CONVEXITY OVER THE CRITICAL BAND AND RIEMANN HYPOTHESIS

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Abstract. The study of peculiar first and second partial derivatives drawn from Dirichlet's Eta function enables us to confirm the Riemann hypothesis.

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1. Introduction

"Mathematics consists in proving the most obvious thing in the least obvious way." George Pólya.

Indeed, the mathematical literature abounds in clues in favour of Riemann's hypothesis [1]. One of those is the strict adherence to the hypothesis of billions of zeroes obtained by numerical evaluation. Limiting yourself to

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the accounting of the first zeros, regardless of their number however, does not give any general property which enables to infer unquestionably a rule for those coming next.

Here we will expand the study on all points of the critical strip in order to encompass the zeros themselves. Luckily, due to a specific symmetry, the search can be reduced on the lower half of the critical band including the critical line.

Our investigation starts and is based on one among the various analytical extensions of the Riemann's series: the Dirichlet Eta function. It establishes the existence of a lower boundary for an indicative function of (positive) convexity deduced from it, resulting in the impossibility of symmetrical Riemann zeros on either side of the critical line.

The proof consist first to establish the property numerically on a (finite) part of the domain of definition of the said indicative function. Additional arguments then will allow us to extent the property to the whole (infinite) domain. A remarkable relationship between extrema and zeros of the hereafter studied expression is crucial to lead up to the desired conclusion.

Trying to address a wide audience, many graphic illustrations are given here to make the thread of ideas as accessible and clear as possible. Despite all these additions, the article remains relatively short. Can its content then be worth a million of others?

2. Analytic continuations

Let us have $s = a + i.b$ some complex number. The parameters a, b and s are used in the same context throughout this presentation. The Riemann Zeta function is defined for $Re(s) > 1$ by the entire function:

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$$

The function diverges roughly in the form of two exponentially growing sinusoids (its real and imaginary parts) for $a = Re(s) \leq 1$, and the zeros of this function, called here (non-trivial) zeros of Riemann, correspond to numbers s such as the middle axis of this sinusoid aligns asymptotically with the axis $y = 0$.

Note that it is impossible to find precisely the zeros of this function by exploiting only this remark.

Riemann's Zeta function admits, for $Re(s) > 0$, an analytical continuation based on Dirichlet's entire function $\eta(s)$.

$$\eta(s) = (1 - 2^{1-s}) \cdot \zeta(s) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^s}$$

This equality shows that the zeros of Dirichlet's Eta function are the union of zeros of $1 - 2^{1-s}$ and zeros of Riemann's Zeta function. We call the first nominees, the Dirichlet's zeros. So, we get the solutions' sets:

$$\{\text{Eta\_zeroes}\} \equiv \{\text{Dirichlet\_zeroes}\} \cup \{\text{Riemann\_zeroes}\}$$

The Dirichlet zeros are equal to

$$s = 1 + i \cdot 2\pi k / \ln(2)$$

where k describes the relative integers' set Z. These zeros, with constant real value ($a = 1$), are genuine Siamese brothers of Riemann zeros as we showed in another article (see reference [6]). The formers are inseparable from the latter and allow us to anticipate the behaviour of Riemann's zeros. They are the trivial image of the veracity of Riemann's hypothesis.

That said, let us now introduce the functional equation (see reference [2]):

$$\zeta(s) = 2^s \cdot \pi^{s-1} \cdot \sin(\pi s/2) \cdot \Gamma(1-s) \cdot \zeta(1-s)$$

This further analytical continuation of the Riemann function introduces, due to the sinus function, additional zeros $-2n$ for any natural (thus positive) integer, called trivial zeros, that are absent in previous functions. This last continuation is useful to our argument because we can state the following theorem:

Theorem 1. *The non-trivial Riemann zeros are symmetrical to the axis $s = 1/2$ within the critical band.*

Proof. Let us have $\xi(s) = (1/2)\cdot$ s $\cdot$ (1-s) $\cdot\pi^{-s/2} \cdot \Gamma(s/2) \cdot \zeta(s)$. We get (see reference [3]) immediately $\xi(s) = \xi(1-s)$. □

Note 1. The modulus of the complex number $\zeta(a + i.b)$ is expected to decrease with parameter a as the series' term is m^{-a}. Therefore it would seem obvious that the right and left side of the critical line cannot be equal at same b value. But if some constant b is close enough to b_r such as $\zeta(1/2+i.b_r) = 0$, then when a is varying from 0 to 1, the decrease of the modulus of $\zeta(a+i.b)$ fails around $a = \frac{1}{2}$. Moreover, one finds readily an exception to that decrease for values $b < 6.3$ without any near zero and we wish to already highlight in this note this peculiar exception as it will unsurprisingly show up again (at similar low range of b values) below although in a totally different context.

Theorem 2. *If the set of all Riemann's zeroes such as $0 < a < 1/2$ is empty, then Riemann's zeros are all on the $1/2$ axis.*

Proof. This is a trivial consequence of theorem 1. □

In 1896, Hadamard and La Vallée-Poussin [3] independently proved that no zero could be on the $Re(s) = 1$ line, and therefore that all non-trivial zeros should be in the interior of the critical band $0 < Re(s) < 1$. For this reason, we chose previously to write $0 < a < 1/2$ instead of $0 \le a < 1/2$, although this second way doesn't in any way hinder us here, quite the contrary, since it allows us to confirm the work of the authors cited simply by examining case $a = 0$ (which is actually done in this article).

3. Explicit equations of the Eta function and related functions

Let us have $\ln(x)$ the Napierian logarithm of x. From the Eta function expression

$$\eta(s) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^s} \tag{1}$$

we get

$$\eta(s) = \sum_{m=1}^{\infty} (-1)^{m-1} \cdot m^{-a} \cdot \cos(b.\ln(m)) + i \cdot \sum_{m=1}^{\infty} (-1)^{m-1} \cdot m^{-a} \cdot \sin(b.\ln(m))$$

The search for the zeros of $\eta(s)$ is therefore tantamount to solve simultaneously the two equations:

$$\sum_{m=1}^{\infty} (-1)^{m-1} \cdot m^{-a} \cdot \cos(b.\ln(m)) = 0$$

and

$$\sum_{m=1}^{\infty} (-1)^{m-1} \cdot m^{-a} \cdot \sin(b.\ln(m)) = 0$$

Let us have

$$C_0(a,b) = \sum_{m=1}^{\infty} (-1)^{m-1} \cdot m^{-a} \cdot \cos(b.\ln(m)) \tag{2}$$

and

$$S_0(a,b) = \sum_{m=1}^{\infty} (-1)^{m-1} \cdot m^{-a} \cdot \sin(b.\ln(m)) \tag{3}$$

Then the cancellation of $\eta(s)$ is equivalent to the cancellation of:

$$(C_0(a,b))^2 + (S_0(a,b))^2 = 0$$

Theorem 3. *Let us have:*

$$D_0(a,b) = \frac{(C_0(a,b))^2 + (S_0(a,b))^2}{2}$$

If the partial first derivative of $D_0(a,b)$ never cancels over the domain $0 \leq a < 1/2$ and the partial second derivative of $D_0(a,b)$ is strictly positive over the domain $0 \leq a \leq 1/2$, both partial derivatives being taken versus the parameter a, then Riemann's hypothesis is true.

Proof. By our hypothesis, the $D_0(a,b)$ function is convex over the said domain. If the partial first derivative never cancels over the same domain except for $a = a_0 = 1/2$, the positive expression $D_0(a,b)$ is necessary increasing when a decreases from $1/2$ to 0. The expression $D_0(a,b)$ can then be null only for $a_0 = 1/2$. $\square$

We note the expressions of successive partial derivatives of $C_0(a,b)$ and $S_0(a,b)$ versus a as follows:

$$C_k(a,b) = \sum_{m=1}^{\infty} (-1)^{m-1+k} (\ln(m))^k \cdot m^{-a} \cdot \cos(b.\ln(m)) \tag{4}$$

and

$$S_k(a,b) = \sum_{m=1}^{\infty} (-1)^{m-1+k} (\ln(m))^k \cdot m^{-a} \cdot \sin(b.\ln(m)) \tag{5}$$

Theorem 4. *Sums and products of $C_k(a,b)$ and $S_k(a,b)$ are convergent series over the critical band, the $a=0$ boundary excluded.*

Proof. Since $(Ln(m))^k$ is negligible asymptotically in front of m^a when $a > 0$, $C_k(a,b)$ and $S_k(a,b)$ are convergent sums as is initially $\eta(s)$. In addition, sums and products of convergent series are also convergent. □

This allows us to write successive partial derivatives, versus a, of $D_0(a,b)$ as follows:

$$D_1(a,b) = 2(C_0(a,b) \cdot C_1(a,b) + S_0(a,b) \cdot S_1(a,b)) \tag{6}$$

and

$$D_2(a,b) = 2(C_0(a,b) \cdot C_2(a,b) + S_0(a,b) \cdot S_2(a,b) + (C_1(a,b))^2 + (S_1(a,b))^2) \tag{7}$$

Our objective is to prove that $D_2(a,b) > 0$ for $0 \leq a \leq 1/2$.
There is a trivially positive part to $D_2(a,b)/2$ namely $P_2(a,b) = (C_1(a,b))^2 + (S_1(a,b))^2$. It ought to be compared to the complementary part $Q(a,b) = C_0(a,b) \cdot C_2(a,b) + S_0(a,b) \cdot S_2(a,b)$. As long as $Q(a,b)$ is positive, everything is fine. If $Q(a,b)$ is negative and we still have $\mid Q(a,b) \mid < P(a,b)$, then the $D_2(a,b)$ expression remains positive and Riemann's hypothesis stems from it. It is therefore appropriate to examine the evolution within the lower critical band of the ratio:

$$R2(a,b) = \frac{C_0(a,b) \cdot C_2(a,b) + S_0(a,b) \cdot S_2(a,b)}{(C_1(a,b))^2 + (S_1(a,b))^2} \tag{8}$$

From this argument results immediately the following theorem equivalent to theorem 3:

Theorem 5. *If $D1(a,b) \neq 0$ for $0 \leq a < 1/2$ and if $R2(a,b) > -1$ for $0 \leq a \leq 1/2$, b any given real number, then Riemann hypothesis is true.*

Note 2. This is a sufficient (and not necessary) condition: A contradictory b (ending with $R2(a,b) \leq -1$) only excludes the desired result for that value b and its immediate vicinity. We will see below that, indeed, there are b values such as the expression $R2(a,b)$ is less than the -1 value, for $0 \leq a \leq 1/2$, at abscissas smaller than that of the first Riemann zero (and the first Dirichlet zero).

Note 3. It should also be noted that because of the symmetry of the functional equation, we only survey the $b \geq 0$ values, the arguments being in any way identical in the $b \leq 0$ case.

4. The D1 problem

Let us start with the partial first derivative hypothesis $D_1(a,b) \neq 0$.

Proposition 1. *The function $D_1(a,b)$ is strictly negative and increasing monotonously with respect to the parameter a on the left side of the critical line.*

Argument. The monotony of the $D1(a,b)$ is an immediate result of $D2(a,b) > 0$ which will be the main object of this article later (see theorem 8). The strict negative values and monotonous evolution of the expression $D_1(a,b)$ from $b = 3$ up to $b = 100000$ is easily checked numerically. A few samples are given in figure 1. The term b is only present within the cosine and sine functions, functions which are limited to the small interval of values -1 to 1. Thus a very large cover of all intermediary possibilities from this numerical sample. Moreover it is well-known that there are no non-trivial zeroes well above the value $b = 10^5$, namely at least up to 10^{22}, confirming somewhat the sample's result to a much greater extent. However, it doesn't make a strong mathematical argument as zeros may nevertheless be reached before the $a = 1/2$ limit. Thus the need of our study on $D2(a,b)$ which will ultimately show that this particular limit value is effectively a threshold with a specific divergence type phenomena (acting back on $D1(a,b)$).

