Number Theory / ***Théorie des nombres***

Imaginary values of the Zeta function zeros.

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Résumé The study of Riemann Zeta function zeros is essentially a lot of speculations. Here is a little more.

Valeurs imaginaires des zéros de la fonction Zêta.

Abstract Tout ce qui relève des zéros de la fonction Zêta de Riemann est essentiellement emprunt de spéculations. En voilà un peu plus..

Status Preprint.
Date V1 : 12/02/2015

Sommaire

1.Context.

The Riemann Zeta function is defined for Re(s) > 1 by the series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} \qquad (1)$$

Elle admet, pour Re(s) > 0, un prolongement analytique reposant sur la série entière Eta de Dirichlet η(s).

$$\eta(s) = (1-2^{1-s}).\zeta(s) \qquad (2)$$

Les zéros de η(s) sont ceux de ζ(s), mais aussi ceux de $1-2^{1-s}$ dont les zéros sont égaux à

$$s = 1+i.2\pi.k/Ln(2) \qquad (3)$$

where k is any relative integer. ζ(s) is not defined at s = 1, the zero corresponding to the value k = 0 should therefore be dismissed. We will call Schaetzel zeros the numbers 1+i.2π.k/Ln(2) because of its later practical use.

To get the zeros of η(s) means to solve the two equations :

$$\sum_{m=1}^{\infty} m^{-a}.(-1)^{m-1}.\cos(b.\ln(m)) = 0 \qquad (4)$$

and

$$\sum_{m=1}^{\infty} m^{-a}.(-1)^{m-1}.\sin(b.\ln(m)) = 0 \qquad (5)$$

This cancellation is written in a single equivalent equation using squares :

$$T_\infty(s) = T_\infty(a+i.b) = \left(\sum_{m=1}^{\infty} m^{-a}.(-1)^{m-1}.\cos(b.\ln(m))\right)^2 + \left(\sum_{m=1}^{\infty} m^{-a}.(-1)^{m-1}.\sin(b.\ln(m))\right)^2 = 0 \qquad (6)$$

For the first square, one gets so one term in cosine brought to the square and two terms otherwise, which are cos(b.ln(r).cos(b.ln(s))) and cos(b.ln(s).cos(b.ln(r)). We can therefore sum up by choosing r > s and adding a multiplicative factor of 2. Thus we have :

$$T_\infty(s) = \sum_{i=1}^{\infty n} \sum_{j=1}^{i} (i.j)^{-a}.(-1)^{i+j}.\cos(b.\ln(i/j)).si(i=j,1,2) \qquad (7)$$

Various hypotheses exist in the mathematical literature on that subject (Lindelöf hypothesis, Mertens hypothesis, conjecture of the correlated pairs, Hilbert-Polya conjecture, quantum chaos and link to a Hamiltonian operator). None of these aspects is addressed here. We develop two themes underneath, the first one being certainly just anecdotal.

2.Relative oscillations relatives of general terms.

For a series to converge, it is necessary that the general term converges to 0. We have here several choices of general terms, in particular the sum over j. Thus, we have for our zeros :

$$\lim_{i \to +\infty} \sum_{j=1}^{i} (i.j)^{-a}.(-1)^{i+j}.\cos(b.\ln(i/j)).if(i=j,1,2) = 0 \qquad (8)$$

Let us note general term as follows :

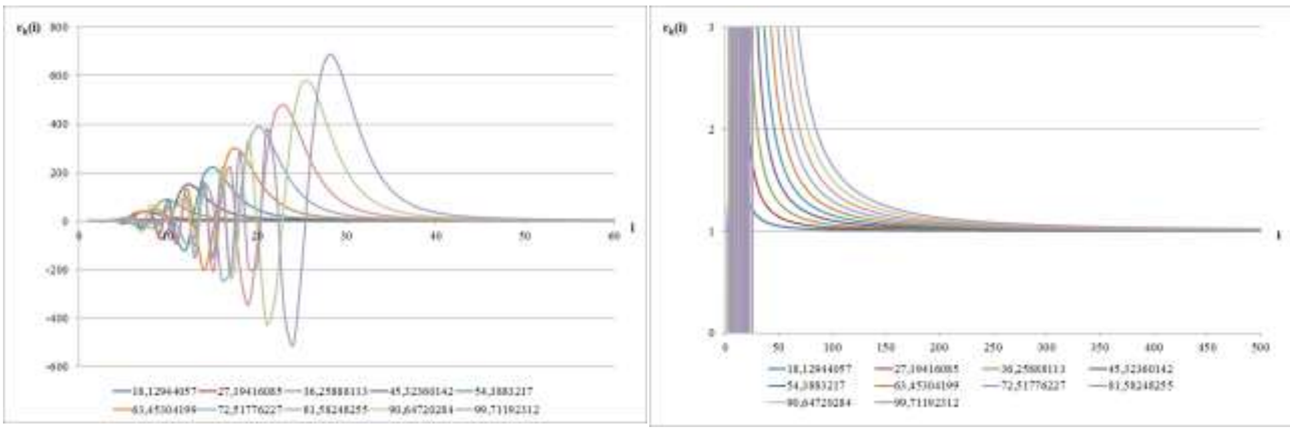
$$PT_i(s) = \sum_{j=1}^{i} (i.j)^{-a}.(-1)^{i+j}.\cos(b.\ln(i/j)).if(i=j,1,2) \qquad (9)$$

