the intervals for b as a function of a, forming the minimums of R2 are as follows, the isopleths being here of the type of that of graphic 31 while showing that the herby example is more complex than the basic model :

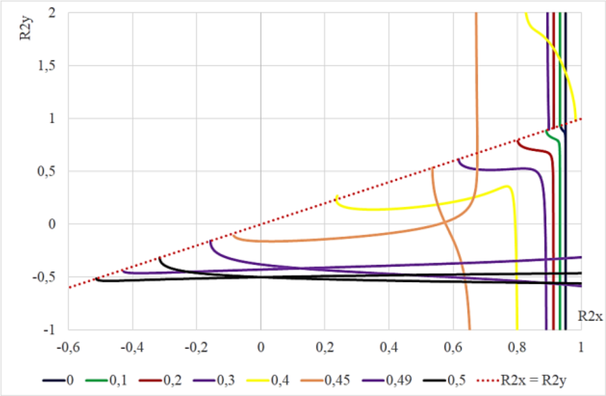
a	b	
	abs_r_M-	abs_r_M+
0	78974,685273	78974,995526
0,1	78974,702881	78975,015107
0,2	78974,707522	78975,020820
0,3	78974,700331	78975,010600
0,4	78974,656833	78974,986167
0,41	78974,653390	78974,968362
0,42	78974,650501	78974,961723
0,43	78974,648114	78974,954352
0,45	78974,753824	78974,936901
0,47	78974,756795	78974,914576
0,49	78974,767718	78974,885999
0,5	78974,773018	78974,875051
0,51	78974,767028	78974,880243
0,53	78974,751497	78974,905664
0,55	78974,745422	78974,926883
0,6	78974,490328	78974,963988
0,7	78974,448655	78975,004068
0,8	78974,430485	78975,019176
0,9	78974,433970	78975,014951



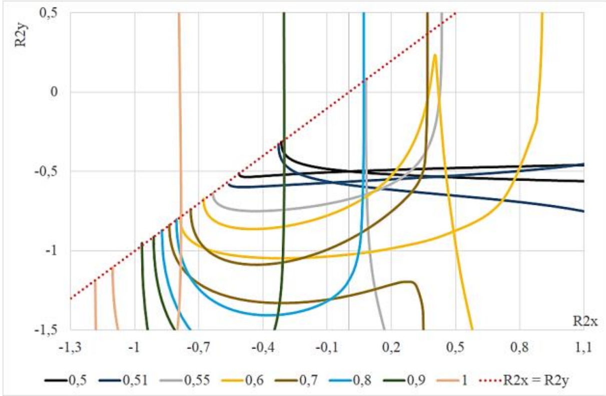




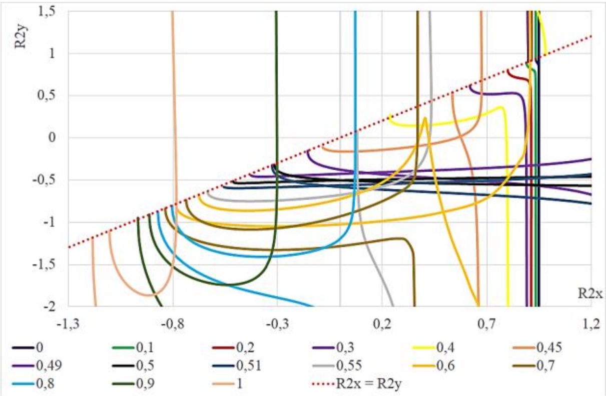
Graphic 91	***Graphic 92***
a = 0 to 1, Δa = 0.1, b ∈]0, 4], Δb = 0.005	a = 0 to 1, Δa = 0.1, b ∈]0, 4], Δb = 0.005



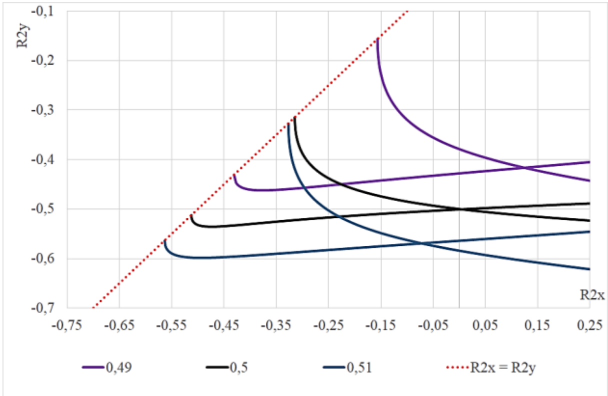
Graphic 93
a = 0 to 1, Δa = 0.1, b ∈]0, 4], Δb = 0.005



Graphic 94
a = 0 to 1, Δa = 0.1, b ∈]0, 4], Δb = 0.005



Graphic 95
a = 0 to 1, Δa = 0.1, b ∈]0, 4], Δb = 0.005

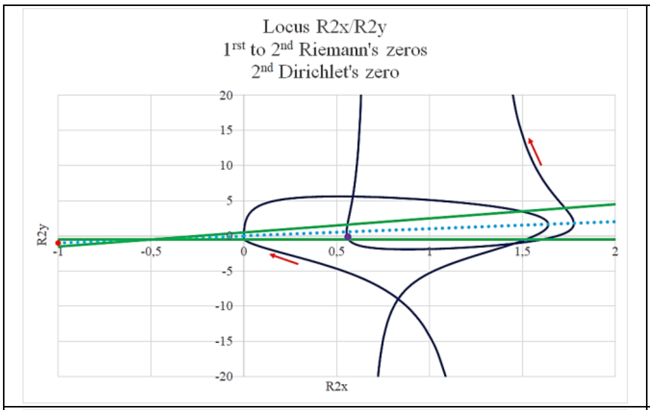


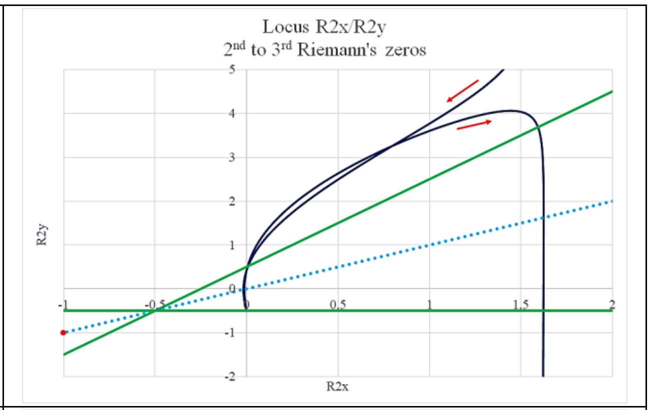
Graphic 96
a = 0 to 1, Δa = 0.1, b ∈]0, 4], Δb = 0.005

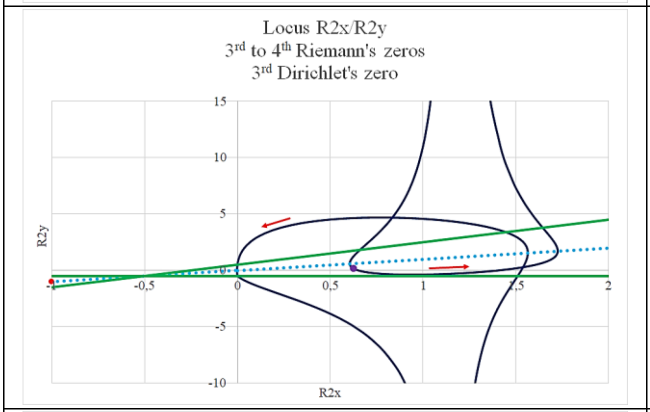
As usual, what interests us is the line R2x = R2y of the effective solutions of R2. This line corresponds at the same time to the minimum values of R2.
In the present case, which is that corresponding to a peak of great magnitude, when parameter a describs the range 0 to 1, the solutions R2 are, roughly by decreasing values, in the approximate interval [-1.184574789, 0.950381679]. The value R2 = -1 is finally reached only when the parameter a exceeds 0.9215, so quite far from a = 1/2.

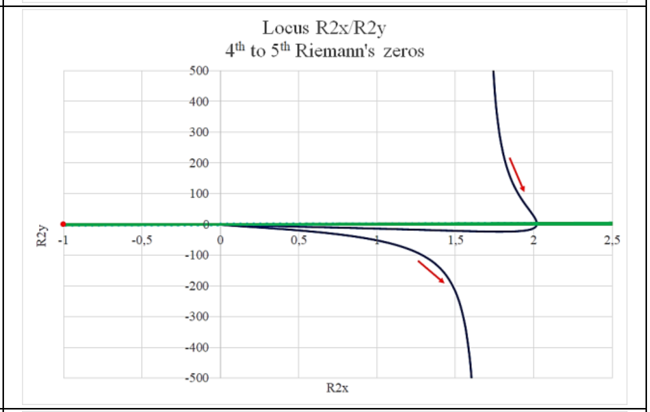
Appendix 11 : R2(a,b) paths.

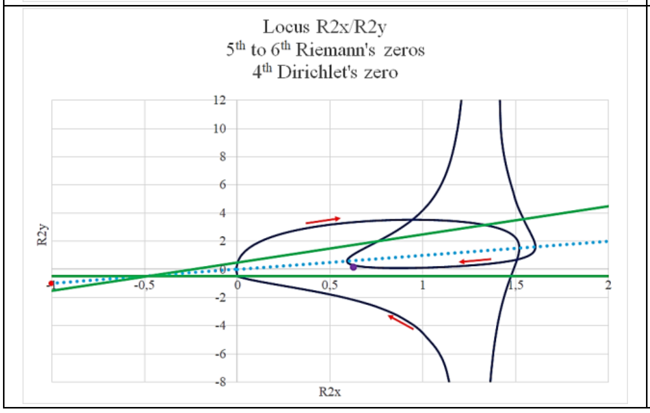
These graphs represent R2y as a function of R2x between a Riemann zero and its successor. Since (C0,S0) = (0,0) for a Riemann zero, this results in (R2x, Ryx) = (0,-1/2) on these abscissas $b = b_R$. The trajectory between two zeros thus begins in the graphs below at (0,-1/2) and ends at (0,-1/2). The direction of the increasing abscissas b is indicated each time. The only relevant solutions, by construction (see body text) are those on the axis R2x = R2y which does not hinder us from looking at the curves themselves. The Dirichlet abscissas $b_D = k.2\pi/Ln(2)$ are represented by purple dots. These points are usually close to the line R2x = R2y and on a branch sending points to infinity (i.e. two values of b such as C1. S2-C2. S1 = 0).