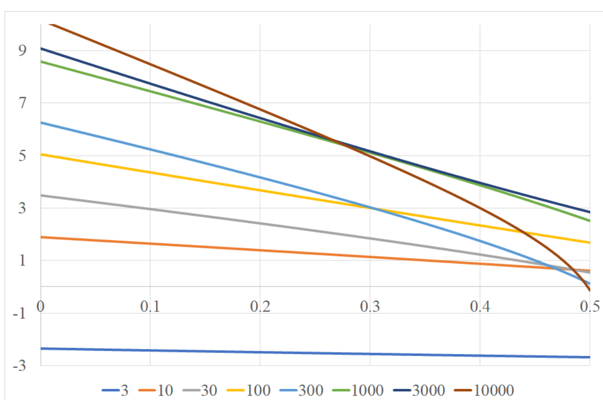

FIGURE 1. Samples of $Ln(-D1(a,b))$ for various values of given b and $a \in [0, 1/2]$.

Note 4. When the chosen b value corresponds to a non-trivial zero, the expression $Ln(-D1(a,b))$ will of course plunge towards $-\infty$, as a tends towards $1/2$, which none of the illustrations in figure 1 do.

5. Preliminary lemmas

The only needed result here is lemma 3 that we will provide below. It is about proving that some expressions, positive by construction, never vanish. This result is however sufficiently important, as it provides crucial confirmation on continuity topics, to spend quite some time on technical details.

That said, let us start by comparing the relative positions of the following functions

$$SC_k(a,b) = (C_k(a,b))^2 + (S_k(a,b))^2,$$

based on $C_k(a,b)$ and $S_k(a,b)$ as defined previously by the expressions 4 and 5, as k is incremented.

Lemma 1. *If $a \in [0,1/2]$, we have almost everywhere $SC_{k+1}(a,b) > SC_k(a,b)$ as soon as $b > 20$.*

Proof. For $a = 0$, it is a straightforward numerical verification. From theorem 4, we know that the $SC_k(a,b)$ are convergent series for $a > 0$ whatever the value of b. Then, using $\cos(\phi - \psi) = \cos(\phi)\cos(\psi) + \sin(\phi)\sin(\psi)$ and $\ln(m_1) - \ln(m_2) = \ln(m_1/m_2)$, the general term of $SC_k(a,b)$ is equal to

$$(-1)^{m_1+m_2}.(\ln(m_1))^k.(\ln(m_2))^k.(\cos(b\ln(m_1/m_2)))/m^{2a}.$$

The effect of $(\ln(m_1))^k.(\ln(m_2))^k$ is a scaling factor and therefore the $SC_k(a,b)$ and $SC_{k+1}(a,b)$ curves will be in a blurred nesting positions one to the other. The ratio of the general term of $SC_{k+1}(a,b)$ to that of $SC_k(a,b)$ is equal to $\ln(m_1).\ln(m_2)$ and therefore greater then 1 as soon as $m_1 > 2$ and $m_2 > 2$, thus providing a greater than 1 scaling at some step. Numerical evaluations, like the example for $a = 1/2$ (and close values) given in figure 2, provide the abscissas b where this greater than 1 scaling starts effectively. This abscissa b is in the range value of the imaginary part of the first Riemann non-trivial zeroes and therefore of no harmful effect on the rest of the present analysis (knowing that there are no Riemann zero exception to $a = 1/2$ found up to billions of them). □

Note 5. It is obvious that infinite sums cannot be calculated up to infinity and therefore the art of truncations has to be mastered. The Eta function value oscillates, with here and there wide jumps, before the convergence process starts. The maximum m_{max} (instead of $+\infty$) of the integer parameter m in the expression (1) has to be carefully chosen and it is easy to prove (see appendix A) that

$$m_{max} = \lfloor 2b/\pi \rfloor \tag{9}$$

is some minimum value of the m truncation to adopt. What is true for the Eta function is of course also for any derived function and more precise evaluations may sometimes require much larger m_{max} then initially expected (when near 0 values and ratios are involved).

This said, the graphics in figures 2 and 3 clearly show that, for $b \gtrsim 20$, the $SC_{k+1}(a,b)$ curves are above the $SC_k(a,b)$ curves in nesting positions one to the other. These figures are drawn for $a = 1/2$. One gets the same nesting positions for any parameter a sample within $[0, 1/2[$ as k increases provided that again $b \gtrsim 20$. As the phenomena is progressive one has only to check the additional cases $a = x/10$, $x = 0, 1, 2, 3$ and 4.

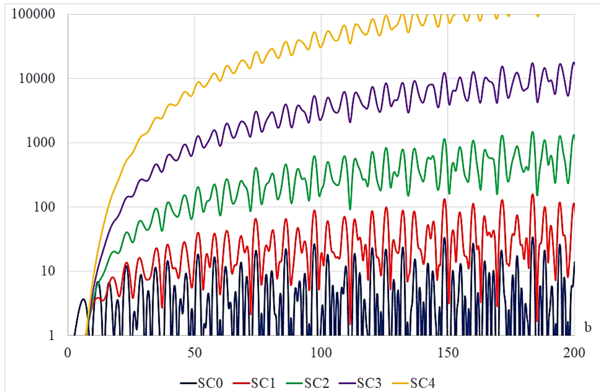

FIGURE 2. Function $SC_{k+1}(a, b)$ versus b, $a = 1/2$, $k = 0$ up to 4.

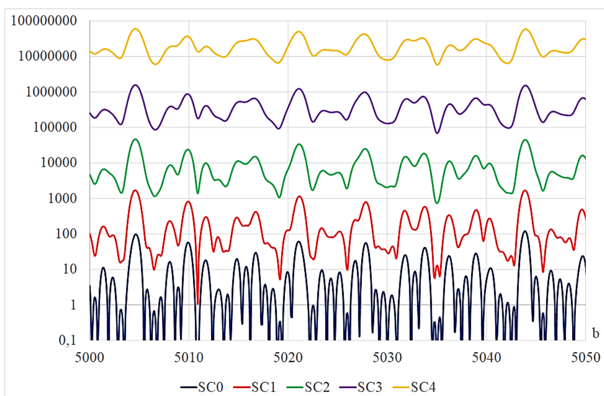

FIGURE 3. Function $SC_{k+1}(a, b)$ versus b, $a = 1/2$, $k = 0$ up to 4.

Note 6. The curves are given with a standard step $\Delta b = 1/10$. The downwards peaks are not necessarily fully formed here. Nevertheless, one can see clearly the peak of $SC_1(a, b)$ at the level of Riemann's zero corresponding to $b \approx 5010.9331981$ for example in figure 3. The nesting does not prevent in any way to have, close to the Riemann zeros, very small (non-negative) values for $SC_k(a, b)$, whatever $k > 0$, and this especially for $SC_1(a, b) = (C_1(a, b))^2 + (S_1(a, b))^2$. This effectively allows us to find increasingly larger $R2(a, b)$ values here or there, since $SC_1(a, b)$ is the denominator of that expression.

Note 7. The terminology "almost everywhere" is not that of the probability theory. The term only means what is meant, thus without any notion of density, as the study is not completed at this stage.

Let us have a zoom to understand what happens when $SC_k(a,b)$ gets closer to $SC_{k+1}(a,b)$. In figures 4, at the downwards peaks' levels, Δb is taken here equal to 1/10000 and smaller. For SC_0, at Riemann's zeros level, both peaks take on lower and lower values (since the possible limit here is 0). For SC_1, the peak progresses to lower values between truncations with 10000 terms up to 50000 terms. This progression then stops. The minimum value statements are 0.00097147 for 10000 terms, 0.00053732 for 50000 terms, 0.00058222 for 150000 terms, nothing in fact prohibiting a higher value in the final instance as accuracy increases. We see above the attraction that constitutes two narrow peaks for $SC_k(a = 1/2, b)$ on the expression $SC_{k+1}(a = 1/2, b)$ in the hereby $k = 0$ case. As two peaks create a peak above them, the phenomenon may occur frequently only up to the $SC_1(1/2, b)$ level. An imposing peak for $SC_2(1/2, b)$ is certainly rare, requiring 3 very close Riemann zeros. A significant downwards spike is undoubtedly exceptional when $k > 2$.

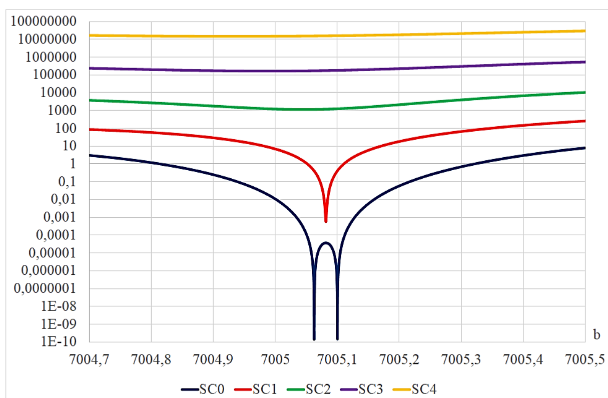

FIGURE 4. Function $SC_{k+1}(a,b)$ versus b, $a = 1/2$, $k = 0$ up to 4.

Lemma 2. *If* $a \in [0, 1/2]$ *and* $b > 20$ *then systematically* $SC_{k+1}(a,b) > SC_k(a,b)$.

Proof. The numerical study clearly shows that the inequality is true except possibly near the downwards peaks' positions and moreover the critical case to examine is that of the relative position of $SC_0(a,b)$ and $SC_1(a,b)$, higher k cases being even more obvious. So, let us place ourselves at a peak for $SC_1(a, b_{peak})$. The expression $SC_0(a, b_{peak})$ presents, at this abscissa, a partial derivative versus b necessary further close to 0 as the two curves get more narrow. It can be written, with the notations for partial derivatives given in paragraph 7.3, $\partial_b((C_0(a,b))^2 + (S_0(a,b))^2) = 2(C_0(a,b) \cdot S_1(a,b) - S_0(a,b) \cdot C_1(a,b))$ with value to be taken at $b = b_{peak}$. Hence the approximate equality $C_0(a,b_{peak}).S_1(a,b_{peak}) \approx S_0(a,b_{peak}).C_1(a,b_{peak})$. Let us simplify the entries by failing to repeat the coordinates (a,b_{peak}). The C_0 and S_0 functions are non-zero since they are not placed at a Riemann zero. We then have at the peak of SC_1, the ordinate difference between SC_1 and SC_0 equal to $C_1^2+S_1^2-(C_0^2+S_0^2) \approx (C_0.S_1/S_0)^2+S_1^2-(C_0^2+S_0^2) = ((S_1/S_0)^2-1)\cdot(C_0^2+S_0^2)$, and in the same way, $C_1^2+S_1^2-(C_0^2+S_0^2) \approx (C_1^2+(C_1.S_0/C_0)^2-(C_0^2+S_0^2) = ((C_1/C_0)^2-1)\cdot(C_0^2+S_0^2)$. These expressions, as sums of continuous functions, are continuous. Thus, for the difference $C_1^2+S_1^2-(C_0^2+S_0^2)$ to become negative, it must first be able to go to zero. The coordinate point (a,b_{peak}) being intermediate between two Riemann zeros, we get $C_0^2+S_0^2 \neq 0$. This means that the nullity of $C_1^2+S_1^2-(C_0^2+S_0^2)$ results in the joint nullity of $(C_1/C_0)^2-1$ and $(S_1/S_0)^2-1$, or simultaneously $C_1 \to C_0$ and $S_1 \to S_0$ near the abscissa of the peak. So let us examine that situation. The difference between the general terms of C_0 and C_1 on one hand and of S_0 and S_1 on the other hand is the multiplicative factor $ln(m)$. Referring again to theorem 4, we know that $abs(1/m^a) < abs(ln(m)/m^{a+\epsilon})$, for any $\epsilon > 0$ when m is large enough. That means $abs(cos(b.ln(m))/m^a) \approx abs(ln(m).cos(b.ln(m)))/m^a$ (taking the example of C_0 and C_1) in the specific context of the asymptotic shape of these curves with increasing m. The term $ln(m)$ doesn't change the curvature (that is the second derivative) is a noticeable way, in other words, the greatest effect is a linear effect, that is a shift plus a scaling. The shift can be disregarded and let us focus on the scaling by some global constant scaling factor. The probability for it to be exactly 1 is equal to 0 for the C_1/C_0 ratio as $ln(m)$ goes to infinity when m goes to infinity.