Numerical applications show that from a certain rank i, $PT_i(s)$ tends toward 0 by keeping the same (that is negative) sign. We then compare the different convergence with the convergence of the first of the zeros, for the Schaetzel zeros on the one hand, for the Riemann zeros on the other hand, that is we write ratios where i is incremented :

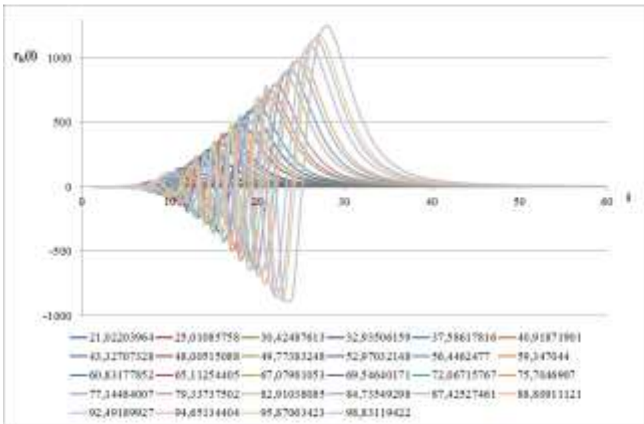
$$\frac{\sum_{j=1}^{i} (i.j)^{-ak}.(-1)^{i+j}.\cos(b_k.(\ln(i/j))).if(i=j,1,2)}{\sum_{j=1}^{i} (i.j)^{-a1}.(-1)^{i+j}.\cos(b_1.(\ln(i/j))).if(i=j,1,2)} \qquad (10)$$

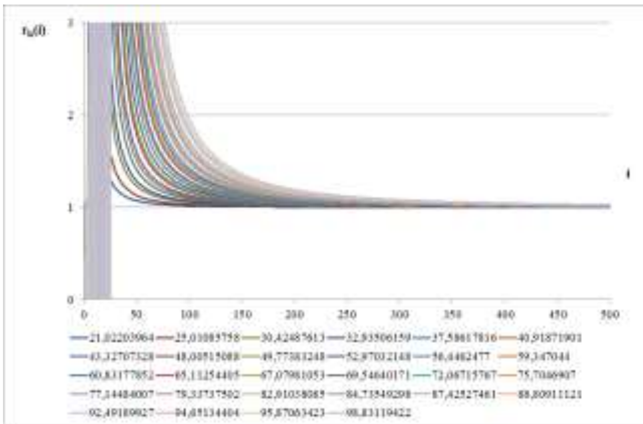
The allure of the curves for the ratios, noted $r_k(i)$ for the k^{th} zero and i^{th} ratio, obtained for the zeros in the interval [0,100] is remarkable. Here we have of course respectively $a_k = 1$ and $a_k = 1/2$ for Schaetzel and Riemann zeros in this interval.

<u>Schaetzel zeros</u>



Riemann zeros





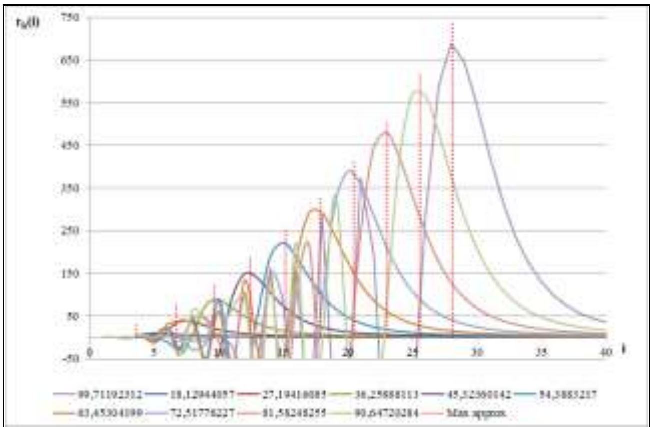
We note that after positive and negative excursions, the ratio $r_k(i)$ tends towards 1 (zooming in on figures as appropriate).
Summarizing