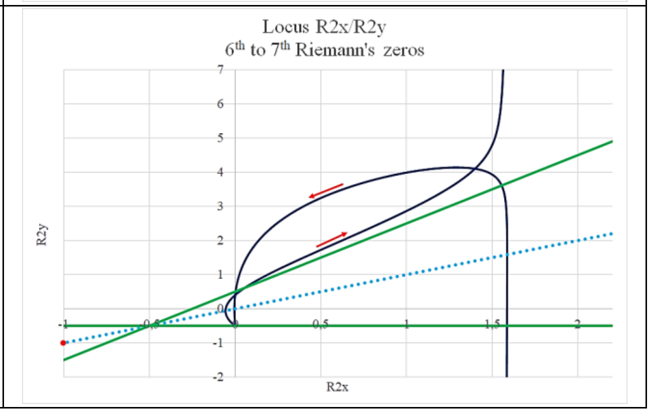


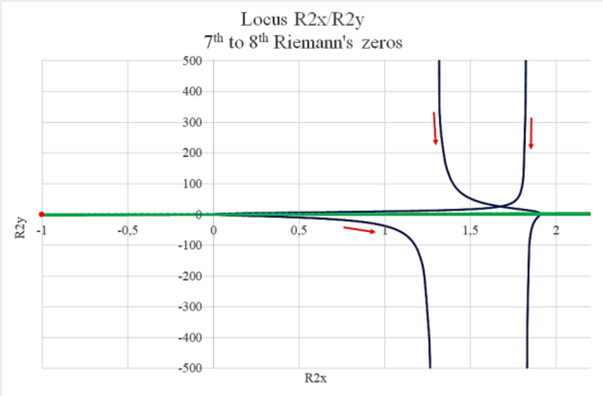


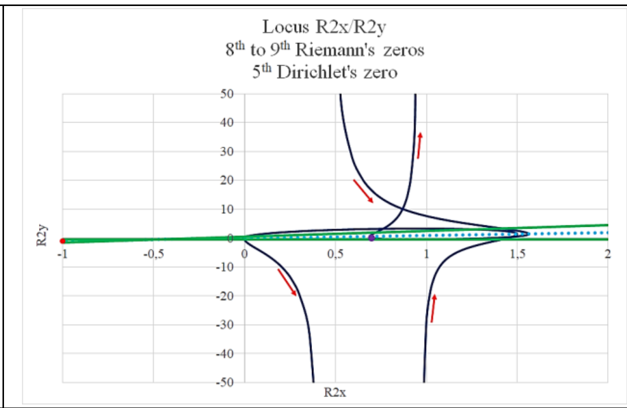


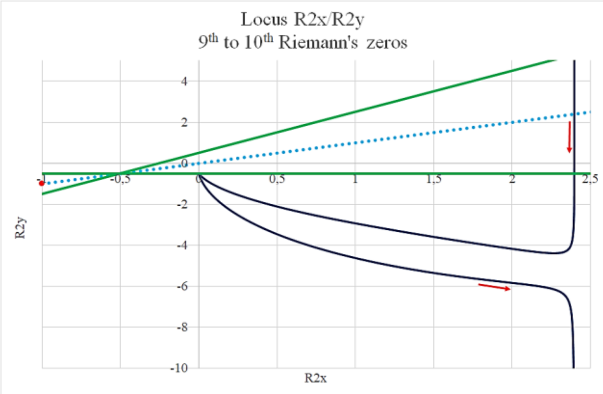


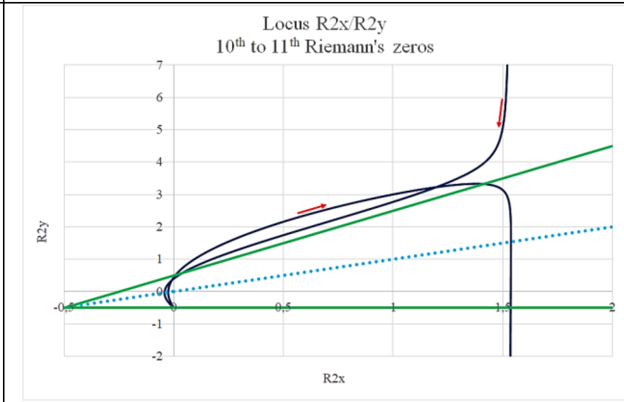


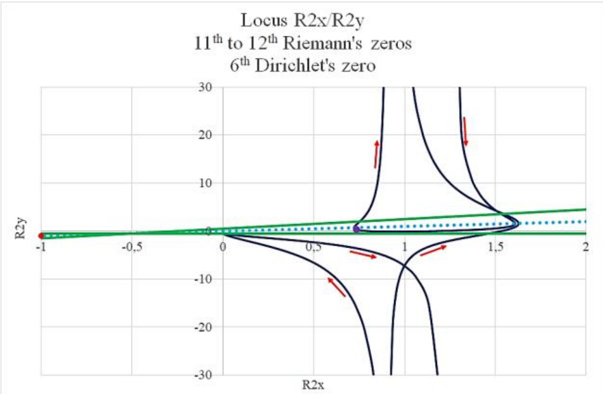


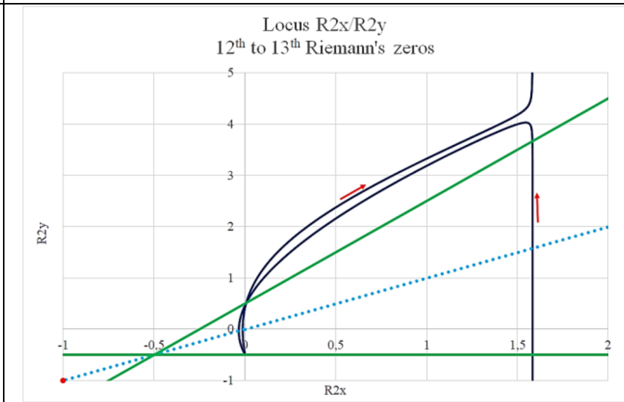

Locus R2x/R2y
7^{th} to 8^{th} Riemann's zeros
R2x
R2y

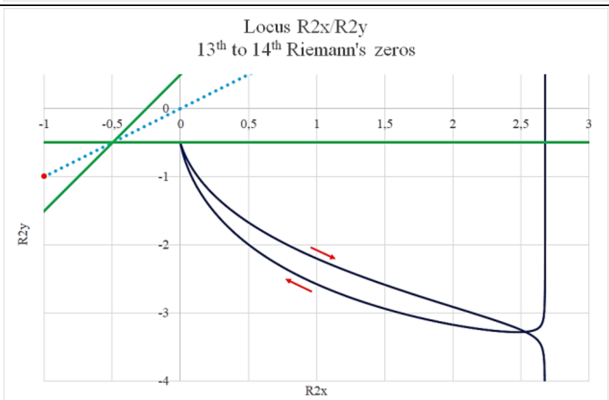

Locus R2x/R2y
8^{th} to 9^{th} Riemann's zeros
5^{th} Dirichlet's zero
R2x
R2y

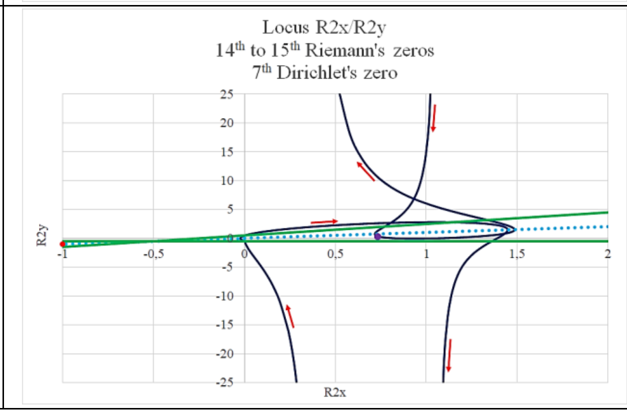

Locus R2x/R2y
9^{th} to 10^{th} Riemann's zeros
R2x
R2y

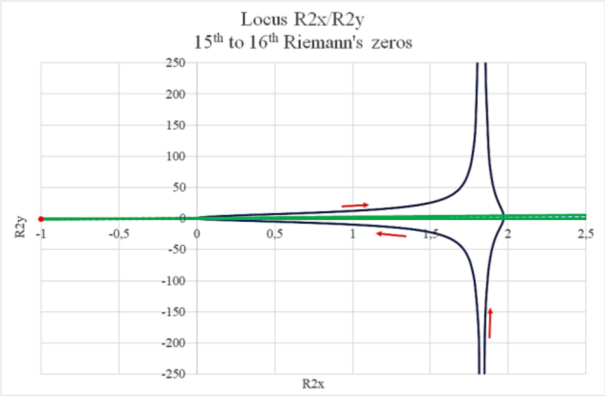

Locus R2x/R2y
10^{th} to 11^{th} Riemann's zeros
R2x
R2y

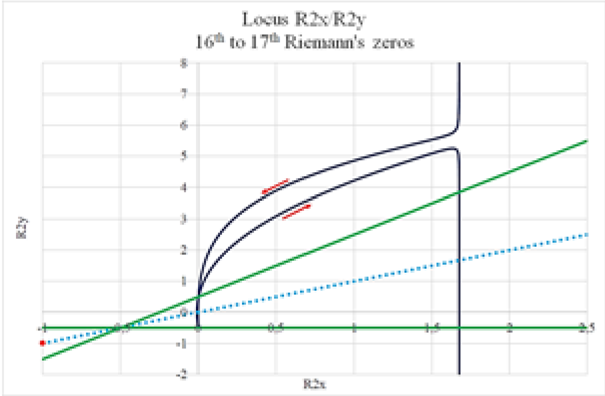

Locus R2x/R2y
11^{th} to 12^{th} Riemann's zeros
6^{th} Dirichlet's zero
R2x
R2y

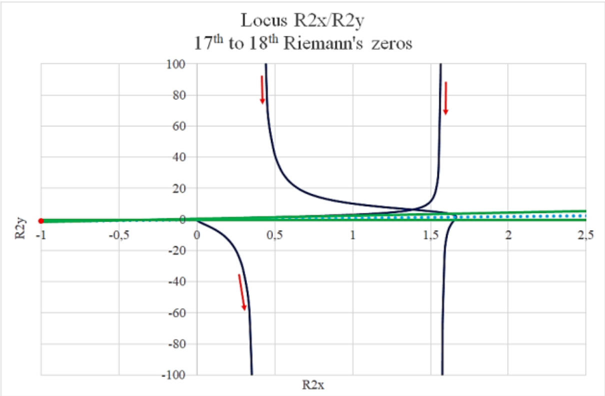

Locus R2x/R2y
12^{th} to 13^{th} Riemann's zeros
R2x
R2y

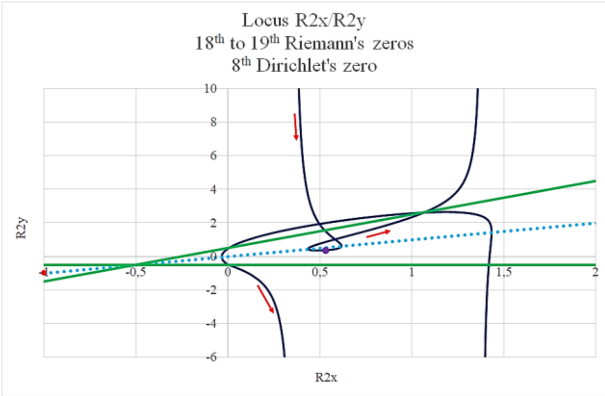

Locus R2x/R2y
13^{th} to 14^{th} Riemann's zeros
R2x
R2y

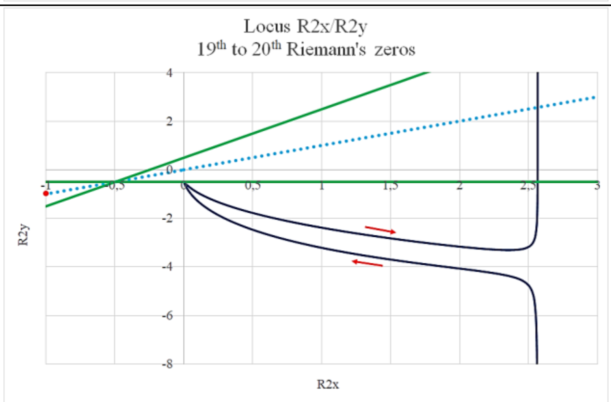

Locus R2x/R2y
14^{th} to 15^{th} Riemann's zeros
7^{th} Dirichlet's zero
R2x
R2y

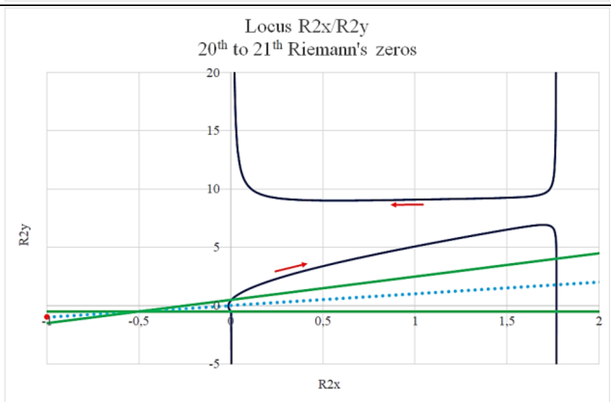

Locus R2x/R2y
15^{th} to 16^{th} Riemann's zeros
R2y
R2x

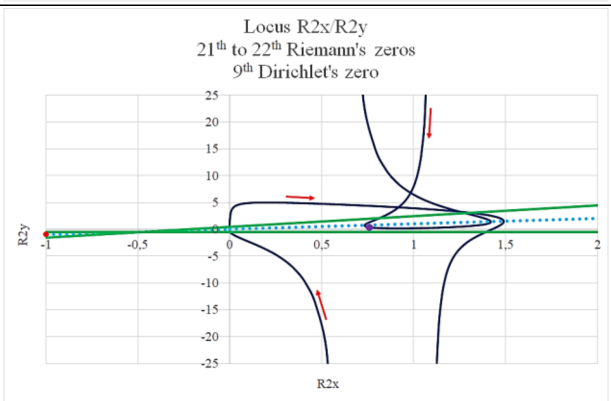

Locus R2x/R2y
16^{th} to 17^{th} Riemann's zeros
R2y
R2x

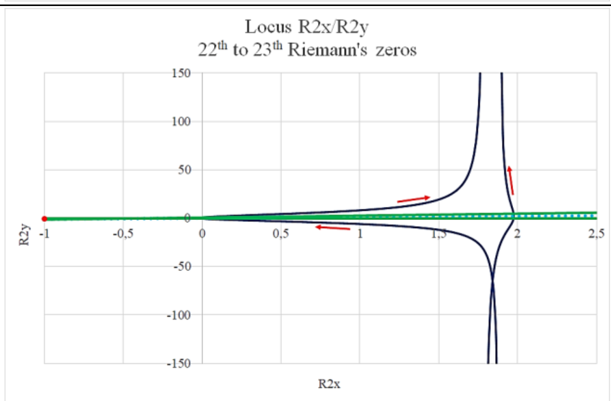

Locus R2x/R2y