But, let us consider anyway the possibility of such event and check by the most tangible way, that is a numerical example, what would happen then. Note however that the case $C_0 \to C_1$ and $S_0 \to S_1$ cannot be reproduced easily. There has been no attempt to do so here as this additional justifying constraint is not necessary as the reader will see. That said, let us focus on the peak nearby the abscissa $b_{peak} \approx 7005.08168$. We do have $C_1^2 + S_1^2 - (C_0^2 + S_0^2) \to 0$ (see figure 5). But $(C_1/C_0)^2 - 1 \approx (S_1/S_0)^2 - 1 \approx 58$ (see figures 6 and 7, which is a reminder of the topic of the constant scaling factor) as long as we take a truncation between 800 and 2300 terms, case where $C_1^2 + S_1^2 - (C_0^2 + S_0^2)$ does not yet converge towards 0. When this convergence finally begins with the sufficient number of terms (here above 2500 in figure 5), the ratios C_1/C_0 and S_1/S_0 suddenly enter an unstable phase due to the low values of C_0, S_0, C_1 and S_1 (in figures 6 and 7) oscillating around the previous value 58. These oscillations seem to remain here regardless of the number of terms. But let us remind again that we are addressing a numerical case. If one does dispose of a sufficient number of terms and a sufficiently exact calculating tool (here we rely only on 14 significant digits tool), we might observe a diminishing oscillations so long the two relevant curves do not touch. The observed messy oscillations are simply a preview, due to the imperfection of any numerical evaluation, of the actual phenomena occurring within infinite number of terms and infinite precision. This actual phenomena is that when $C_1^2 + S_1^2 - (C_0^2 + S_0^2) = 0$ effectively, the deviation would go to infinity, that is C_1/C_0 and S_1/S_0 would be equal to ∞ which is absurd as we started on a peak's position which is not a non-trivial zero's spot. $\square$

Note 8. Another way to bring up the former contradiction is to observe in figure 4 that the event of the curves SC_0 and SC_1 touching each other will happen precisely when the two near zeros of the curve SC_0 will meet as a unique point. But that concurring event is also the one with no significant downwards peak's effect (as the whole thing depends precisely on the existence of a pair).

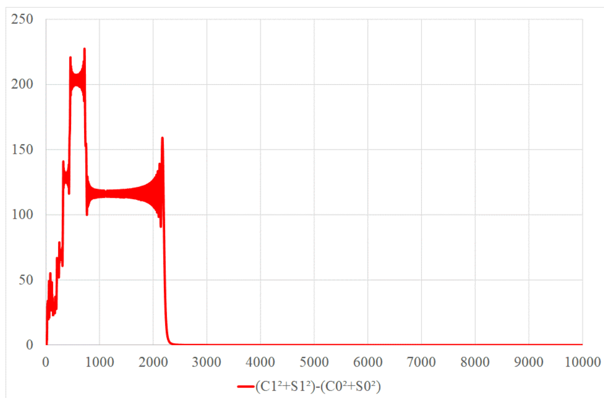

FIGURE 5. Function $C_1^2 + S_1^2 - (C_0^2 + S_0^2)$ versus m. Truncation results up to $m_{max} = 10000$, $a = 1/2$, $b_{peak} \approx 7005.08168$. Oscillations ending after $m \approx 2500$.

Lemma 3. *For* $0 \leq a \leq 1/2$ *and* $k > 0$*, we get inequality:*

$$SC_k(a, b) > 0 \tag{10}$$

Proof. The property is check numerically directly near $b = 0$. Otherwise, this is an immediate result of lemma 2. $\square$

Note 9. The SC_0 ordinate at the intermediate abscissa b_{peak} is, a priori, statistically lower (using a logarithmic scale to represent values approaching 0) as two Riemann zeros are closer. We return to this point in paragraph 6.2.

6. Numerical evidences and approximations on $R2(a, b)$

6.1. **Numerical evidences.** Figure 8 shows the chaos in the variations of $R2(a, b)$ when $a > 1/2$. On the opposite, the trend towards the asymptotic value $R2(a \to -\infty, b) \to 1$ is quickly activated on the side $a \leq 1/2$. Hence

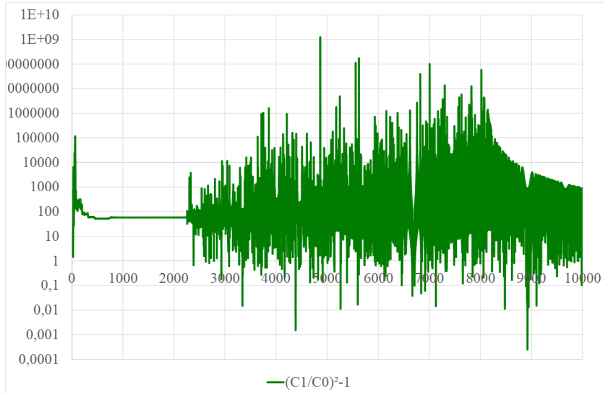

FIGURE 6. Function $(C_1/C_0)^2 - 1$ versus m. Truncation results up to 10000, $a = 1/2$, $b_{peak} \approx 7005.08168$. Strong permanent oscillations starting after $m \approx 2500$.

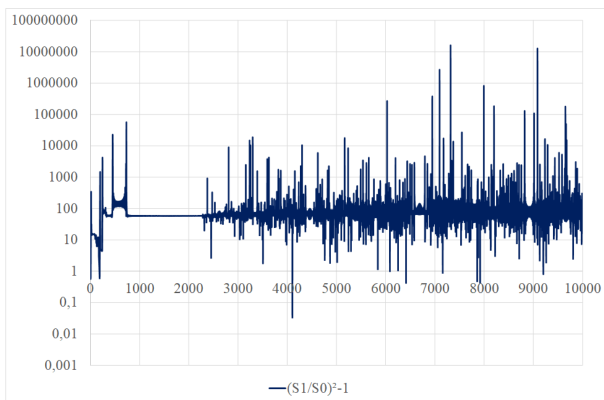

FIGURE 7. Function $(S_1/S_0)^2 - 1$ versus m. Truncation results up to 10000, $a = 1/2$, $b_{peak} \approx 7005.08168$. Strong permanent oscillations starting after $m \approx 2500$.

the obvious interest in choosing this side of the critical band for the Riemann hypothesis proof.

The $R2(a = 1/2, b)$ function changes from local minimum to local maximum when b changes. Here we are interested in finding some relationships between a maximum and the two minima that frame it. The figures 9 and 10 give a sample of the values taken by $R2(1/2, b)$ for $b \in [15000, 15250]$. The savvy reader may note, although this is not greatly visible, that an upwards peak also corresponds to negative value spikes on either side of this peak.

A closer look of the phenomena stands within the two figures 11 and 12. It is as if the rise towards the high values r_{peak} of $R2(a, b)$ requires a spring force acting from under ordinate 0. Indeed, roughly, the higher

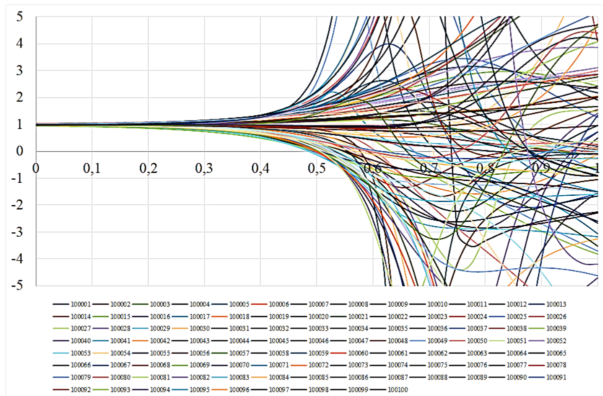

FIGURE 8. Function $R2(a, b)$ versus a. Samples $b = 100001$ up to 100100 by unit increment, $min(R2(1/2, b)) \approx -0.128374$, $max(R2(1/2, b)) \approx 3.027709$.

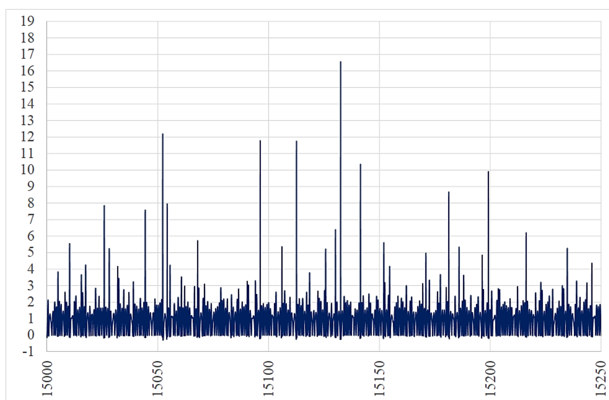

FIGURE 9. Function $R2(a, b)$ versus b. $a = 1/2$, $b \in [15000, 15030]$.

a peak stretches, the greater in absolute value the negative values r_M of $R2(a, b)$ surrounding it.

The graph of figure 13 shows the coordinates (a, b) of the minima of $R2(a, b)$ surrounding the peak located between the 106073^{rd} and the 106074^{th} Riemann's zeros ($b \approx 78974.79335$ and $b \approx 78974.82196$). It shows the complexity of the line of minima and suggest that the peak value may not be only the mere result of the two minima at $a = 1/2$. The shape of the surrounding of the peaks, next to the peak studied here, on both outside sides of this line of minima (at the abscissa $a = 1/2$), instead of being inwards will be outwards. Therefore, as the figure 11 shows, "high" negative values on one side may induces only two mild spikes as their spring momentums are already affected in the middle peak. Indeed, representing r_{peak} as a function of r_M, where r_{peak} is the value of a given peak, r_M the average between the

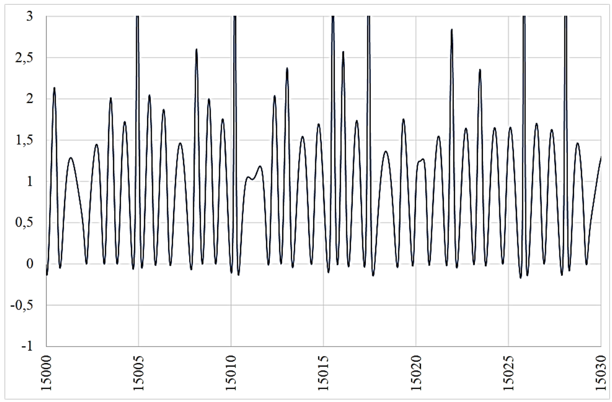

FIGURE 10. Function $R2(a,b)$ versus b. $a = 1/2$, $b \in [15000, 15030]$.