$$PT_i(s_1) < 0 \text{ from a certain row } i$$

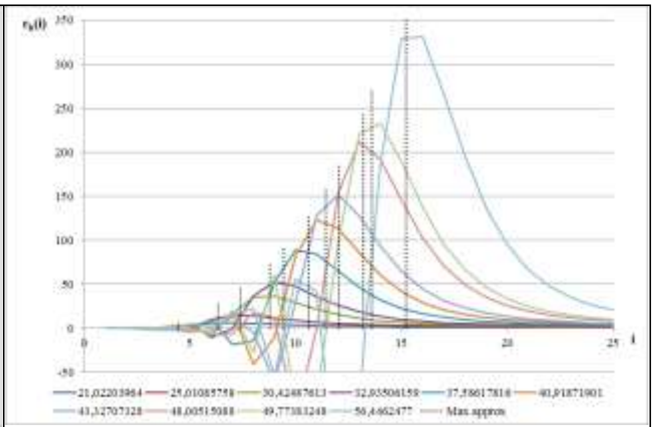
et

$$\lim_{i \to +\infty} PT_i(s_k)/PT_i(s_1) = 1 \qquad (11)$$

We note also for the final trip to the stabilization to 1 that the order of the curve over k is respected. In addition, for the curves relating to the Schaetzel zeros, we easily visualize some regularity of distances between curves. Similarly, the relative gaps between zeros of Riemann are found in related gaps between curves.
We took below as reference point the last maxima :



Schaetzel zeros



Riemann zeros

Here, we show the curves in their raw form (without smoothing as previously) since the calculations are discrete (i is incremented).
The abscissas, marked for maxima, follows :
- for the Schaetzel zeros, the equation $0{,}509.vs_k^{0,89}$, where vs_k is the imaginary value of the k^{th} Schaetzel zero,
- for the Riemann zeros, the equation $0{,}419.vr_k^{0,89}$, where vr_k is the imaginary value of the k^{th} Riemann zero.

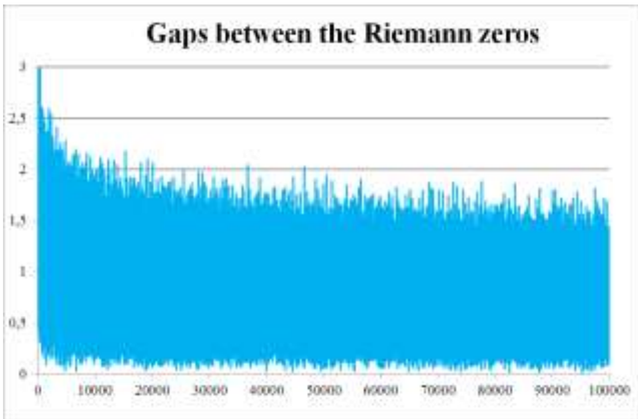
Here the 0.89 power is common. This is not however the best approximation. Indeed, as k increases, it is likely that the space corresponding to the Schaetzel zeros tends towards a constant and thus the power 1 (linearity) is surely the most adapted. We have adopted a different power to better stick to the early maxima.

3.A network as a barrier.

This second theme says nothing about what actually the Riemann zeros imaginary values are. It says that they may not be.

The zeros of Riemann and Schaetzel zeros are distinct (which we admit it).
Before generalizing this point, let us look at some numerical data.

The graph below, developed with data from Andrew Odlyzko [2], shows the gaps between the values of the adjacent Riemann zeros for the first 100,000 of them. Beyond the first 20 000 of them, the fluctuations are, with few exceptions, contained in the interval [0,2].