17^{th} to 18^{th} Riemann's zeros
R2y
R2x

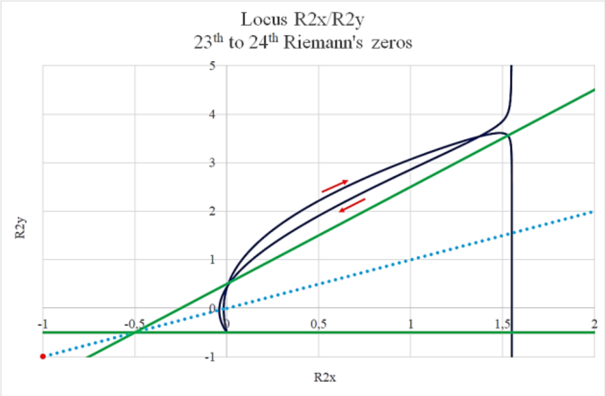

Locus R2x/R2y
18^{th} to 19^{th} Riemann's zeros
8^{th} Dirichlet's zero
R2y
R2x

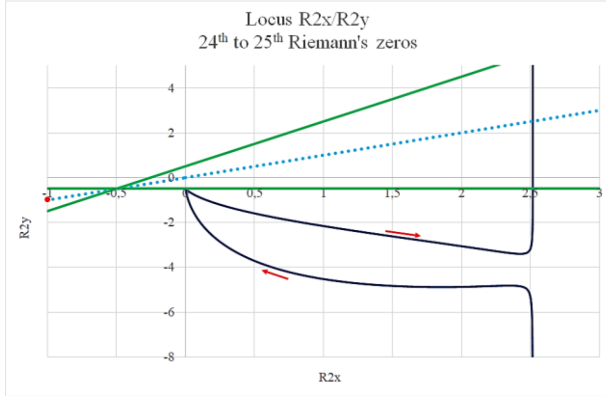

Locus R2x/R2y
19^{th} to 20^{th} Riemann's zeros
R2y
R2x

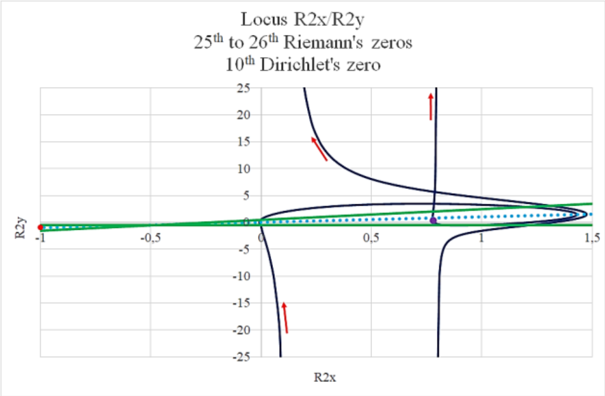

Locus R2x/R2y
20^{th} to 21^{th} Riemann's zeros
R2y
R2x

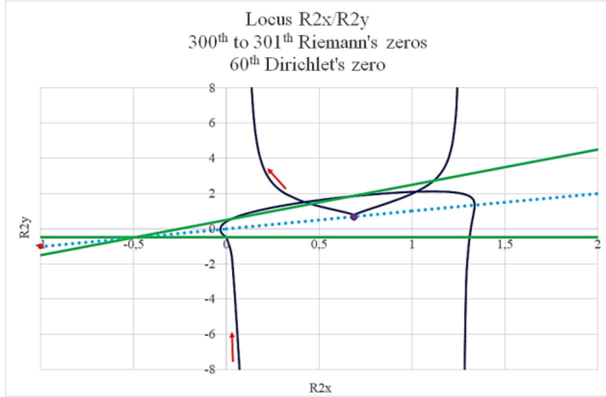

Locus R2x/R2y
21^{th} to 22^{th} Riemann's zeros
9^{th} Dirichlet's zero
R2y
R2x

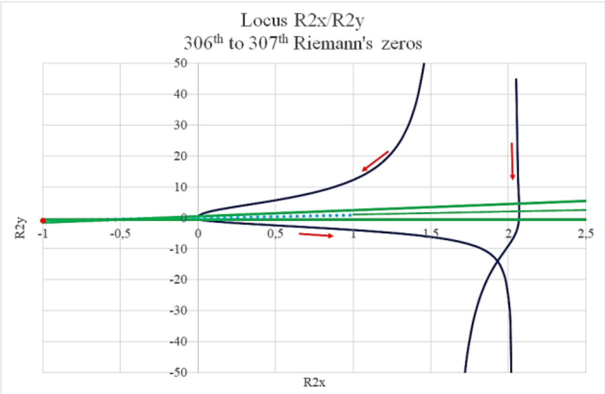

Locus R2x/R2y
22^{th} to 23^{th} Riemann's zeros
R2y
R2x

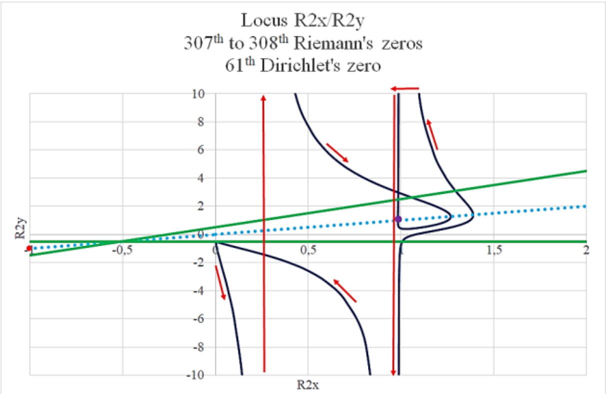

Locus R2x/R2y
23th to 24th Riemann's zeros
R2y
R2x

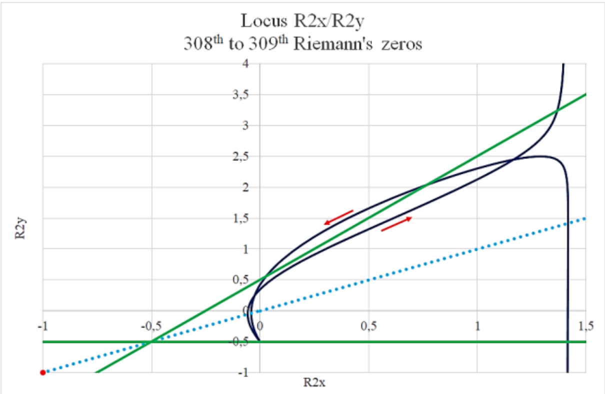

Locus R2x/R2y
24th to 25th Riemann's zeros
R2y
R2x

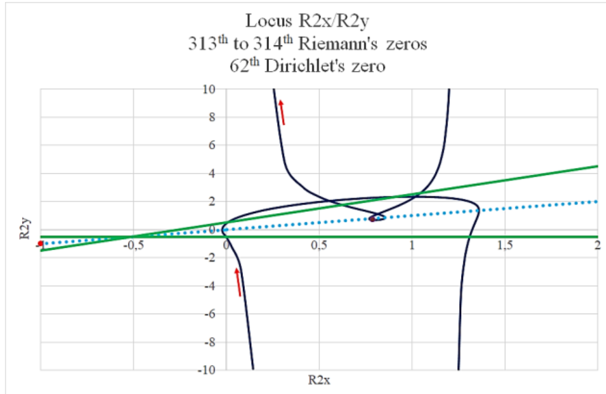

Locus R2x/R2y
25th to 26th Riemann's zeros
10th Dirichlet's zero
R2y
R2x

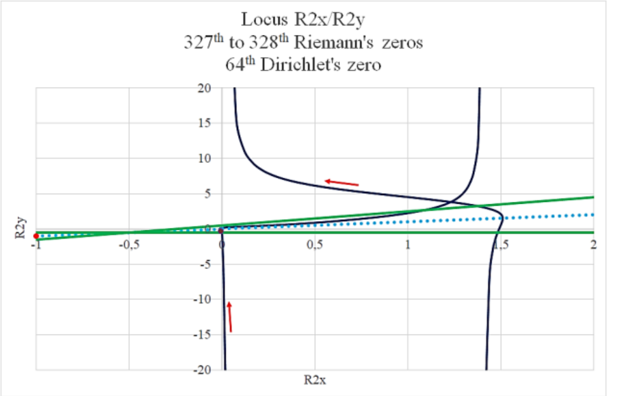

Locus R2x/R2y
300th to 301th Riemann's zeros
60th Dirichlet's zero
R2y
R2x

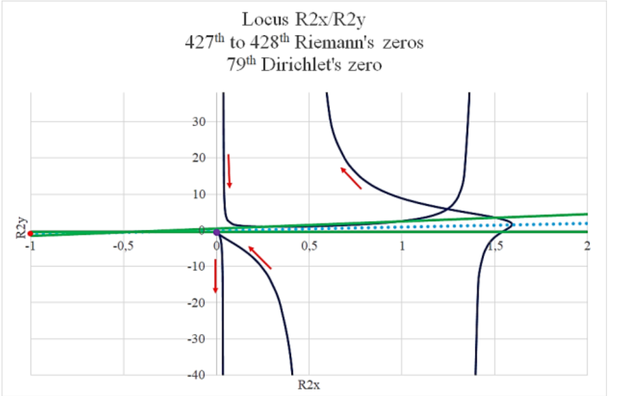

Locus R2x/R2y
306th to 307th Riemann's zeros
R2y
R2x

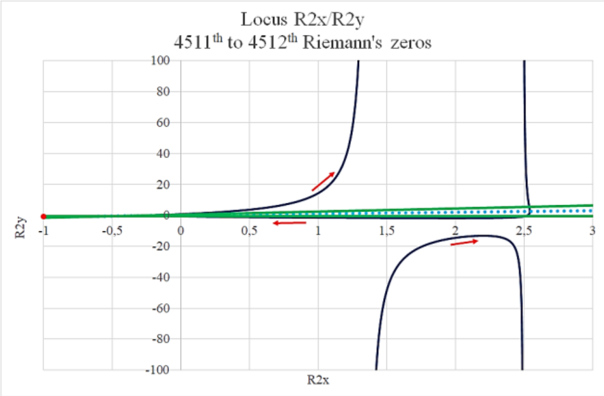

Locus R2x/R2y
307th to 308th Riemann's zeros
61th Dirichlet's zero
R2y
R2x

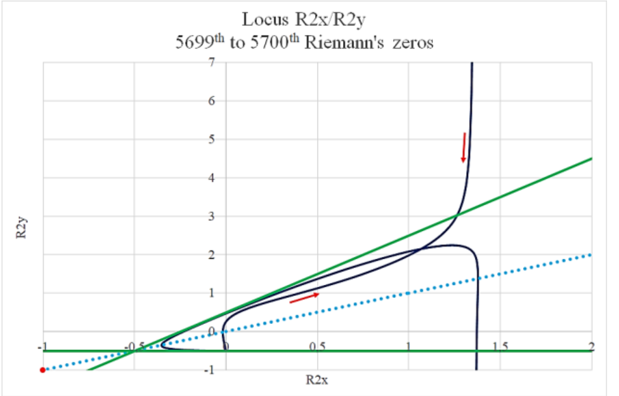

Locus R2x/R2y
308th to 309th Riemann's zeros
R2y
R2x

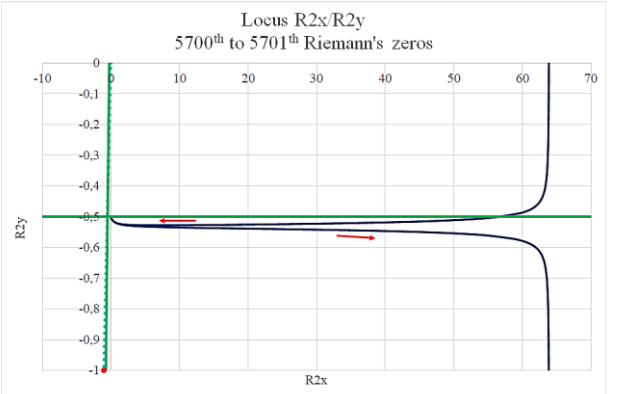

Locus R2x/R2y
313th to 314th Riemann's zeros