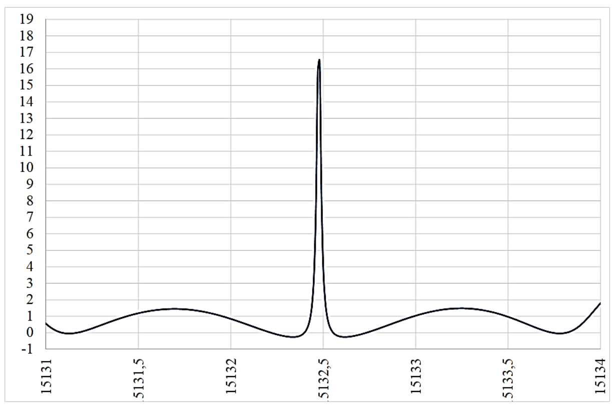

FIGURE 11. Function $R2(a,b)$ versus b. $a = 1/2$, $b \in [15131, 15134]$.

two lower values r_{M-} and r_{M+} on either side,

$$r_M = (r_{M-} + r_{M+})/2,$$

we necessarily get a "parasitic" branch. This is what is shown in figure 14.

The reader will note that this graphic was made by aggregating the data provided in intervals $b \in$ [3000, 3250], [6000, 6250], [9000, 9250], [12000, 12500], [15000, 15250] with a $\Delta b = 1/100$ step.

The "parasitic" branch is the one extending horizontally, the left side being the result of some blurred mix of two dominant phenomena. It won't provide any additional useful information to that provided by the ascending branch. The overriding character of the ascending branch develops when r_M reaches values under -0.125 (a quarter of the -0.5 limit value of r_M we will consider below).

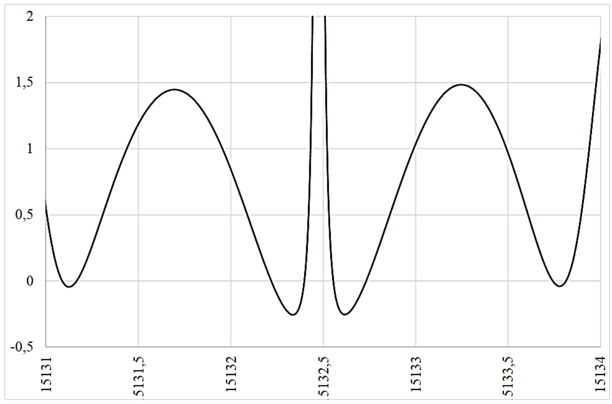

FIGURE 12. Function $R2(a,b)$ versus b. $a = 1/2$, $b \in [15131, 15134]$.

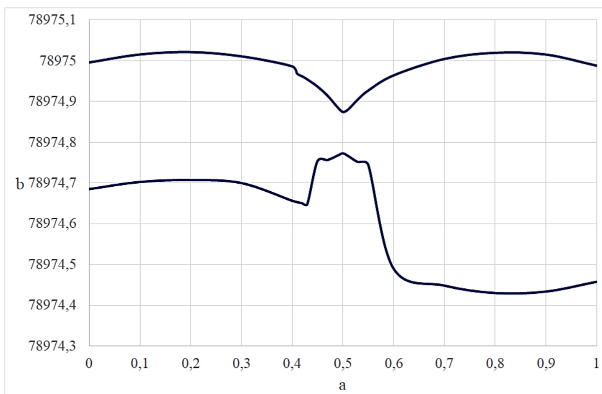

FIGURE 13. Function r_{peak} versus r_M. Main branch.

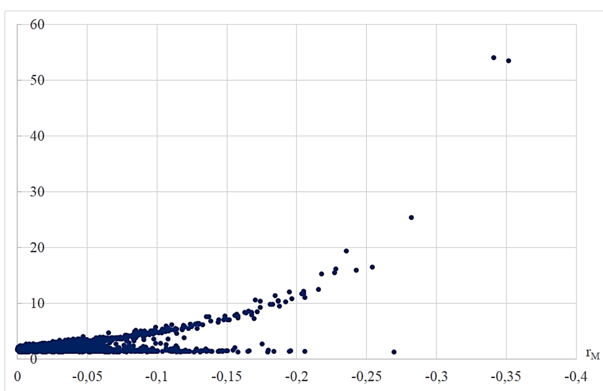

FIGURE 14. Function r_{peak} versus r_M

The numerical illustration, based on figures 11 and 12, leads also to an important remark. The $\Delta b = 1/10$ step is much to wide to get a good accuracy of the actual values of the peaks r_{peak}. It is imperative to do a precise point-by-point study. We thus provide in [7] Appendix 3 Table 7 the complete data of the figures 15 and 16 (which are the same data with simply a logarithmic scale for the y-axis in the second chart) one gets by a precise study.

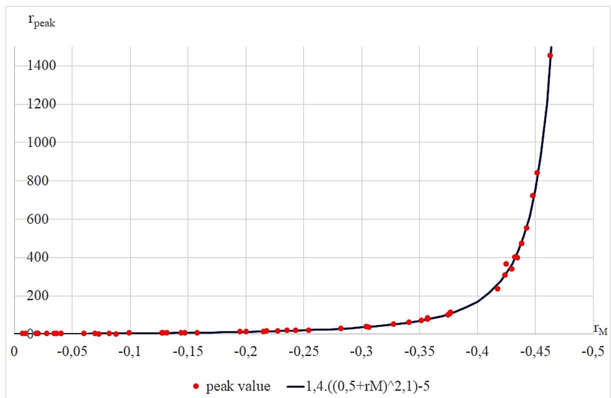

FIGURE 15. Function r_{peak} versus r_M. Main branch.

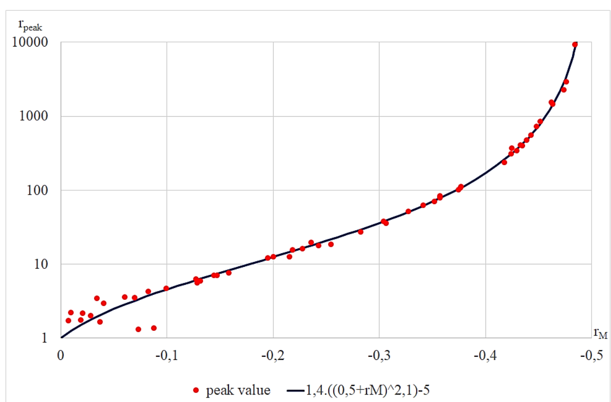

FIGURE 16. Function r_{peak} versus r_M. Main branch ($r_M < -0,1$).

The resulting collected points in figure 15 show an increasingly rapid divergence beyond the value $r_M \approx -0.35$. The second figure 16 shows that this increase gets faster then exponential. The data near the origin are more erratic because of the combination of the ascending and horizontal branches within figure 14.

6.2. **Numerical approximations.** The interpolation function that one can use here is:

$$r_{peak} \approx \frac{1.4}{(0.5 + r_M)^{2.1}} - 5 \tag{11}$$

The adjustment parameters, except 0.5 in front of r_M, are approximate. We incline for an exponent on the denominator equal to 2 instead of 2.1, but our data to date indicates the adjustment proposed here. For lack of better, we let it that way.

The important point is that this function diverges at $r_M = -0.5$ which means a bumper value impossible to exceed (as soon as -0.5 instead of -1) because of the continuity of $R2(a, b)$ demonstrated in paragraph 7.2. This then is a first hint confirming theorem 3.

Note 10. The term r_M is an average of 2 terms. Nothing prevents one of them from being smaller than -0.5. However, since the two r_M coordinates around a peak are outside on either side of the two Riemann zeros, and therefore such that $r_{M-} < 0$ and $r_{M+} < 0$, the limit $r_M > -0.5$ means that neither of the two r_M can be less than -1.

A random search for high-amplitude peaks of $R2(a, b)$ would require enormous computational resources without the existence of a sufficiently simple tracking. Fortunately, an apparent link between the gap of two consecutive Riemann zeros and the height of the intermediate peak makes the search quite easy thanks to the database referenced in [5].

As it turns out to be, a peak is generally all the more ample as the gap between two consecutive Riemann zeros (at abscissas noted *zero_R−* and *zero_R+*) is smaller. We get the following approximate relationship, where b_r is the peak abscissa, Δb_r the gap between two Riemann's zeroes.

$$r_{peak} \approx 1 + \frac{5}{\Delta b_r^2 \cdot b_r^{\frac{1}{4}}} \tag{12}$$

According to this relationship, the amplitude of the peak tends towards infinity when the gap Δb_r tends towards zero. These cases may be more and more common for very high-value abscissas since the average difference between zeros is asymptotically in $2\pi/\ln(abscissa\_zeroR)$. The presence of the logarithm and the counterpart-balancing effect of the term b_r , however, makes it difficult to find many cases with very high values here. Notably, there is no $r_{peak} > 10000$ example for the first 500000 Riemann zeroes.

Figure 17 represents the numerical results and evaluation by an interpolation formula without taking into account the abscissa of the peak b_r ($br^{1/4}$ term at denominator obliterated). In figure 18, this additional factor is introduced giving more precise interpolation at some smaller values of the peaks.

For large peaks, we can neglect the constants -5 and 1 in the relations 11 and 12. In order to compensate for the power 2.1 in relations 12 that we wish to reduce to 2, we increase somewhat the constant in front of the

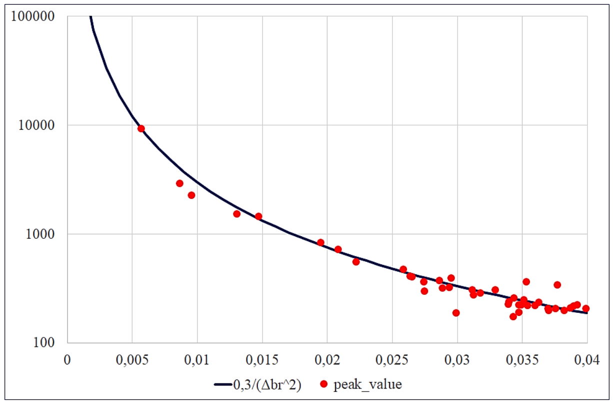

FIGURE 17. Function r_{peak} versus Δb_r.

