Number Theory / *Théorie des nombres*

The Collatz matrices.

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Abstract   Any strictly positive integer results in 1 by Collatz algorithm. This is the conclusion we reach here by completing the study of the team of Departamento de Matemática do Instituto Superior Técnico in Lisbon carried out in 2004/2005 and based on Jacobi's formula for the derivative of a determinant.

Les matrices de Collatz.

Résumé   Tout entier strictement positif aboutit à 1 par l'algorithme de Collatz. C'est la conclusion à laquelle nous aboutissons ici en achevant l'étude de l'équipe de Departamento de Matemática do Instituto Superior Técnico de Lisbonne réalisée en 2004/2005 et reposant sur la formule de Jacobi pour les dérivées de matrices.

1   Introduction.

Collatz conjecture or Syracuse problem, which has many other names, is fairly known not to have to go back at length on its expression and we could find in the margin enough room to write our proof thanks to the work already carried out by the authors cited in references [1] and [2]. However, in order to prevent the reader from tedious diggings, some full-page developments are given here. The conjecture claims that any positive integer x_0 results in $x_n = 1$ by the repeated application of $3x_i+1$ if x_i is odd and $x_i/2$ if x_i is even. The team of the Departamento de Matemática do Instituto Superior Técnico in Lisbon [1] has introduced the possibility of solving this problem by demonstrating the invariance of the determinant of particular matrices M_k, $k \geq 2$, an idea recently revived by the authors of reference [2], without full complete proof, and which have aroused our curiosity.

2   Two theorems.

Theorem 1

Let us have $M_k = (m_{i,j})$ the square matrices of rank k defined by
- $m_{i,i} = 1$, i = 1 to k,
- $m_{i,i/2} = x$, if i = 0 mod 2 and i ≤ k,
- $m_{i,(3i+1)/2} = x$, if i = 1 mod 2 and (3i+1)/2 ≤ k,
- $m_{i,j} = 0$, otherwise.

Then if for all $k \geq 2$, $\det(M_k) = 1-x^2$, the Collatz conjecture is true.

Illustration

The Collatz matrix is given below at rank 8.

$$\begin{pmatrix} 1 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & x & 0 & 0 & 0 \\ 0 & x & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & x \\ 0 & 0 & x & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & x & 0 & 0 & 0 & 1 \end{pmatrix}$$

Proof

The proof is at pages 279 to 280 of reference [1].

Note

The authors of both references attempted the proof by assessing the determinant of M_k from the determinant of M_{k-1}. However, as the k = 8 example shows here, in cases k = 2 modulo 6, we have x simultaneously on the last line and the last column. This

requires for this evaluation to remove column or line inside the matrix, which gradually for large rows causes the thread of the exam to be lost. We propose a simple and outright alternative avoiding these withdrawals.

<u>Theorem 2</u>

Let us have $M_k = (m_{i,j})$ a Collatz matrix of rank at least 2.
Then the determinant of the matrix is invariant and equal to $1-x^2$ for fixed x.

<u>Proof</u>

Adding a linear combination of other columns to a given matrix's column does not change the value of the determinant. Our proof is based on this property of invariance. The representation of the example at rank 23 below makes it easy to visualize the evolution to a lower triangular matrix of the initial matrix. The 0's are not posted. The process is initiated from below and right side. The initial numbers are shown in black and the successive arrows indicate the evolutions after making a linear combination with a given line. When the combination does not change the value of an item, the evolution is not indicated. This is usually the case for $1 \rightarrow 1$ and $0 \rightarrow 0$. It is easy to verify that the hypothesis is true for all matrices of ranks between 2 and 7. Let us have then M_k a given matrix of rank k greater than 8. The first item of value x to be examined is at the position $(1+2n, 2+3n)$ where $n = \lfloor (k-2)/3 \rfloor$. The line subject to evolution is therefore $1+2n$ and the line with which we make the combination is line $2+3n$ line. The subtraction of the latter multiplied by x gives $-x^2$ at $(1+2n, (2+3n)/2)$ if n is even and leaves the previous value at 0 otherwise. It should be noted here that the position $(1+2n, (2+3n)/2)$ is to the left of $(1+2n, 1+2n)$ and is therefore indeed in the lower triangular part of the matrix. This procedure is applied to the line $(1+2n)-2$, then $(1+2n)-4$, etc. until reaching line 3. For all matrices (rank $k \geq 8$) each x in upper triangular part and column superior or equal to 8 turns into 0 and the positions $(4+3n, 5+4n)$ are filled with $-x^2$ on a line between the x line of the lower triangular and the line of the 1's of the main diagonal.

The only irregularities to this repetitive pattern are in the upper left part that we reproduce below :

The application of the process to line 5 using line 8 is consistent with what preceded giving $-x^2$, but at line 3, we then get at first in (3, 4) the value $0-1.x.(-x^2) = x^3$ that remains in the upper triangular part. A second treatment, using line 4, however, allows the value to be transferred to the position (3, 2), this time landing in the lower triangular part, reducing the previous value to 0. Finally, we treat the first line and the switch from x to 0 in (1, 2) gives $1-x^2$ at (1, 1). We end up with a lower triangular matrix with 1 on the main diagonal everywhere except $1-x^2$ in the first line. The determinant of the initial matrix is therefore $1-x^2$.

3 Conclusion.

The Collatz's algorithm in N* consistently ends up at 1.
The proof is based on the arguments of other authors who have achieved the difficult part of the study. This point is analogous and follows another article [4] of our making leading to exactly the same conclusion for the conjecture (i.e. its veracity) the difficult work having been at that time Riho Terras's achievement [3].

References

[1]   João F. Alves, Mario M. Graça, M.E. Sousa Dias. José Sousa Ramos. A linear algebra approach to the conjecture of Collatz. Elsevier. Received 17 March 2004, accepted 27 July 2004. Linear Algebra and its Applications 394 (2005) 277–289.

[2]   Louis H. Kauffman. Pedro Lopes. On the orbits associated with the Collatz conjecture. ArXiv:2005.13670. Submitted on 27 May 2020 (v1), last revised 1 Jan 2021 (v3).

[3]   Riho Terras. A stopping time problem on the positive integers. Acta Arithmetica 30 (1976), 241–252

[4]   https://sites.google.com/site/schaetzelhubertdiophantien/ Syracuse trees.

[5]   https://sites.google.com/site/schaetzelhubertdiophantien/ Altitude flight with Collatz numbers : the genesis of a Pascal trihedron.

[6]   https://sites.google.com/site/schaetzelhubertdiophantien/ Geography of the Pascal trihedron

[7]   https://hubertschaetzel.wixsite.com/website