62th Dirichlet's zero
R2y
R2x

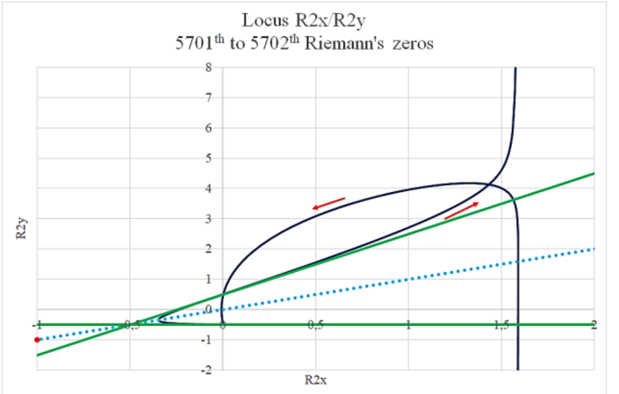

Locus R2x/R2y
327th to 328th Riemann's zeros
64th Dirichlet's zero
R2x
R2y

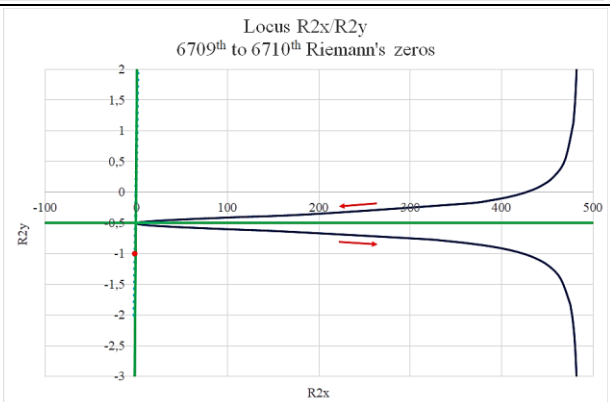

Locus R2x/R2y
427th to 428th Riemann's zeros
79th Dirichlet's zero
R2x
R2y

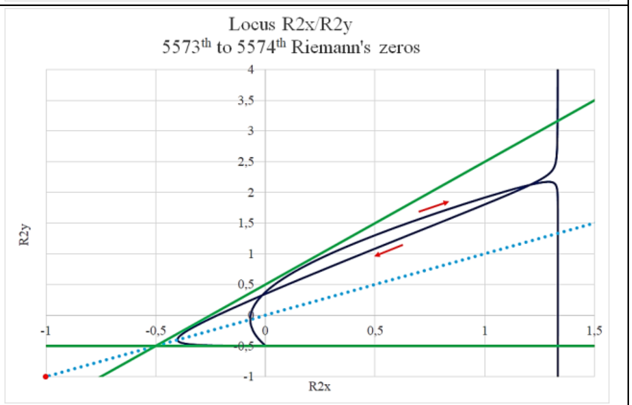

Locus R2x/R2y
4511th to 4512th Riemann's zeros
R2x
R2y

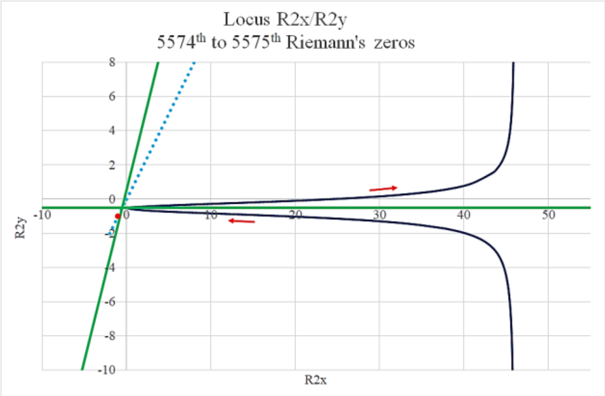

Locus R2x/R2y
5699th to 5700th Riemann's zeros
R2x
R2y

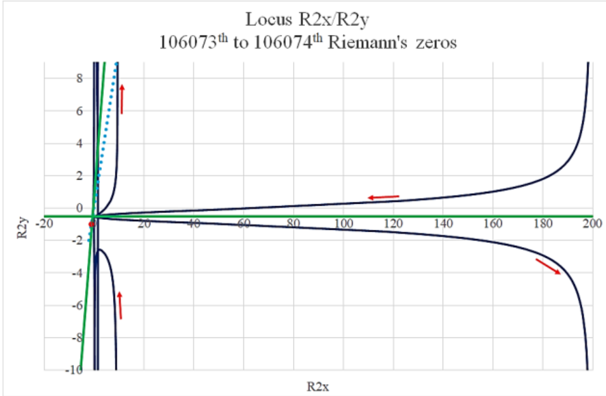

Locus R2x/R2y
5700th to 5701th Riemann's zeros
R2x
R2y

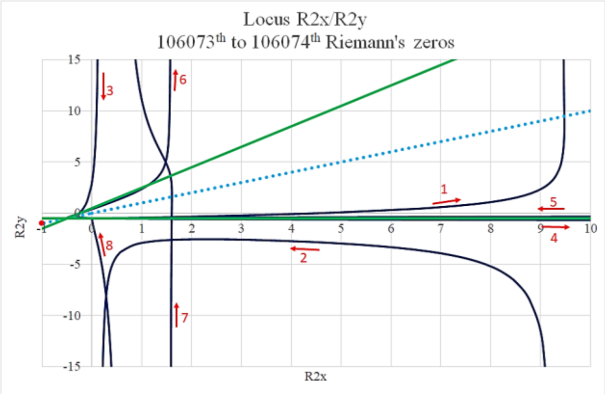

Locus R2x/R2y
5701th to 5702th Riemann's zeros
R2x
R2y

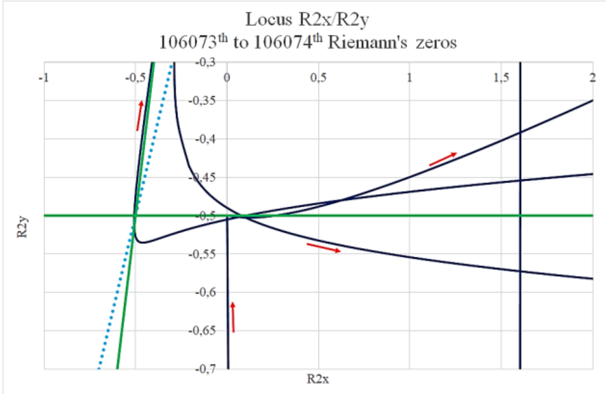

Locus R2x/R2y
6709th to 6710th Riemann's zeros
R2x
R2y


Locus R2x/R2y
5573th to 5574th Riemann's zeros
R2x
R2y





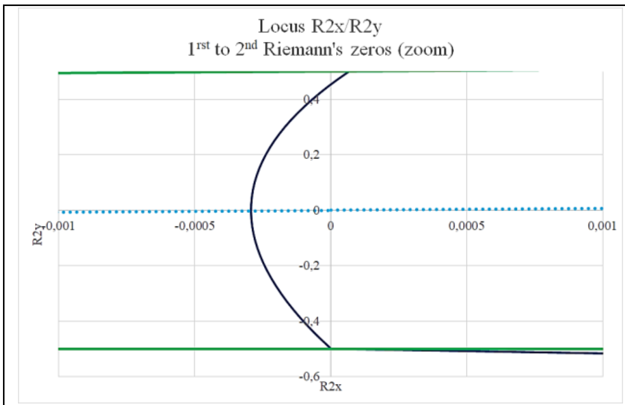




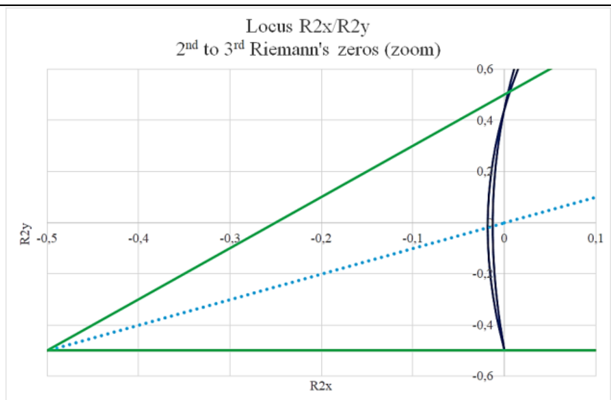
Now let us take a close-up view of the triangular area that is the most interesting.
We observe two main types of graphs between two Riemann zeros :

- Type 1 when the trajectory includes an even number of divergences due to C1. S2-C2. S1 = 0.
- Type 2 when the trajectory includes an odd number of divergences due to C1. S2-C2. S1 = 0.

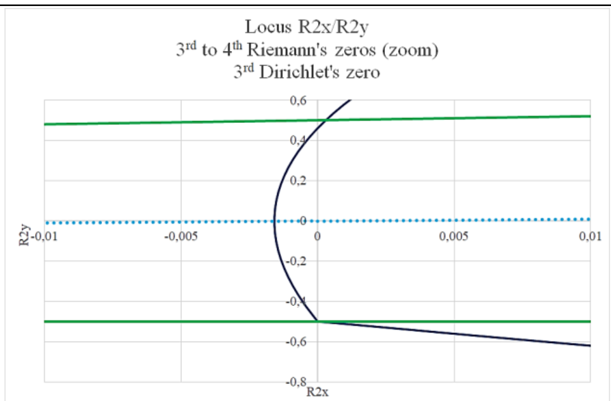
Note: 1 divergence corresponds to a one-way trip $\pm\infty$ plus a return trip $\mp\infty$.