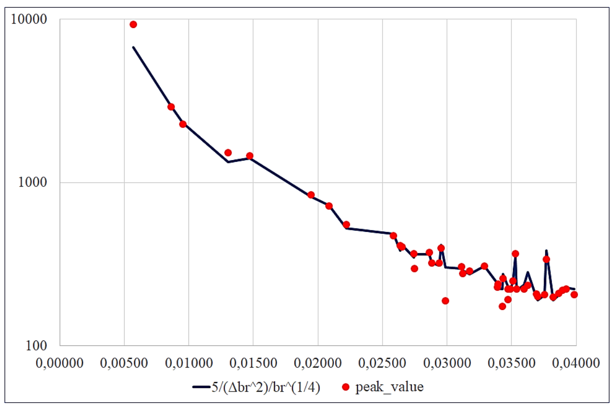

FIGURE 18. Function r_{peak} versus Δb_r and b_r.

fraction. Then equating relations 11 and 12, therefore eliminating r_{peak}, we get:

$$r_M \approx -\frac{1}{2} \cdot (1 - 1.2 \cdot \Delta b_r \cdot b_r^{\frac{1}{8}}) \tag{13}$$

Since $\Delta b_r \cdot b_r^{\frac{1}{8}} > 0$, the term r_M is greater than -0.5. For values x close to 0, $\exp(-x) \approx 1 - x$, and thus:

$$r_M \approx -\frac{1}{2} \cdot \exp(-1.2 \cdot \Delta b_r \cdot b_r^{\frac{1}{8}}) \tag{14}$$

In the range of numerical values examined, we also have the simpler alternative formula:

$$r_M \approx -\frac{1}{2} \cdot \exp(-5\Delta b_r) \tag{15}$$

which leads to the figures 18 and 19.

Below the critical value of the gap (approximatively $\Delta b_r = 1/2$), the points align quite perfectly, the only selection criterion having been to take

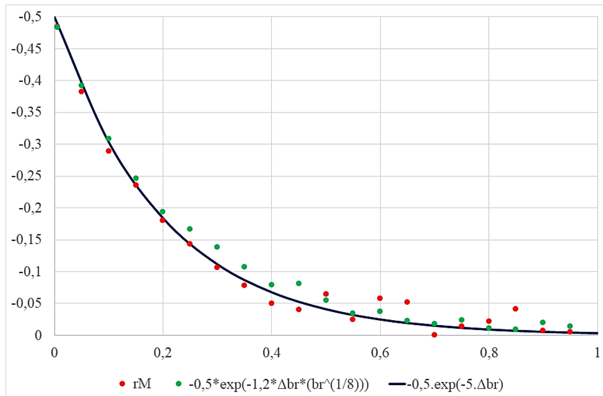

FIGURE 19. Function r_M versus Δb_r and b_r.

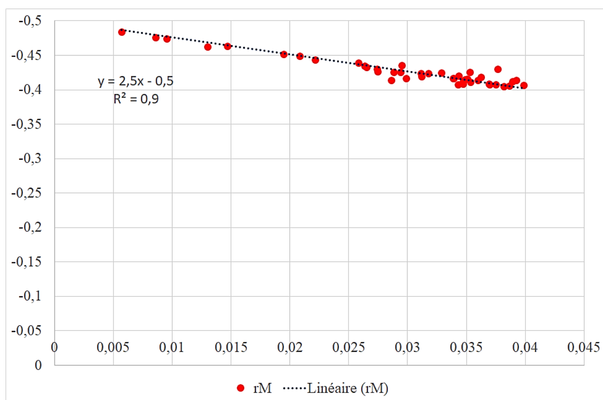

FIGURE 20. Function r_M versus Δb_r.

the gap Δb_r among the first 100000 zeroes such as Δb_r is closest by higher value of 0.05, 0.10, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40 and 0.45, (for the construction of figure 19), a selection that gives only an almost uniform spacing in abscissa but in no way any predisposition on the value of the ordinate. We added also to the chart the lower Δb_r gap solution that exists among Riemann's first 500000 zeroes.

Close to the origin ($\Delta b_r < 0.05$), the approach towards $r_M = -0.5$ is quasi-linear as one can see in figure 20.

7. Theorems related to $R2(a, b)$

7.1. **Geodesics of $R2(a, b)$.**

Theorem 6. *The local maximum value of $R2(a, b)$ is related to the minimums' paths in the vicinity of this peak.*

Proof. The extrema of $R2(a, b)$ are determined by the cancellation of the two partial derivatives $\partial_a R2$ and $\partial_b R2$. This means, using relations 17 and

18 that we will establish later on, that $(C_1^2+S_1^2+2C_0\cdot C_2+2S_0\cdot S_2)\cdot(C_1\cdot C_2+S_1\cdot S_2)+(C_1^2+S_1^2)\cdot(C_0\cdot C_3+S_0\cdot S_3)=0$ and $(C_1^2+S_1^2+2C_0\cdot C_2+2S_0\cdot S2)\cdot(C_1\cdot S_2-S_1\cdot C_2)+(C_1^2+S_1^2)\cdot(S_0\cdot C_3-C_0\cdot S_3)=0$. Thus $(C_1\cdot C_2+S_1\cdot S_2)\cdot(S_0\cdot C_3-C_0\cdot S_3)=(C_1\cdot S_2-S_1\cdot C_2)\cdot(C_0\cdot C_3+S_0\cdot S_3)$. This last equation is common to local minimums and maximums, hence the obvious link. □

Note 11. The common equation explains the link between a peak of $R2(a,b)$ and the minima on either side of that peak observed in the illustrations in the previous paragraph. In fact, what produces the value of the peak is not only the two values on either side, where the parameter $a=1/2$ is set in advance, but the entire minimum geodesic "surrounding" that peak in the general coordinate (a,b). However, the average of the two values examined above is already, when the peak has a significant value above 1, a good representation of the said neighbourhood and thus allows to anticipate the peak's value.

7.2. **Continuity of $R2(a,b)$.**

Theorem 7. *The $R2(a,b)$ function is continuous in interval $0\le a\le 1/2$.*

Proof. It is sufficient to prove that the $R2(a,b)$ denominator, i.e. $SC_1(a,b)$ $=(C_1(a,b))^2+(S_1(a,b))^2$ does not cancel. This is an immediate result of lemma 3. □

Note 12. Obviously, the function is then also continuous outside the indicated interval.

7.3. **Partial derivatives linked to $R2(a,b)$.** In this text, the functions are generally dependent on two variables a and b. The handling of the objects is simplified by writing F instead of $F(a,b)$. The partial derivative writing of F, versus parameter a, $\partial/\partial_a(F(a,b))$ is simplified as $\partial_a F$. The same goes for b.

7.3.1. *Partial derivatives of $C_k(a,b)$ and $S_k(a,b)$.* Let us write again relations 4 and 5:

$$C_k(a,b)=\sum_{m=1}^{\infty}(-1)^{m-1+k}(\ln(m))^k\cdot m^{-a}\cdot\cos(b.\ln(m))$$

and

$$S_k(a,b)=\sum_{m=1}^{\infty}(-1)^{m-1+k}(\ln(m))^k\cdot m^{-a}\cdot\sin(b.\ln(m))$$

We get then immediately

$$\partial_a C_k=\sum_{m=1}^{\infty}(-1)^{m+k}(\ln(m))^{k+1}\cdot m^{-a}\cdot\cos(b.\ln(m))$$

$$\partial_a S_k=\sum_{m=1}^{\infty}(-1)^{m+k}(\ln(m))^{k+1}\cdot m^{-a}\cdot\sin(b.\ln(m))$$

$$\partial_b C_k = \sum_{m=1}^{\infty} (-1)^{m+k} (\ln(m))^{k+1} \cdot m^{-a} \cdot \sin(b. \ln(m))$$

and

$$\partial_b S_k = \sum_{m=1}^{\infty} (-1)^{m-1+k} (\ln(m))^{k+1} \cdot m^{-a} \cdot \cos(b. \ln(m))$$

In other words:

$$\begin{aligned} \partial_a C_k &= C_{k+1} \\ \partial_a S_k &= S_{k+1} \\ \partial_b C_k &= S_{k+1} \\ \partial_b S_k &= (-1) \cdot C_{k+1} \end{aligned} \tag{16}$$

All of these functions, as sums of continuous functions are continuous.

7.3.2. *Partial derivatives of* $R2(a, b)$. Let us rewrite equation 8 in a simplified way:

$$R2 = \frac{C_0 \cdot C_2 + S_0 \cdot S_2}{C_1^2 + S_1^2}$$

It follows using identity $(u/v)' = (u'.v - u.v')/v^2$ and introducing

$$RSC_0 = (C_1^2 + S_1^2 + 2C_0C_2 + 2S_0S_2)(C_1C_2 + S_1S_2),$$

the two expressions:

$$\partial_a R2 = \frac{RSC_0 + (C_1^2 + S_1^2)(C_0C_3 + S_0S_3)}{(C_1^2 + S_1^2)^2} \tag{17}$$

$$\partial_b R2 = \frac{RSC_0 + (C_1^2 + S_1^2)(S_0C_3 - C_0S_3)}{(C_1^2 + S_1^2)^2} \tag{18}$$

Note 13. The two previous partial derivatives are continuous due to the fact that $(C_1(a, b))^2 + (S_1(a, b))^2$ doesn't cancel (see again lemma 3).

7.4. **The impossible equality** $R2(a, b) = -1$**.** Making sure that $R2 = -1$ is out of reach, we get at the same time a broader impossibility, that is $R2 < -1$ since $R2(a, b)$ is continuous according to both coordinates a and b (as proven in subsection 7.2) and therefore the passage by this intermediary step is an absolute necessity. In addition, we place ourselves in the conditions $a \in [0, 1/2]$ and $b \in [3, +\infty[$.

From relation 8, we get by definition $R2 = (C_0C_2 + S_0S_2)/(C_1^2 + S_1^2)$, so that also $(C_0C_2 + S_0S_2) = R2 \cdot (C_1^2 + S_1^2)$. Relation 18 becomes then:

$$\partial_b R2 = \frac{(1 + 2R2)(C_1S_2 - S_1C_2) + (S_0C_3 - C_0S_3)}{(C_1^2 + S_1^2)} \tag{19}$$

We seek the values for which the $R2$ expression is minimal when b varies, thus those values such that $\partial_b R2 = 0$, which according to the former expression means also $R2 = (1/2).((C_0.S_3 - S_0.C_3)/(C_1.S_2 - S_1.C_2) - 1)$. The solutions are hence those for which we have simultaneously:

$$R2x = \frac{C_0C_2 + S_0S_2}{C_1^2 + S_1^2} \tag{20}$$

and

$$R2y = \frac{1}{2}\frac{C_0S_3 - C_3S_0}{C_1S_2 - C_2S_1} - 1 \tag{21}$$

and

$$R2 = R2x = R2y \tag{22}$$

The two figures 21 and 22 are the same, the second being only a close-up view of a particular area. They contain all the $(R2x(a,b),\ R2y(a,b))$ points obtained for $a = 1/2$, $b = 0$ to 20000 and $\Delta b = 1/4$, the actual solutions joining these points by continuity. The point $(R2x,\ R2y) \approx (-0.5122,\ -0.5121)$ for $(a,\ b) = (1/2,\ 78974.87502)$ corresponding to the only example found where $R2(a,b) < -1/2$ is also reported on the graphic. The only solutions to retain are on the $R2x = R2y$ axis of this graphic (the light blue dotted line), but the usefulness of spotting all the dots $(R2x,\ R2y)$ is obvious. This makes it possible to visualize in an obvious or even garish way, the minimum abscissa when simultaneous equalities are obtained. We see that the dots are concentrated, for the part below the abscissa $R2x = 0$, in a triangle $R2y = -1/2$, $R2x = 0$, $R2y = 1/2+2R2x$ (green frame). The lower abscissa value of this triangle is $-1/2$. A few points are slightly outside this triangle, but this does not affect our conclusion. One finds these few points mostly above the triangle near the 0^- abscissa.