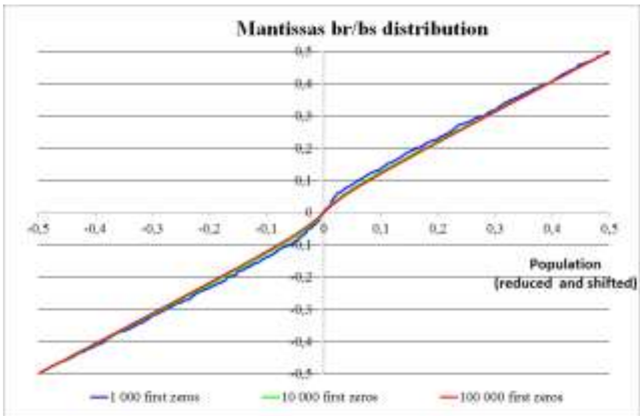
The gaps between zeros weak slowly in the viewed domain.
The density of zeros around T is a well-established datum in the literature. The inverse of its approximate value is $2\pi/\ln(T)$ which is not without remembering the gap between two Schaetzel zeros (when T = 2).

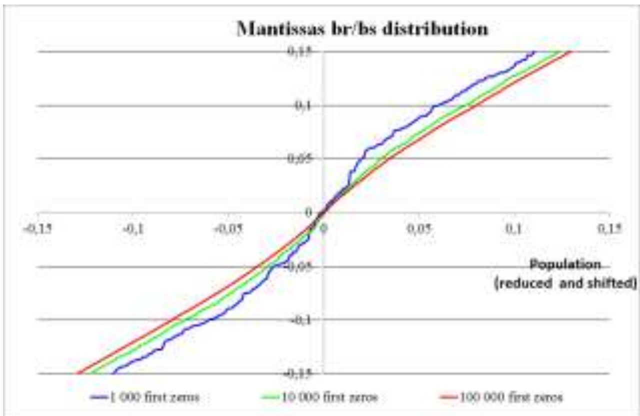
The table below reproduces some results on these gaps :

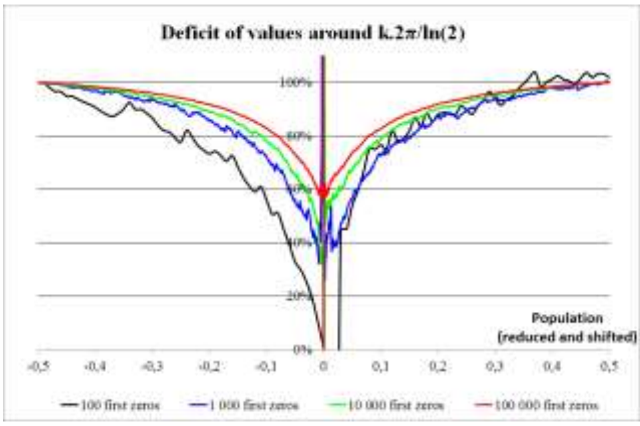
Qt zeros	Max	Mean value	Min	(2π/ln(2))/max	(2π/ln(2))/mean	(2π/ln(2))/min
10	6,8873	3,8836	1,7687	1,316	2,334	5,125
100	6,8873	2,2364	0,7158	1,316	4,053	12,664
1 000	6,8873	1,4063	0,1615	1,316	6,446	56,128
10 000	6,8873	0,9865	0,0377	1,316	9,189	240,453
100 000	6,8873	0,7491	0,0147	1,316	12,101	616,586

Thus, if for small values of the zeros, a Schaetzel zero shows for 2 Riemann zeros, this last number gradually increases and is greater than 10 for a population of 100,000 such objects. The asymptotic ratio is de facto ln(T)/ln(2).

We can ask ourselves the question of whether the two types of zeros interact a certainly way on each other. To do this, we divide the value of the Riemann zeros by $2\pi/\ln(2)$ and assess the mantissas to the nearest integer rounding. We get numbers in the interval [-1/2,1/2]. After ascending sort of resulting numbers, we trace the curve of the results with on x-axis the set of zeros normalized in an interval [-1/2,1/2]. In case of equiprobabilty, we expect a perfectly linear line. The resulting trends are represented below :