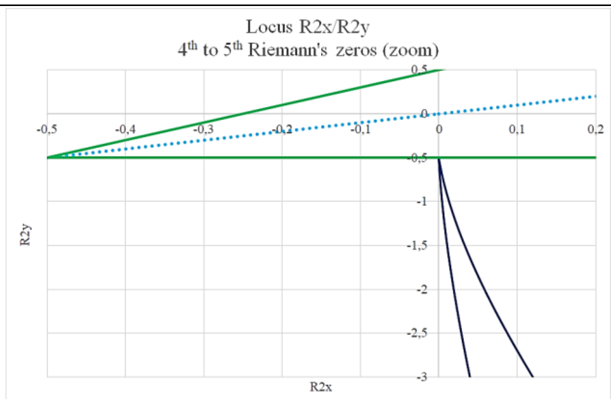
Type 1a : 2 divergences



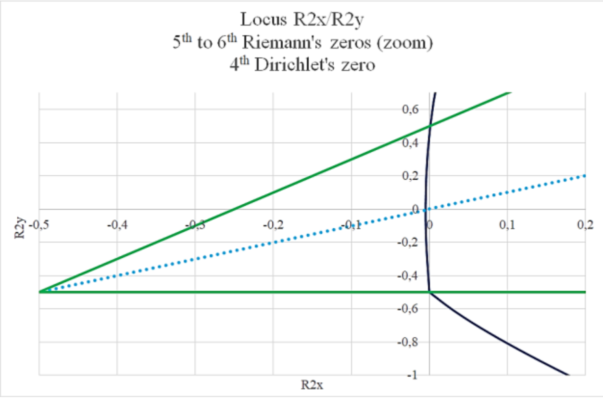
Type 2a : 1 divergence



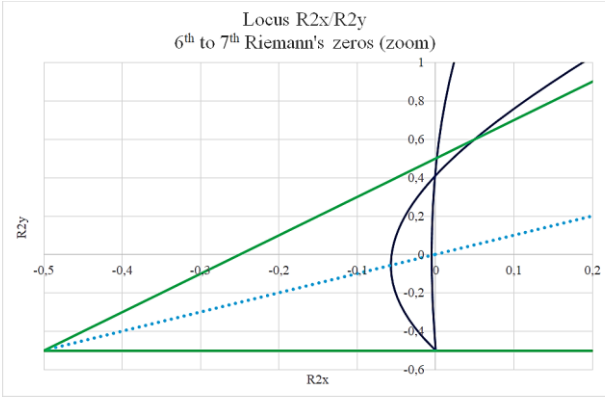
Type 1a : 2 divergences



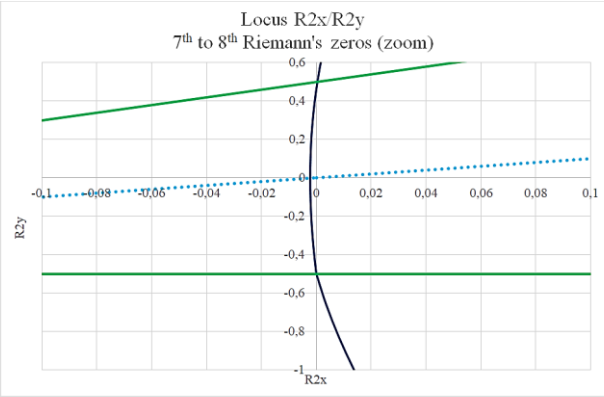
Type 2b : 1 divergence



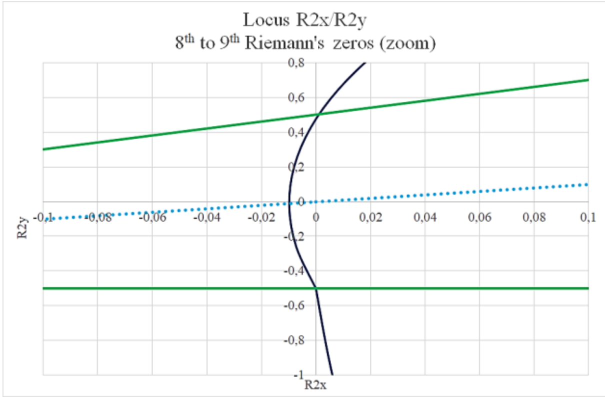
Type 1a : 2 divergences



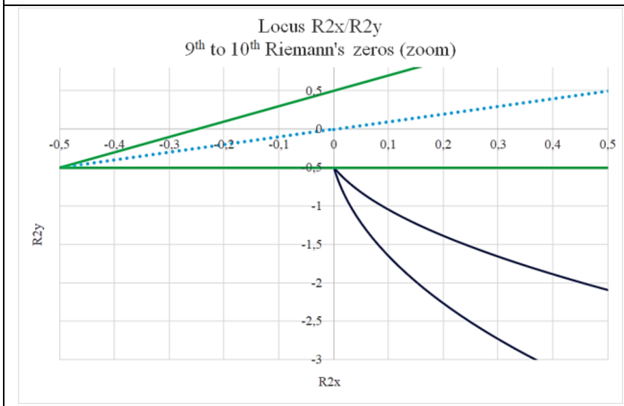
Type 2a : 1 divergence



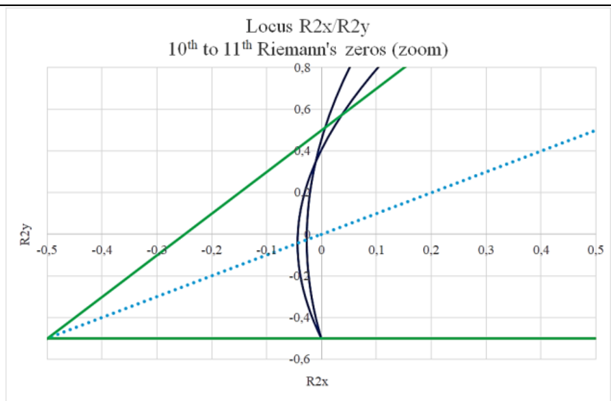
Type 1a : 2 divergences



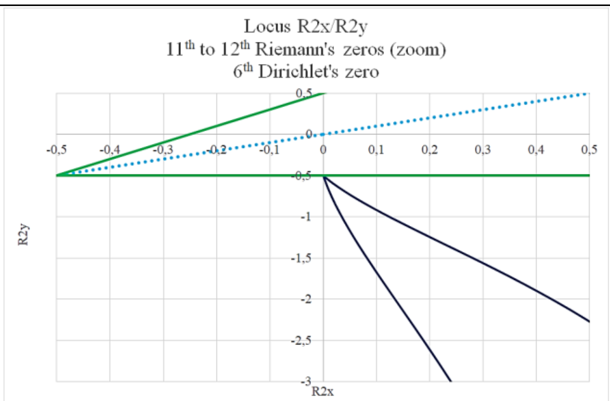
Type 1a : 2 divergences



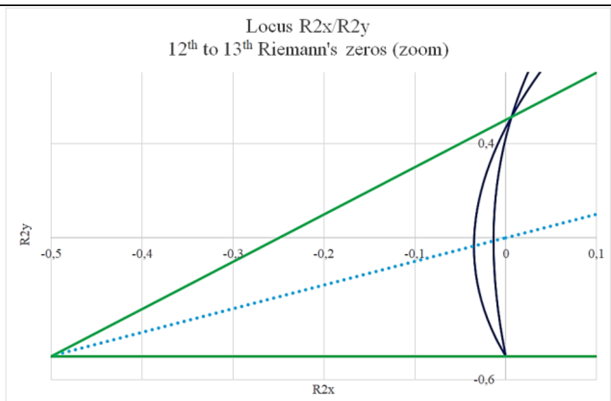
Type 2b : 1 divergence



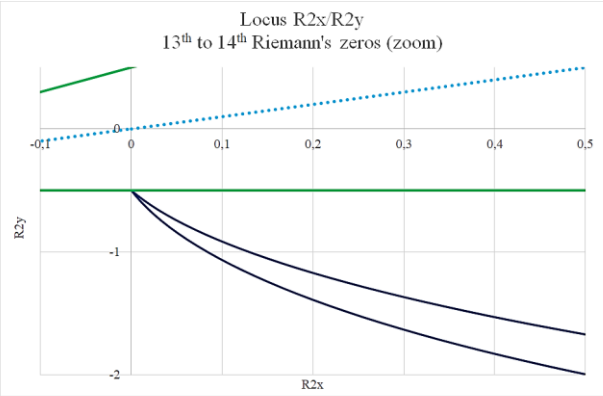
Type 2a : 2 divergences



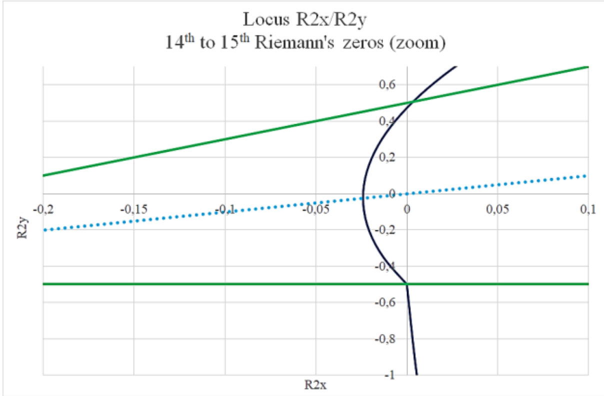
Type 2b : 3 divergences



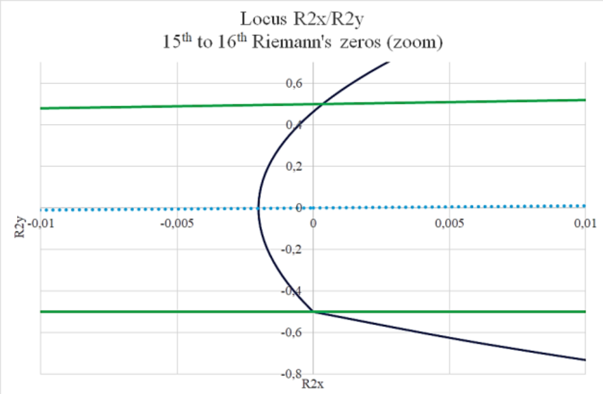
Type 2a : 1 divergence



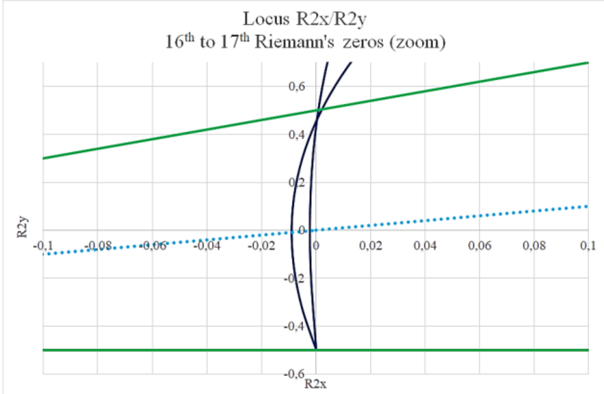
Type 2b : 1 divergence



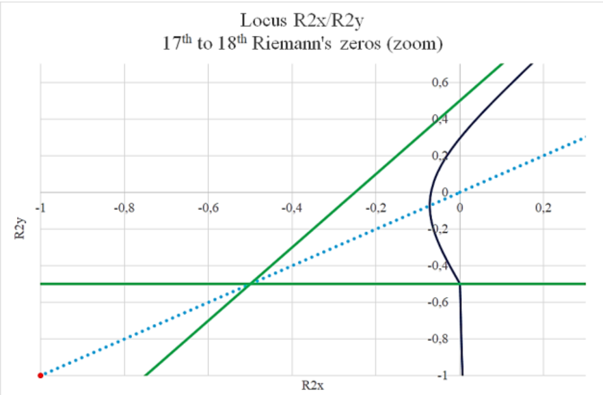
Type 1a : 2 divergences



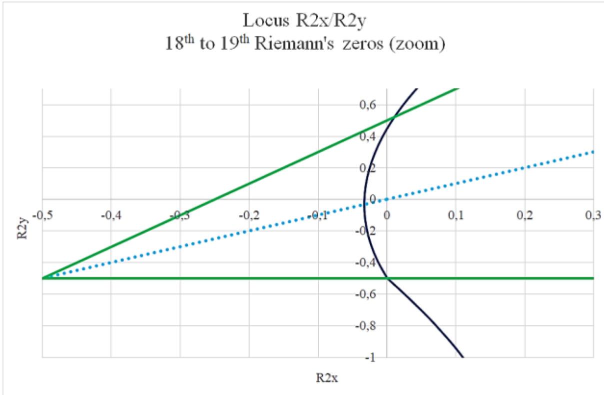