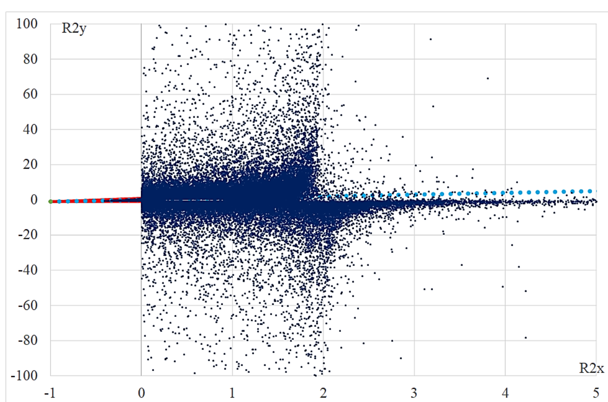

FIGURE 21. $R2y$ versus $R2x$. $a = 1/2$, $b \in]3, 20000]$. Samples b with $\Delta b = 0.25$ spacing.

Such a figure with very sharp boundary lines, although slightly permeable, show the absurdity of points extending far beyond $R2x < -1/2$, namely to a hypothetical $R2x = -1$, for $a = 1/2$.

The graph in figure 21 shows a concentration of coordinates (Rx, Ry) along and below the axis $Ry = -1/2$ when Rx tends towards infinity. However, the effective solutions are those placed on the axis $Rx = Ry$. This means

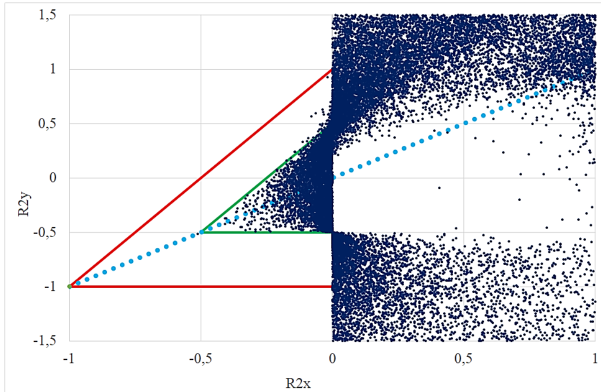

FIGURE 22. $R2y$ versus $R2x$. $a = 1/2$, $b \in]3, 20000]$. Samples b with $\Delta b = 0.25$ spacing. Closer-up of previous figure.

the scarcity of high peak values both because of the scarcity of points along the axis $Ry = -1/2$ and the distance of the line $Rx = Ry$ from the said line. Therefore a double penalty in some way for any attempt to deny the Riemann hypothesis...

The slope of the curve giving $R2y$ as a function of $R2x$ is ∞ on the dotted blue line $R2y = R2x$ as shown in the figure 23's example and the many more that the interested reader may find at reference [7] appendix 11. This is a direct consequence of the construction of this graph which initially uses the relationship $\partial_b R2 = 0$ and therefore results in $\partial_b R2x = 0$ precisely on the line $R2x = R2y$.

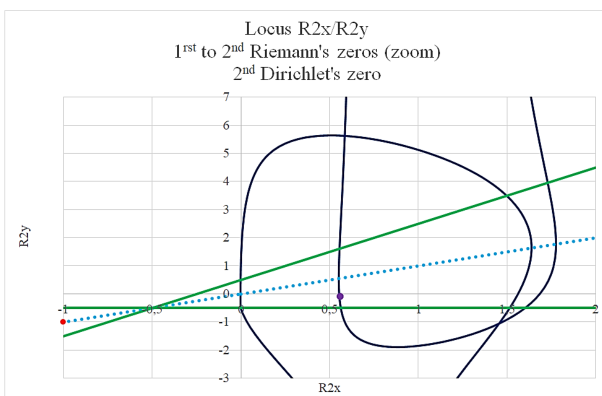

FIGURE 23. Function $R2y$ versus $R2x$. The second Dirichlet zero is here the point in violet color. The red point is the "impossible" target $R2 = -1$.

This slope is also infinite when $R2y$ diverges, that is to say when $C_1.S_2 - C_2.S_1 = 0$ but then there is of course no corresponding point on the curve itself (see again reference [7] appendix 11). The previous graphics thus show that points cannot reach the -1 abscissa (remaining the fact that overruns of $-1/2$ are possible).

Theorem 8. *The minimal value of* $R2(a,b)$, $b>3$, *is* -1 *excluded.*

Proof. From relations 20, 21 and 22, we get the two following equations to be solved:

$$(C_2).C_0+(S_2).S_0-R2.(C_1^2+S_1^2)=0$$
$$(S_3).C_0-(C_3).S_0-(1+2R2).(C_1.S_2-S_1.C_2)=0$$

So that:

$$\begin{pmatrix}C_0\\S_0\end{pmatrix}=\begin{pmatrix}C_2 & S_2\\S_3 & -C_3\end{pmatrix}^{-1}\begin{pmatrix}R2.(C_1^2+S_1^2)\\(1+2R2).(C_1.S_2-S_1.C_2)\end{pmatrix}$$

Then:

$$\begin{pmatrix}C_0\\S_0\end{pmatrix}=\frac{1}{C_2.C_3+S_2.S_3}\begin{pmatrix}C_3 & S_2\\S_3 & -C_2\end{pmatrix}\begin{pmatrix}R2.(C_1^2+S_1^2)\\(1+2R2).(C_1.S_2-S_1.C_2)\end{pmatrix}$$

Let us write

$$\alpha=R2.(C_1^2+S_1^2)$$
$$\beta=(1+2R2).(C_1.S_2-S_1.C_2)$$

Then

$$C_0^2+S_0^2=\frac{\alpha^2.(C_3^2+S_3^2)+\beta^2.(C_2^2+S_2^2)+2\alpha.\beta.(C_3.S_2-C_2.S_3)}{(C_2.C_3+S_2.S_3)^2}$$

which is obviously equivalent, when $C_0^2+S_0^2\neq 0$, to

$$1=\frac{\alpha^2.(C_3^2+S_3^2)+\beta^2.(C_2^2+S_2^2)+2\alpha.\beta.(C_3.S_2-C_2.S_3)}{(C_0^2+S_0^2).(C_2.C_3+S_2.S_3)^2} \tag{23}$$

Posing $R\alpha\beta 2=\alpha^2.(C_3^2+S_3^2)+\beta^2.(C_2^2+S_2^2)$, let us write the ratio:

$$DSC(X)=\frac{R\alpha\beta 2+2\alpha.\beta.(C_3.S_2-C_2.S_3)}{(C_2.C_3+S_2.S_3)^2.(C_0^2+S_0^2)} \tag{24}$$

and

$$DL(X)=\ln(DSC(X))$$

Here we set DSC(X) as all the results of equation 24 with the condition

$$R2(a,b)=X \tag{25}$$

A good understanding of the argument below requires, as before, the distinction between the two cases $R2(a,b)=-1/2$ and $R2(a,b)=-1$, knowing that continuity gives perfectly accessible intermediate values, if necessary, between these two cases.

Case 1: If $R2=-1/2$, we get:

$$DSC(-1/2)=\frac{(C_1^2+S_1^2)^2.(C_3^2+S_3^2)}{4(C_0^2+S_0^2).(C_2S_3+S_2.S_3)^2}$$

and

$$DL(-1/2)=\ln(DSC(-1/2))$$

According to equation 23, the $R2(a,b) = -1/2$ equality is reached when

$$DSC(-1/2) = 1$$

which is equivalent to

$$DL(-1/2) = 0$$

Case 2: If $R2(a,b) = -1$, we get letting in the same time $RSC_1 = (C_1^2 + S_1^2)(C_3^2 + S_3^2) + 2(C_1S_2 - C_2S_1)(C_3S_2 - C_2S_3)$:

$$DSC(-1) = \frac{(C_2^2 + S_2^2)(C_1S_2 - C_2S_1)^2 + RSC_1.(C_1^2 + S_1^2)}{(C_0^2 + S_0^2)(C_2C_3 + S_2S_3)^2}$$

and

$$DL(-1) = \ln(DSC(-1))$$

The $R2(a,b) = -1$ case is reached when

$$DSC(-1) = 1$$

equivalent to

$$DL(-1) = 0$$

Figures 24 and 25 of $DSC(-1/2)$ and $DSC(-1)$ are, in fact, point clouds' variants undergoing a continuous distortion of figure 21 (and 22). They show the same thing in a different form.

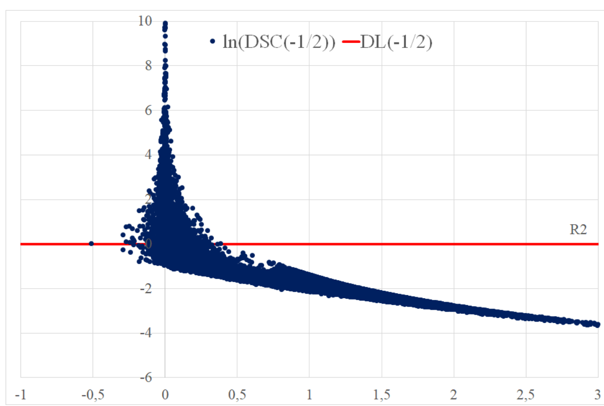

FIGURE 24. Function $\ln(DSC(-1/2))$ versus $R2$. $a = 0.5$ and $b \in]50, 5050]$.

Figure 24 shows the outgrowth of the minimums, on the negative $R2$ side, aligned on the line $\ln(DSC(-1/2)) = 0$. With figure 25, the outgrowth is deflected upwards showing the impossibility of reaching $R2 = -1$ values. On line $\ln(DSC(-1)) = 0$, where this event $R2 = -1$ must be effective to reject Riemann's hypothesis, the intersection is not only above $-1/2$, but much further beyond 0^+.

Finally, let us look at the precise reason why $R2(a,b) = -1$ events are not achieved by local punctual examples. For this, we choose the case for

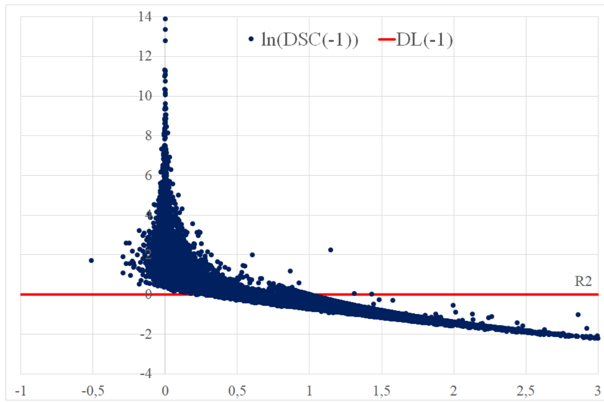

Figure 25. Function $\ln(DSC(-1))$ versus $R2$. $a = 0.5$ and $b \in]50, 5050]$.

which we detected the smallest difference between Riemann's zeroes among the first 500000 of them and a few others.

The graphic in figure 26 shows simultaneously, one on one, the evolution of $R2$ and $\ln(DSC(-1))$. We recall that $\ln(DSC(-1)) = 0$ (or $DSC(-1) = 1$) is the target value right above the $R2$ minimums.