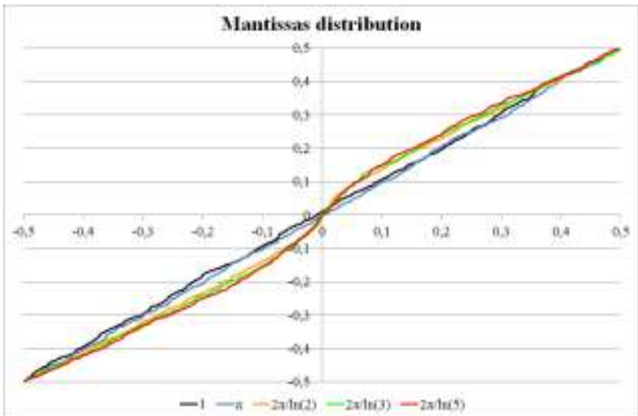


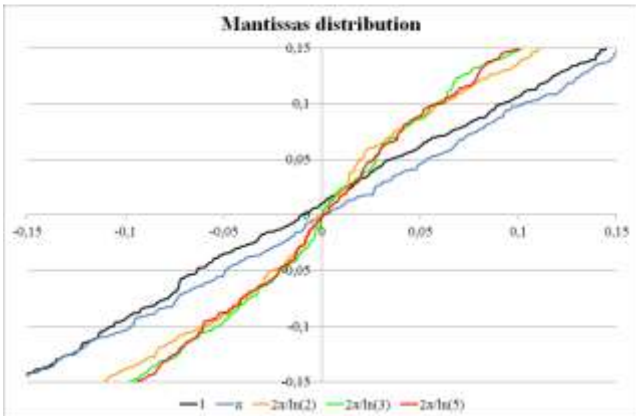
The second graph is just a magnification of the first one. When the Riemann zeros population which is involved increases, we see that the distribution is actually approaching the equiprobability. However near the origin, the gap reduces less. The third chart, that displays the ratio between the values of the x-axis and the y-axis, more clearly shows a deficit of values of type k.2π/ln(2), with k an integer, near the origin. Thus, when all of the Riemann zeros set is taken into account, it is reasonable to assume that the distribution becomes uniform except at 0 with a Dirac at this location.

Euler discovered the following fundamental formula, true for Re(s) > 1 and bearing on the primes :

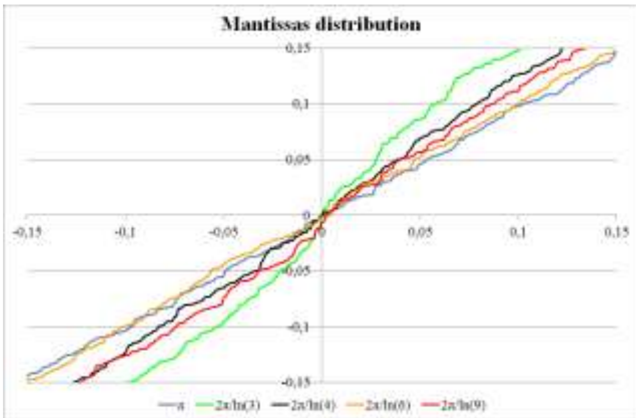
$$\zeta(s) = \prod_{p=2}^{\infty} \frac{1}{1-p^{-s}} \qquad (12)$$

The poles of this expression are equal to s = i.2π.k/ln(2), which we found the imaginary footprint above with the same formula, and s = i.2π.k/ln(p). Would it remain a distorted trace of the poles of Euler's formula, a pole being synonymous with forbidden value for Re(s) < 1 values ? We examined so these values (p > 2) in the highlight of the previously quoted Dirac. The representations below, corresponding to the first 1000 Riemann zeros set, allow to assume that no imaginary value (the real value being equal to 1/2) of Riemann zeros is equal a priori to 2π.k/ln(p).





Previous figures corroborate our intuition for p = 3 and p = 5 (after p = 2 in 2π/ln(p)). They also show that a random choice (here 1 and π) gives closer curves to y = x without particular origin phenomenon. When p is not prime, the position of the curves on the charts is either intermediate (for the p^k type, what is coherent in a 1/k ratio), either in the linear trend (if different from p^k, what is also expected a priori).



Without showing the charts here, we checked the look of curves for 2π.k/ln(p) taking p between 2 and 199. The looks are quite analogous without exception with however a gradual decline. Beyond that, up to p = 499, it remains a semblance of trend. Beyond that, our argument is no more noticeable. It would be necessary to work with a tool with more than 14 digits to possibly remove the doubt. Of course, a real mathematical proof would be more adequate.

That being supposed, $2\pi/\ln(p)$ is inferior to $\varepsilon > 0$ as soon as $p > e^{2\pi/\varepsilon}$. Thus, the set $\{2\pi k/\ln(p) \ / \ k \in Z, p \in P\}$ is dense in the set of the real numbers R. This network is therefore a genuine barrier to the crossing of the y = 0 axis by Riemann curves in cosine and sine, the strategy adopted (in some way) by these cosine and sine sums then being appropriately to cross together the axis y = 0.

References

[1] G.F.B. Riemann. Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse. Monatsberichte der Berliner Akademie. Nov 1859.

[2] http://www.dtc.umn.edu/~odlyzko/zeta_tables/index.html