Type 1a : 2 divergences



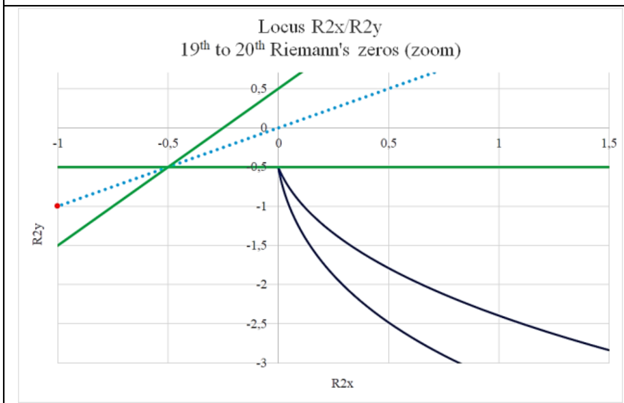
Type 2a : 1 divergence



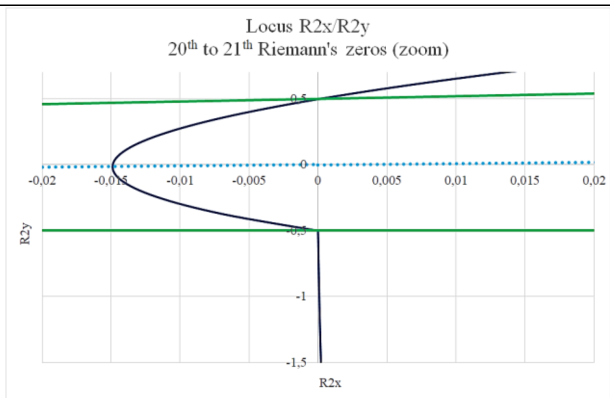
Type 1a : 2 divergences



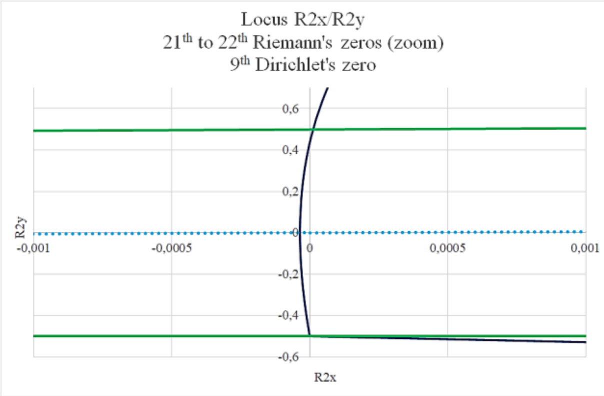
Type 1a : 2 divergences



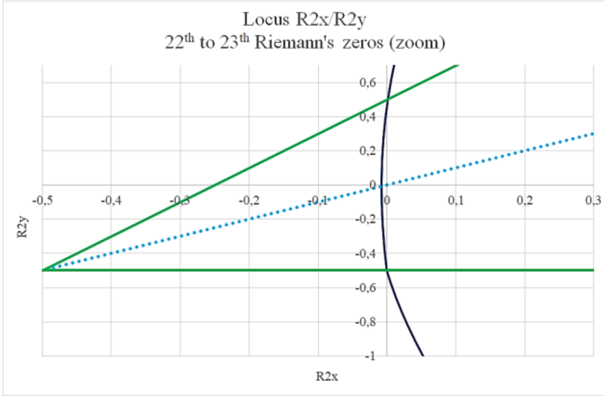
Type 2b : 1 divergence



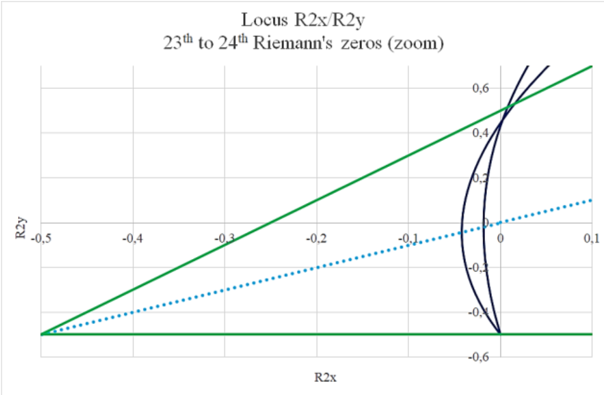
Type 1a : 2 divergences



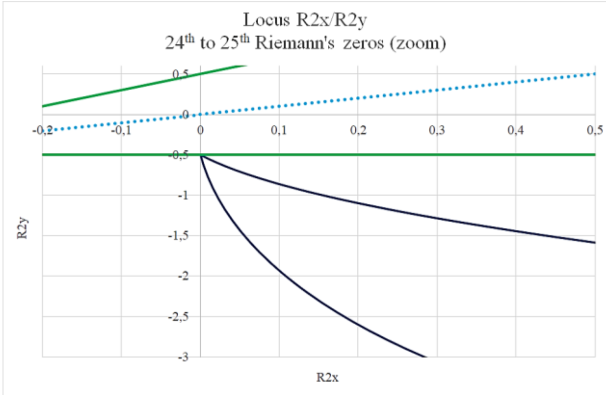
Type 1a : 2 divergences



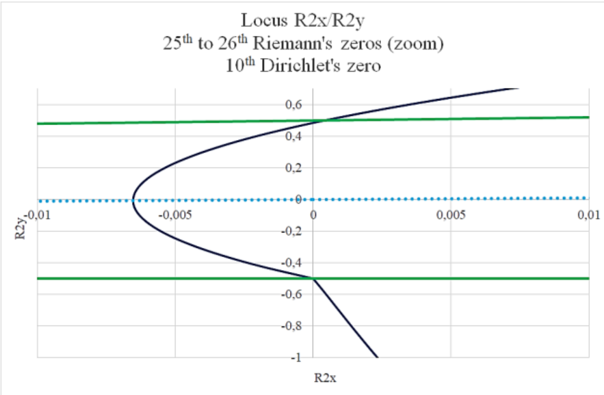
Type 1a : 2 divergences



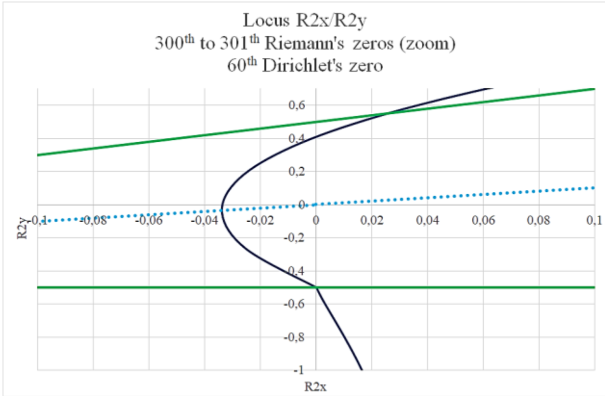
Type 2a : 1 divergence



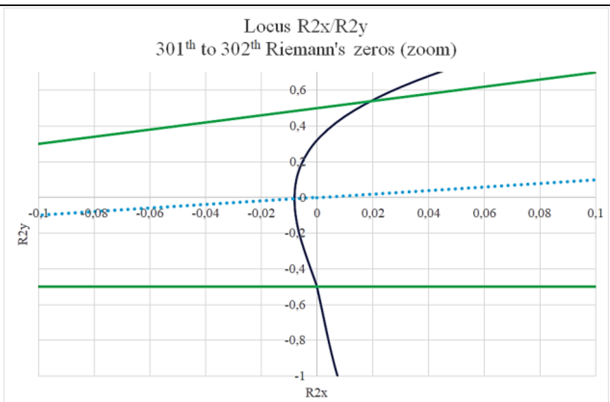
Type 2b : 1 divergence



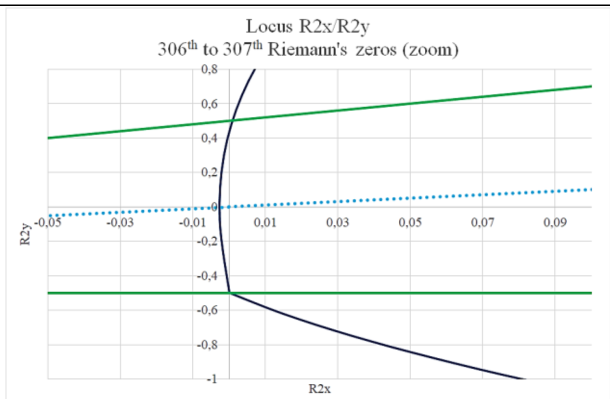
Type 1a : 2 divergences



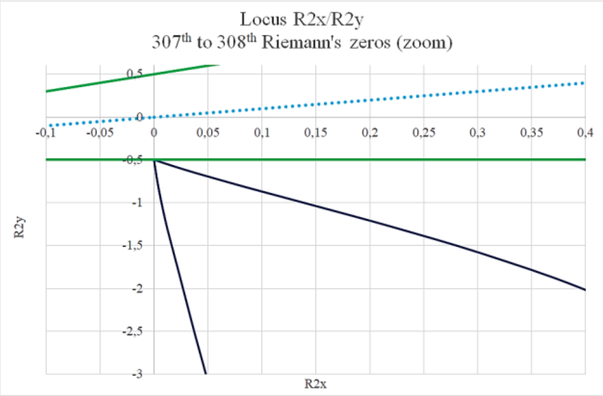
Type 1a : 2 divergences



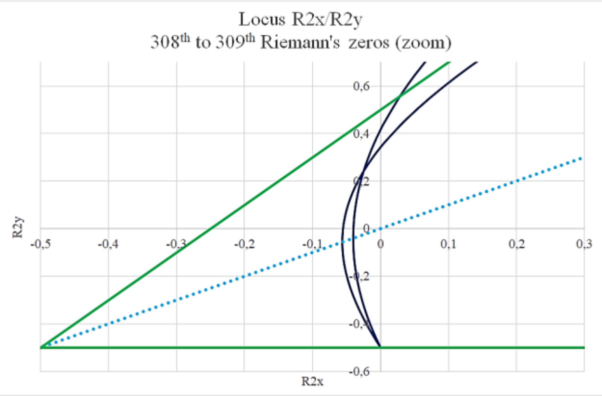
Type 1a : 2 divergences



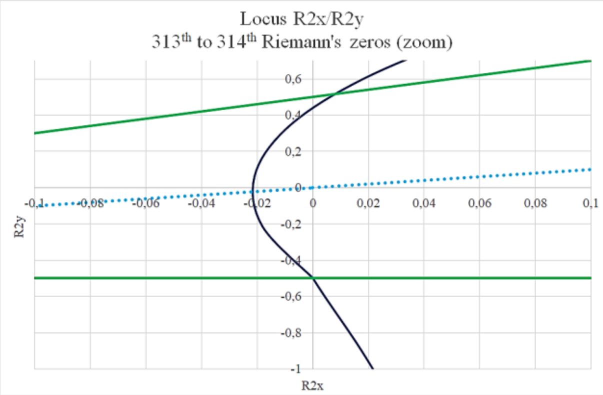
Type 1a : 2 divergences



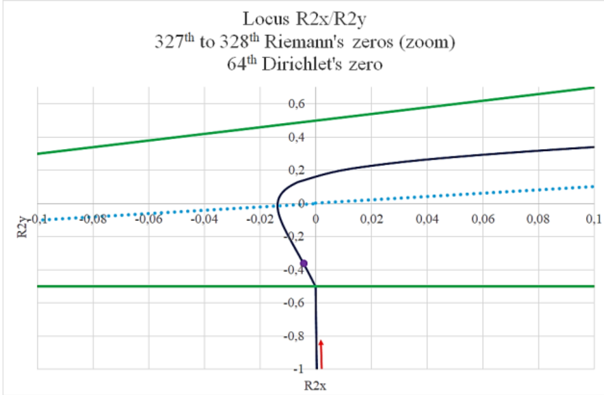