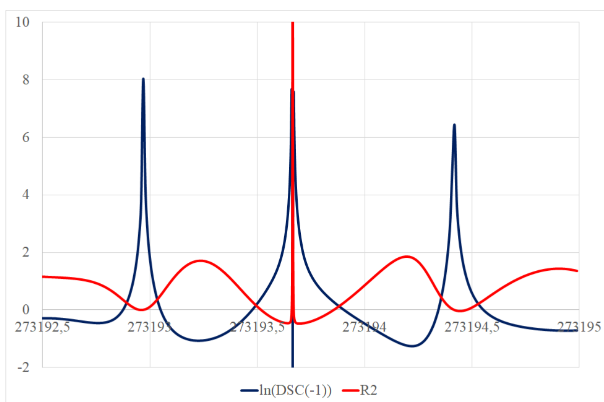

Figure 26. Function $R2$ versus b. Function $\ln(DSC(-1))$ versus b. $a = 0.5$ and $b \in [273192.5, 273195]$.

The initiation of an $R2$ descent induces the initiation of a $\ln(DSC(-1))$ ascent and vice versa as shows the examples of figures 27 and 28. This behaviour is perfectly reproducible, as shown by the two additional figures 29 and 30.

The crossing of $R2$ and $\ln(DSC(-1))$ curves at the approach of a low-$R2$ zone is at the level of ordinate 0 and $\ln(DSC(-1))$ then quickly increases. Let us consider the intersections of the $R2$ and $\ln(DSC(-1))$ curves. We call inner crossings those whose abscissas are between two Riemann's zeros and outer crossings the other two to the right and left (of figure 28).

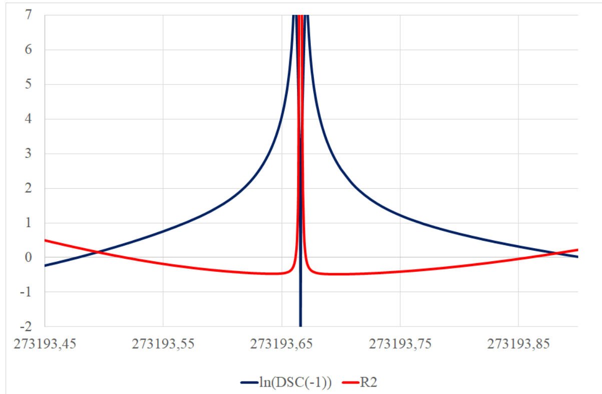

FIGURE 27. Zoom 1: Function $R2$ versus b. Function $\ln(DSC(-1))$ versus b. Outer crossings at ordinates ≈ 0.68 and ≈ 0.695.

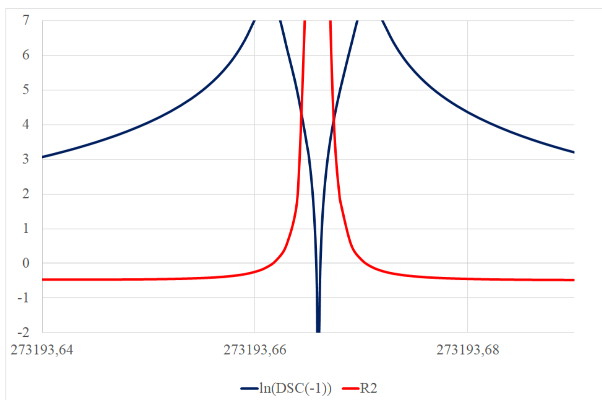

FIGURE 28. Zoom 2: Function $R2$ versus b. Function $\ln(DSC(-1))$ versus b. Inner crossings at ordinates ≈ 0.43 and ≈ 0.415.

The term $\ln(DSC(-1))$ necessarily diverges according to the relationship 24 since $C_0^2 + S_0^2 = 0$ for any Riemann (and Dirichlet) zero. So, the inner crossings are trivially above ordinate 0. The outer crossings are also, a point that however seems difficult to establish. The easiest way is to assess the value of $DSC(-1)$ at the points that matter to us, that is where $R2$ is minimum.

For this, we pick the data used to establish figure 19 with the same selection criterion chosen at that time. Doing so, we get figure 31.

We recorded $DSC(-1)-$ the value of $DSC(-1)$ at the r_{M-} abscissa before the peak and $DSC(-1)+$ the value at the r_{M+} abscissa after the peak. We reported also in the graphics the average $DSC(-1)$ value of these two

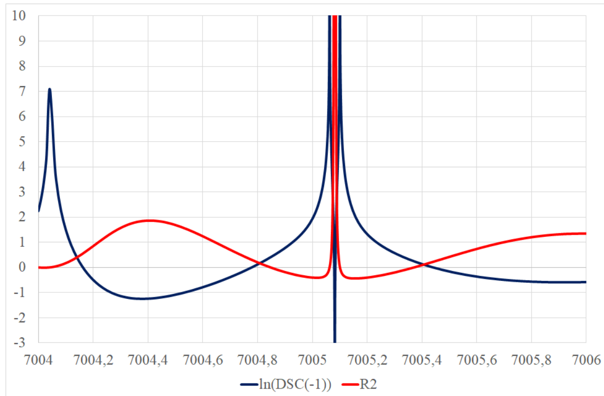

FIGURE 29. Function $R2$ versus b. Function $\ln(DSC(-1))$ versus b. $a = 0.5$ and $b \in [7004, 7006]$. Outer crossings at ordinates ≈ 0.16 and ≈ 0.1. Inner crossings at ordinates ≈ 2.8 and ≈ 2.8.

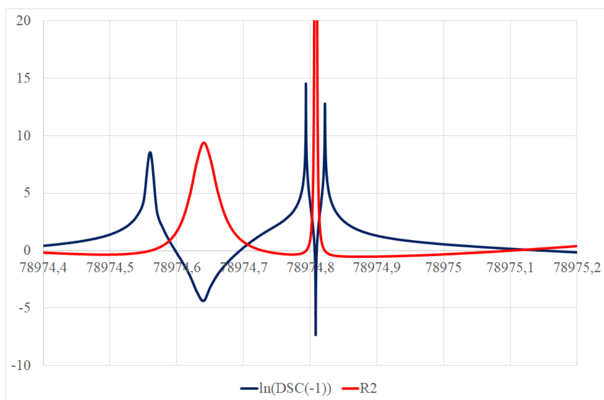

FIGURE 30. Function $R2$ versus b. Function $\ln(DSC(-1))$ versus b. $a = 0.5$ and $b \in [78974.4, 78975.2]$. Outer crossings at ordinates ≈ 0.55 and ≈ 0.1. Inner crossings at ordinates ≈ 2.8 and ≈ 2.85.

values, knowing that what really is important here is rather the minimum value of the two values $DSC(-1)-$ and $DSC(-1)+$.

The alignment of the points only fails for a gap between zeroes of Riemann higher than approximatively $\Delta b_r = 1/2$ in exactly the same way we had found in the case of figure 19, therefore in a region which has no consequence to our purpose.

On the contrary, for the points of interest to us, i.e. points near to the origin of the abscissas (extremely small Δb_r), this region shows an upwards move of $DSC(-1)$ well beyond the critical value $DSC(-1) = 1$. This surge is due to the simple fact that the closer two Riemann zeros are, the more

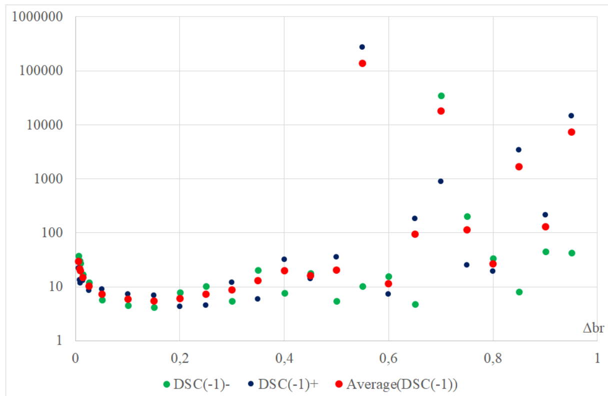

FIGURE 31. Function $DSC(-1)$ versus Δb_r. $a = 1/2$.

pronounced the corresponding peak is and the steeper the flanks of the peak, including until the abscissas of the minimums of $R2$. Thus, the abscissa of a minimum (r_{M-} or r_{M+}) of $R2$ is close to that of its corresponding zero, in other words, when $\Delta b_r \to 0$, then $C_0^2 + S_0^2 \to 0$ at the abscissas r_{M-} and r_{M+} also. But $C_0^2 + S_0^2 \to 0$ is at the $DSC(-1)$ denominator and no term in C_0 or S_0 is within the numerator for compensation. The term $C_1^2 + S_1^2$, and even more $C_2^2 + S_2^2$, will tend to 0 with many decades of delay as shown by the typical example of figure 4, the number of decades increasing rapidly with the lowering of Δb_r. The other terms do not tend in any way towards 0. The compensation remains effective in $DSC(-1/2)$ because of the square of $C_1^2+S_1^2$ in the $DSC(-1/2)$ numerator, but it would take a power of at least 4 effected to $C_2^2 + S_2^2$ in $DSC(-1)$ (plus 3 very close Riemann's zeros at least) to obtain the said compensation. Thus, $DSC(-1)$ necessarily diverges when $\Delta b_r \to 0$ and so in a very steep manner as figure 31 confirms it, hampering the slightest possibility to get a $DSC(-1) = 1$ mishap. □

Note 14. The reader will note that all the examples of this proof are built with $a = 1/2$. There is an obvious and practical reason to that, which is that the numerical results are interesting only for this critical line, as the values of the extrema decline otherwise quite quickly (in the case $a < 1/2$). There are no data outside the said critical line that can be used to contradict our presentation in a mere relevant way.

Note 15. To be exhaustive, the reader will also take in account in the previous proof the argument of paragraph 7.5.3 as well as the important note 10.

Theorem 9. *The asymptotic value of the $R2(a, b)$ minimums is $-1/2$.*

Proof. By the term "asymptotic," we mean the minimums of $R2(a, b)$ when b tends towards infinity (and the parameter a is fixed). In this case, Riemann's zeros are, on average, at a distance of about $2\pi/\ln(n_r)$, where n_r is the number of Riemann zeroes up to abscissa b_r (of some Riemann zero imaginary part), meaning they get on average closer and closer. According to relation 18, the numerator of $\partial_b R2$ is equal to $(C_1^2+S_1^2+2C_0.C_2+2S_0.S_2).(C_1.S_2-S_1.C_2)+(C_1^2+S_1^2).(S_0.C_3-C_0.S_3)$. The cancellation of $\partial_b R2$ occurs for $(C_0.C_2+S_0.S_2)/(C_1^2+S_1^2)=(1/2).((C_0.S_3-S_0.C_3)/(C_1.S_2-S_1.C_2)-1)$, in other words when:

$$R2=\frac{-1}{2}+\frac{C_0.S_3-S_0.C_3}{2.(C_1.S_2-S_1.C_2)}$$

Asymptotically, as we saw in the last part of the proof of the impossibility of $R2=-1$, the terms C_0 and S_0 tend towards 0 much faster than all the terms of the C_k and S_k's type, $k>0$. It immediately follows $R2\to -1/2$. $\square$

Note 16. This result reminds us that negative overruns of -0.5 are possible. These will become more frequent when b increases. But, asymptotically, these overruns will also be more and more restricted to the immediate vicinity of -0.5 and therefore without the possibility of joining -1, thus confirming again theorem 8.

Note 17. At the peak abscissa r_{peak}, the expression $C_1.S_2-S_1.C_2$ necessarily takes values very close to 0, taking away this prerogative from the other two extrema (the minimums).