Type 2b : 3 divergences



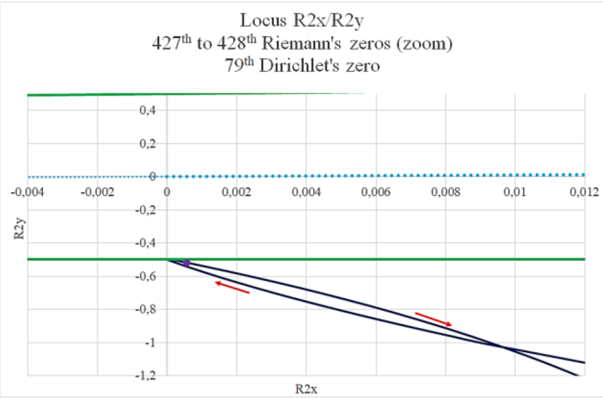
Type 2a : 1 divergence



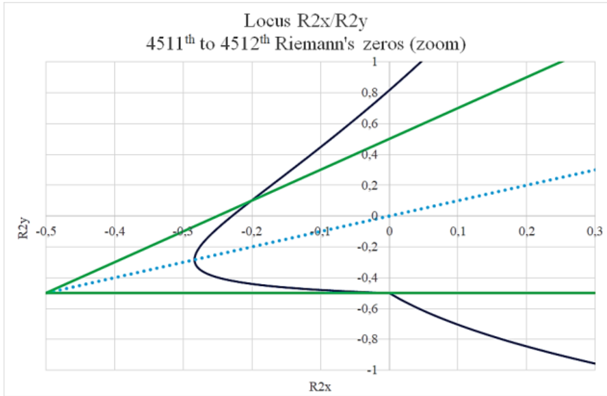
Type 1a : 2 divergences



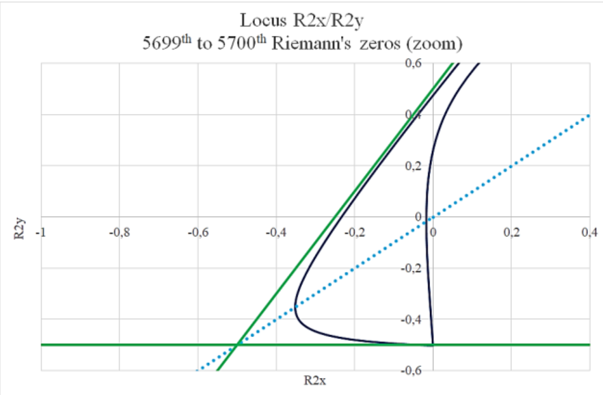
Type 1a : 2 divergences



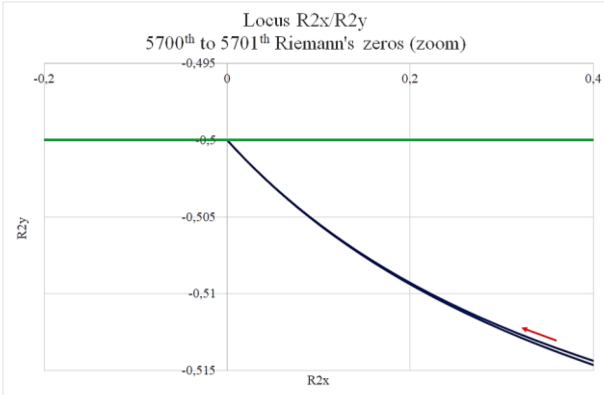
Type 2b : 3 divergences



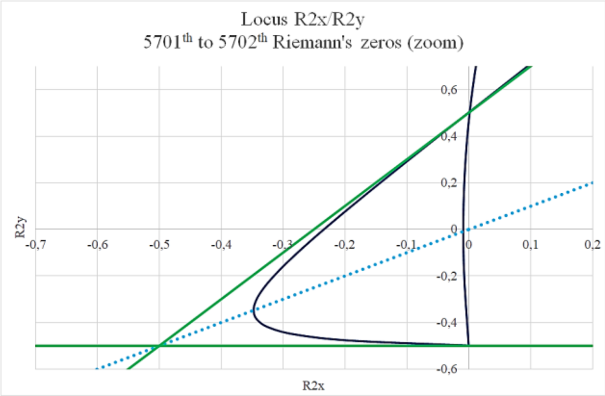
Type 1a : 2 divergences



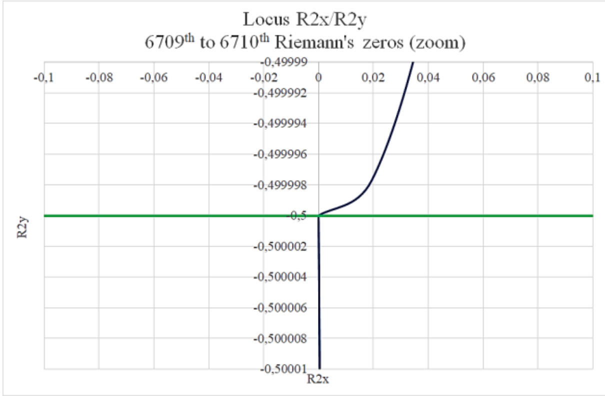
Type 2a : 1 divergence



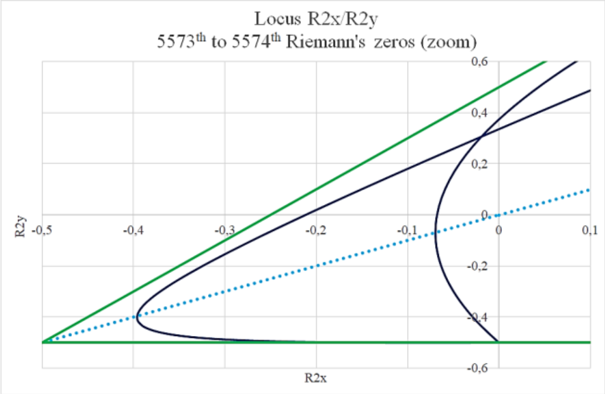
Type 2b : 3 divergences



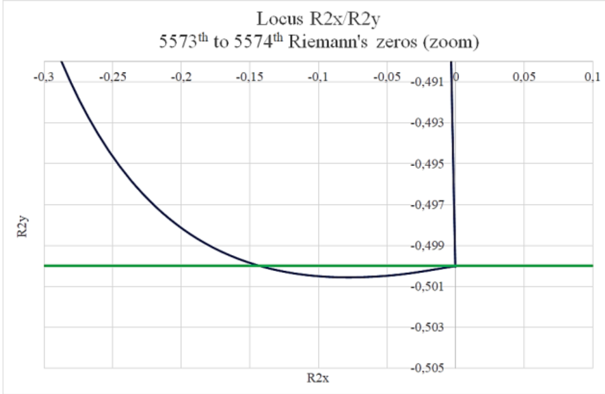
Type 2a : 1 divergence



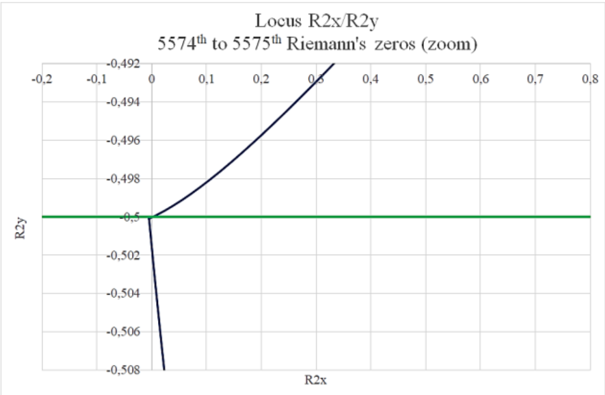
Type 2c : 1 divergence



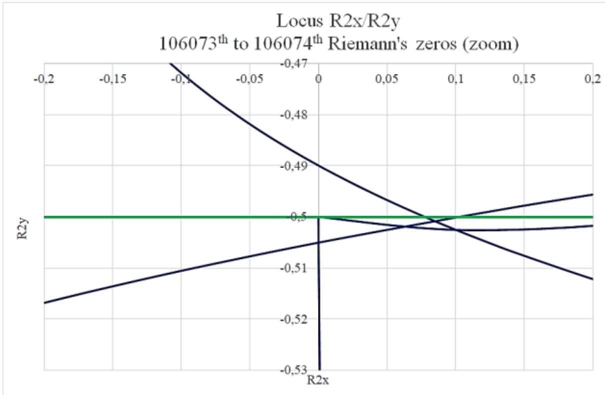
Type 2c : 1 divergence



Type 2c : 1 divergence

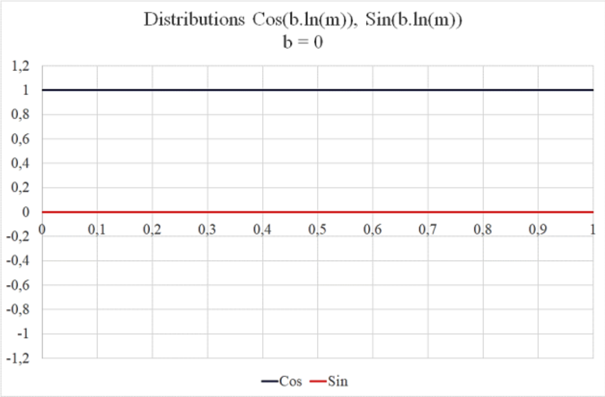


Type 2d : 1 divergence

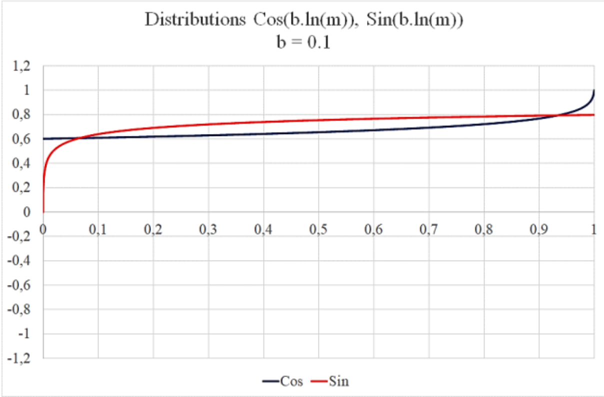


Type 2b : 5 divergences

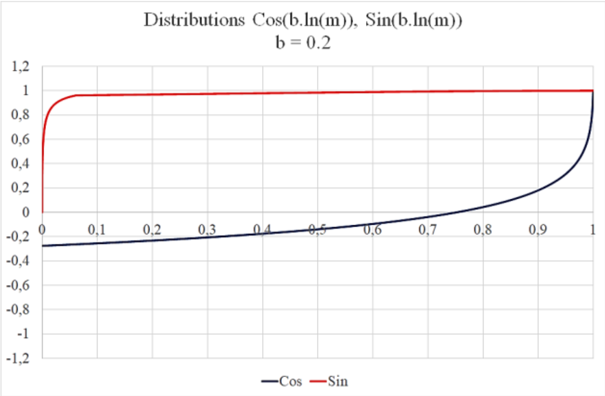
Appendix 12 : Distribution of Cos(b.Ln(m)) and Sin(b.Ln(m)).



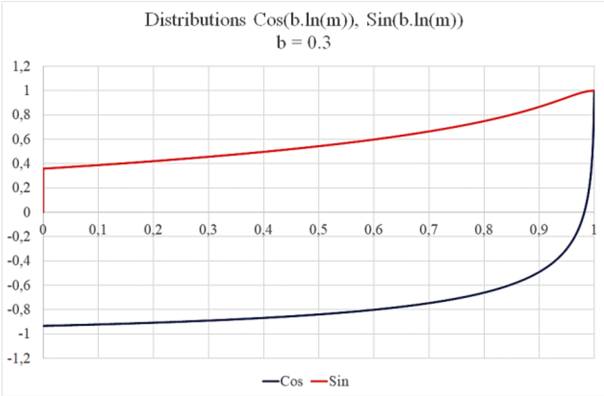
__Graphic 97__
a = 0.5 and b = 0



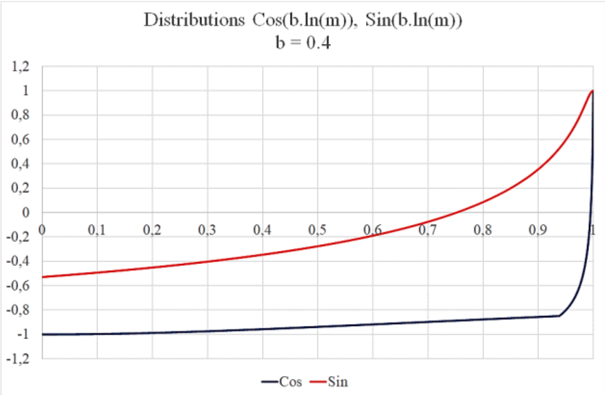
__Graphic 98__
a = 0.5 and b = 0.10



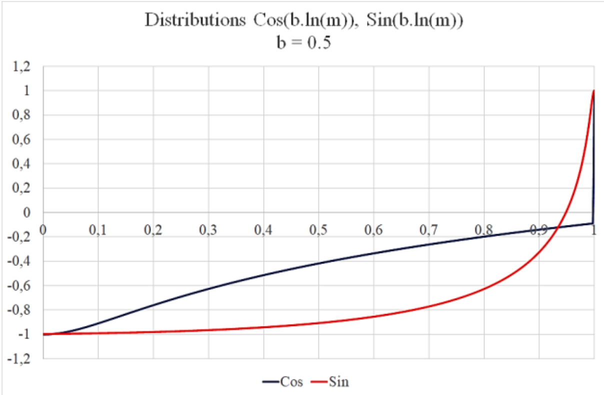
__Graphic 99__
a = 0.5 and b = 0.20



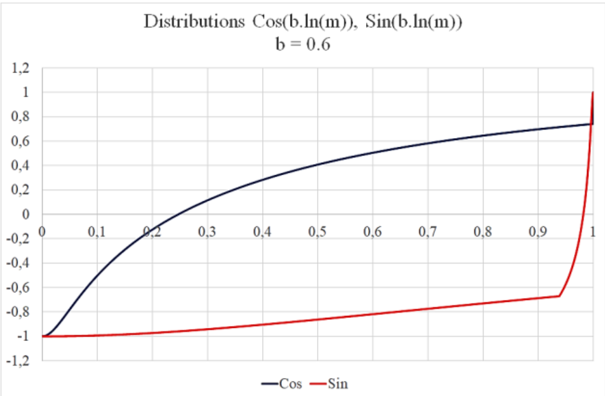
__Graphic 100__
a = 0.5 and b = 0.30



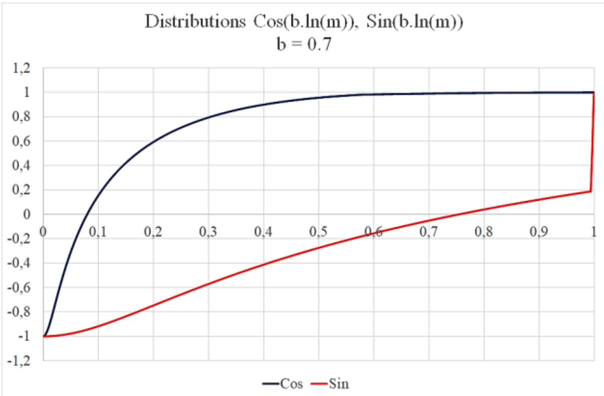
__Graphic 101__
a = 0.5 and b = 0.40



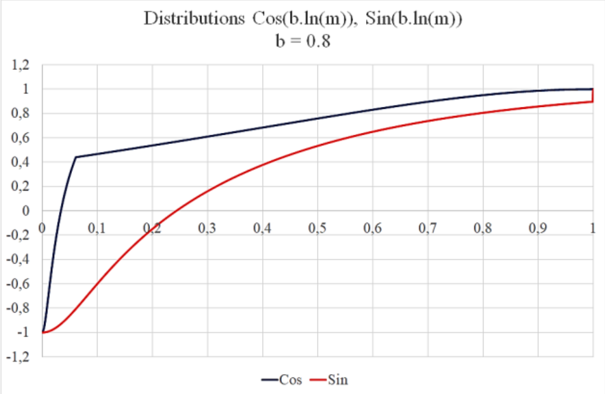
__Graphic 102__
a = 0.5 and b = 0.50



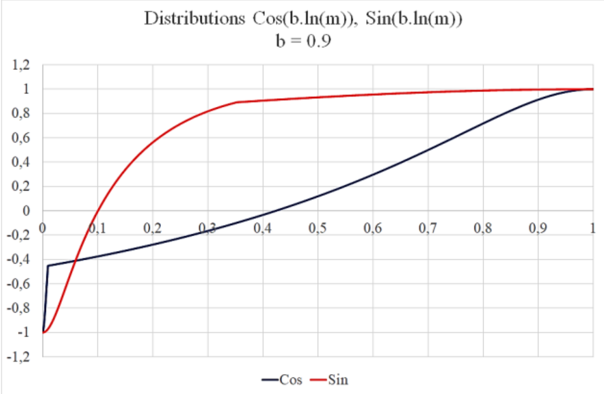
__Graphic 103__
a = 0.5 and b = 0.60



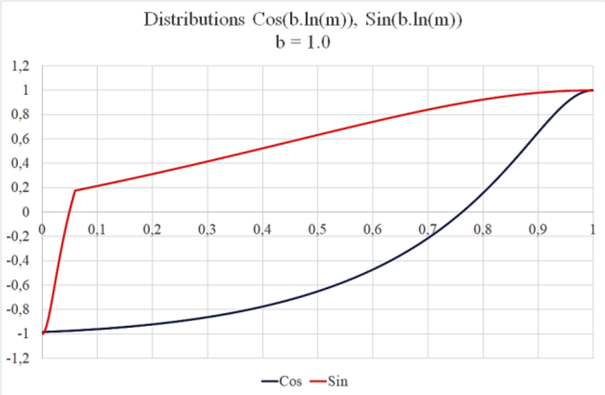
__Graphic 104__
a = 0.5 and b = 0.70



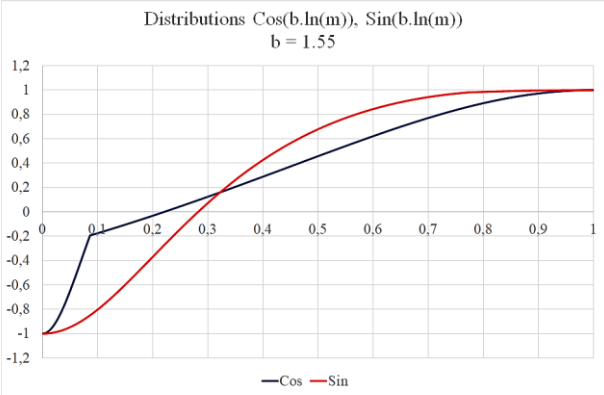
Graphic 105
a = 0.5 and b = 0.80



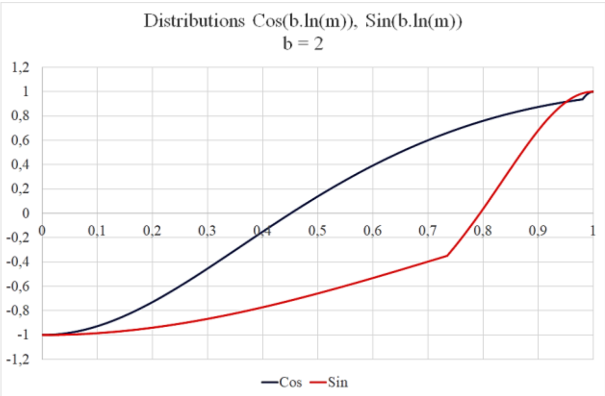
Graphic 106
a = 0.5 and b = 0.90



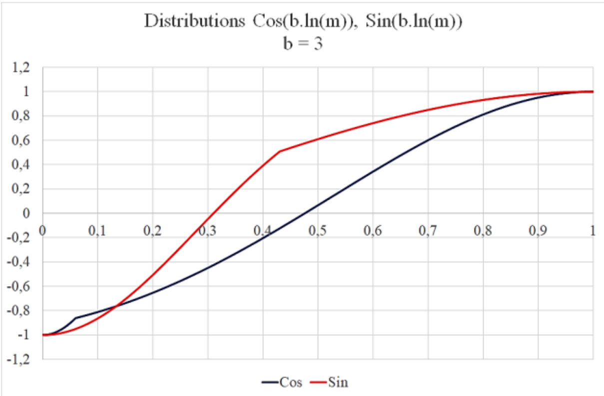
Graphic 107
a = 0.5 and b = 1



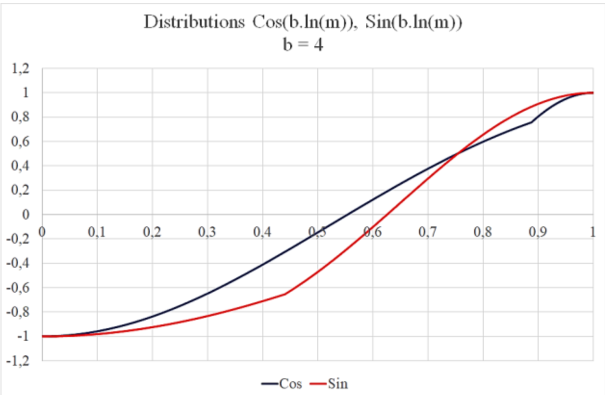
Graphic 108
a = 0.5 and b = 1.55



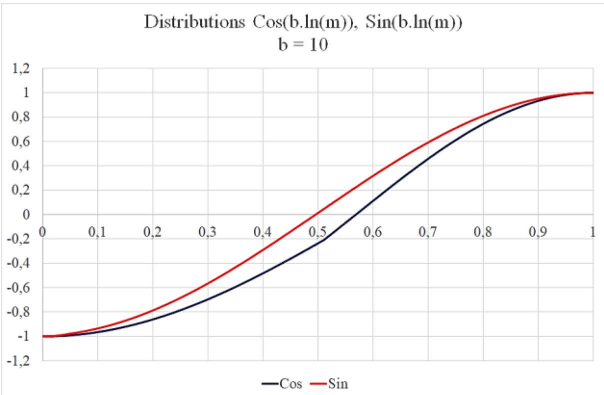
Graphic 109
a = 0.5 and b = 2



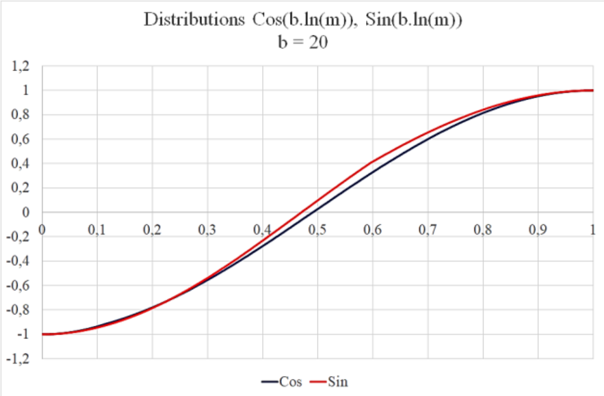
Graphic 110
a = 0.5 and b = 3



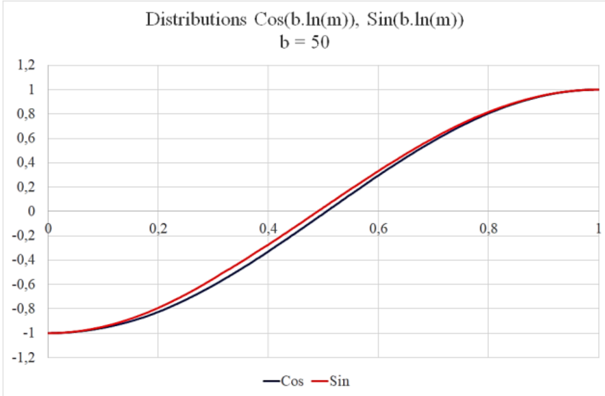
Graphic 111
a = 0.5 and b = 4



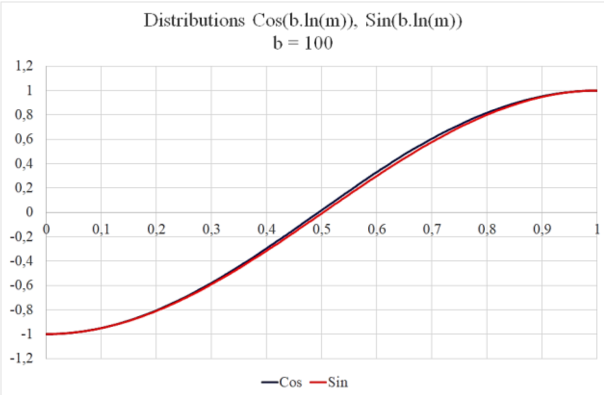
Graphic 112
a = 0.5 and b = 10



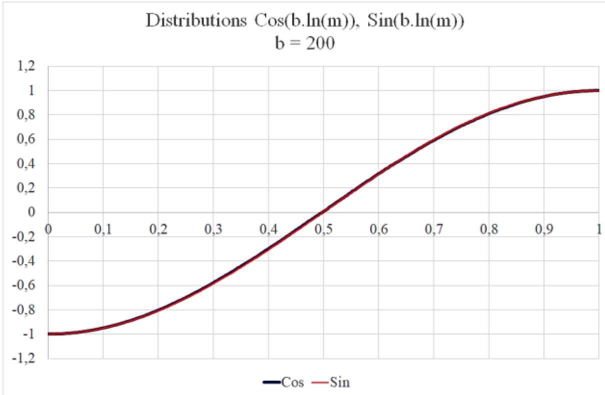
Graphic 113
a = 0.5 and b = 20



Graphic 114
a = 0.5 and b = 50



Graphic 115
a = 0.5 and b = 100



Graphic 116
a = 0.5 and b = 200