Numerical examples.

The examples below are realized thanks to the on-line computer application PARI/GP. The reader can find the digital lines of code at reference [7], Riemann sheet, appendix 9.

These are the three cases with smallest gaps between Riemann zeros of abscissas less than $b=2000000$. It confirms that the minimum values r_M- and r_M+ of $R2$ on respectively the left and right sides of a peak r_{peak} of $R2$ between two close Riemann zeros tend generally both towards -0.5 as expected (thus do not so in a unbalanced way like for example $r_{M-}\approx 0^-$ and $r_{M+}\approx -1^+$).

In these examples we give various values to the truncations in order to enable the reader to see the impact of such choices. Of course, we choose large enough values to start with so that we focus only on "near" finally expected results.

The "theoretical" value of the peak (truncation $\to+\infty$) is obtained from the formula $r_{peak}\approx 1+5/(\Delta b_r^2.b_r^{1/4})$, which we know is only an approximation. It is difficult to obtain the actual precise value of these peaks numerically. Some values concerning the minimums r_M- and r_M+ of $R2$ to the left and right of the Riemann's zeros surrounding a peak may also remain imprecise if the truncation does not include enough terms. The reader will be also able to compare the final truncations used to the numbers of terms $m\approx 1/(\exp(\pi/b)-1)$ corresponding to the last jump of values of the terms $\sum(-1)^{m+k}.(\ln(m))^{k+1}.m^{-a}.\cos(b.\ln(m))$ and $\sum(-1)^{m+k}.(\ln(m))^{k+1}.m^{-a}.\sin(b.\ln(m))$.

The reading is to be done line by line, 3 columns by 3 columns.

Example 1. Gap between zeros = 0.00416113
3271858^{th} Riemann zero, $b = abs\_zeroR- = 1779292.80366586$
3271859^{th} Riemann zero, $b = abs\_zeroR+ = 1779292.80782699$
Number of terms for the last values' jump: 566400.

abs_r$_M$-	trunc.	r$_M$-	abs_r$_M$+	trunc.	r$_M$+	abs_peak	trunc.	value_peak
							+∞ (th.)	7907.52
1779292.7940	1000000	-0.41291	1779292.830	1500000	-0.50279	1779292.805757	3000000	28888.16
1779292.7940	1500000	-0.44111	1779292.830	3000000	-0.51002	1779292.805757	4000000	6634.78
1779292.7940	3000000	-0.47259	1779292.835	5000000	-0.50784	1779292.805757	5000000	6030.39
1779292.7920	5000000	-0.46615	1779292.835	7000000	-0.50774	1779292.805756	6000000	7235.79
1779292.7915	7000000	-0.46839	1779292.830	9000000	-0.50774	1779292.805746	7000000	8360.63
1779292.7920	7000000	-0.468493	1779292.835	9000000	-0.50789	1779292.805747	7000000	8361.39
1779292.7925	7000000	-0.468489	1779292.840	9000000	-0.50647	1779292.805748	7000000	8301.65
							+∞ (th.)	7907.52

Example 2. Gap between zeros = 0.003259290
3637897^{th} Riemann zero, $b = abs\_zeroR- = 1961773.9933561$
3637898^{th} Riemann zero, $b = abs\_zeroR+ = 1961773.9966154$
Number of terms for the last values' jump: 634000.

abs_r$_M$-	trunc.	r$_M$-	abs_r$_M$+	trunc.	r$_M$+	abs_peak	trunc.	value_peak
1961773.979	1500000	-0.50639	1961774.010	1500000	-0.50081	1961773.995	3000000	2476.21
1961773.979	2000000	-0.49447	1961774.010	2000000	-0.48523	1961773.995	4000000	6172.31
1961773.979	4000000	-0.48805	1961774.010	4000000	-0.48598	1961773.995	7000000	3971.08
1961773.979	6000000	-0.49153	1961774.010	6000000	-0.48904	1961773.9949	10000000	20729.31
1961773.979	8000000	-0.49178	1961774.010	8000000	-0.48832	1961773.9949	15000000	15082.05
1961773.978	10000000	-0.49072	1961774.009	10000000	-0.487921	1961773.99497	20000000	4659.89
1961773.979	10000000	-0.49081	1961774.010	10000000	-0.487927	1961773.99498	20000000	13634.89
1961773.980	10000000	-0.49075	1961774.011	10000000	-0.487741	1961773.99499	20000000	11987.72
							+∞ (th.)	12577.56

Example 3. Gap between zeros = 0.002958654
1115578^{th} Riemann zero, $b = abs\_zeroR- = 663318.508310486$
1115579^{th} Riemann zero, $b = abs\_zeroR+ = 663318.511269140$
Number of terms for the last values' jump: 211200.

abs_r$_M$-	trunc.	r$_M$-	abs_r$_M$+	trunc.	r$_M$+	abs_peak	trunc.	value_peak
663318.493	400000	-0.47470	663318.531	400000	-0.49702	663318.509792	2000000	12083.47
663318.493	500000	-0.48127	663318.531	500000	-0.48719	663318.509792	3000000	12857.79
663318.493	600000	-0.47625	663318.531	600000	-0.49116	663318.509792	4000000	30072.08
663318.493	700000	-0.48800	663318.531	700000	-0.49903	663318.509792	5000000	15770.54
663318.493	800000	-0.48891	663318.531	800000	-0.49867	663318.509792	6000000	21286.54
663318.493	900000	-0.48151	663318.531	900000	-0.49167	663318.509792	7000000	27207.50
663318.492	1000000	-0.48170	663318.530	1000000	-0.49245	663318.509792	8000000	25831.65
663318.493	1000000	-0.48176	663318.531	1000000	-0.49245	663318.509792	9000000	19475.39
663318.494	1000000	-0.48161	663318.532	1000000	-0.49234	663318.509792	10000000	20324.01
							$+\infty$ (th.)	20016.79

7.5. **The exception to the rule and solace on $R2(a, b)$.** We mentioned an exception to the minimum rule of -1 very early in this text (see note of theorem 5). It is essential to give the reason for it because, although of no practical importance as it is local and therefore of easily verifiable effect (actually none as there is no Riemann zero around the corresponding b value), it is nevertheless like a thorn in the foot from a theoretical point of view and an explanation is therefore welcome.

7.5.1. *The very peculiar case $b < b_{r1}$.* We examine the case where abscissa b is lower than that of the first Riemann zero and its evolution out from this area. The types of curves and choice of colours are the same as in the graphics 21, 22 and 23. In particular, the dark blue curve represents the points ($R2x$, $R2y$). The only difference here is instead of giving the points ($R2x$, $R2y$) in a discrete way choosing some Δb spacing, we represent the continuous curve.

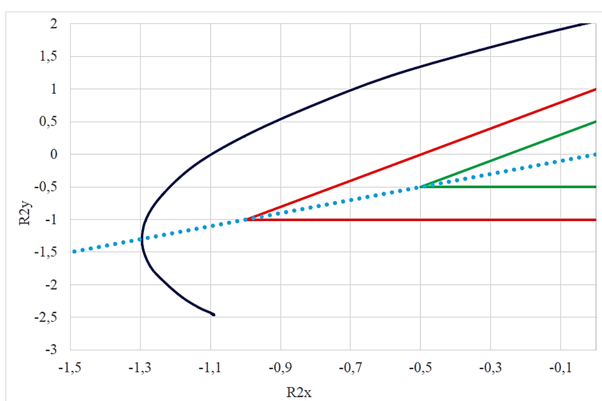

FIGURE 32. Function $R2y$ versus $R2x$. $a = 1/2$ and $b \in]0, 3.5]$.

The reader can see in figure 34 that as soon as the blue curve crosses abscissa $R2x = 0$, it is trapped in the areas described above despite all the restlessness, to say the least, that reigns there.

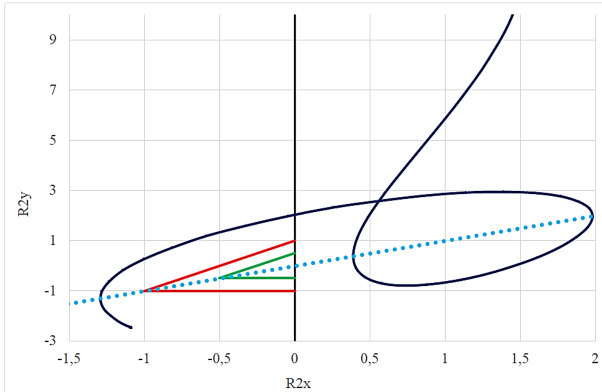

FIGURE 33. Function $R2y$ versus $R2x$. $a = 1/2$ and $b \in]0, 11.5]$.

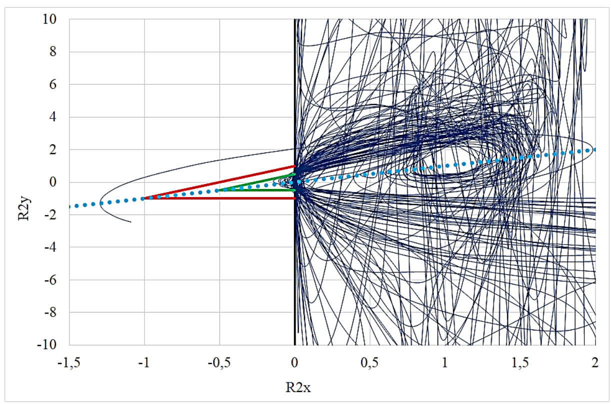

FIGURE 34. Function $R2y$ versus $R2x$. $a = 1/2$ and $b \in]0, 200]$.

7.5.2. *Why can $R2x$ be less than -1 for small values of b ?* Some rule will apply in a context and only in this case. It is not otherwise here. Indeed, let us collect the values of $\cos(b.\ln(m))$ and $\sin(b.\ln(m))$, in the sums C_k and S_k as a function of m, for truncation $m = 1$ to 10000 for two samples of b, $b = 1.55$ and $b = 14$. The two samplings, joining the points by continuity, the values of m in abscissa, are represented in figures 35 and 36.

Let us collect also these values for two additional cases ($b = 0$ and $b = 100$). Then, for these four samplings, let us list the results, not by increasing m, but by increasing function values which enables us to see their function distributions. We get the graphs in figures 37 up to 40. We observe that the distributions are not according to some fixed scheme for small values of b. It gradually tends however, as b increases, towards a unique sinusoidal distribution common to the elements of $\sum \cos(b.\ln(m))$ and those of $\sum \sin(b.\ln(m))$. The minimum -1 rule is necessarily subject to a certain

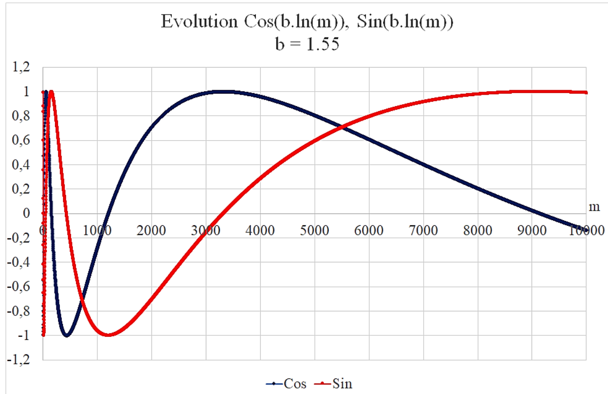

FIGURE 35. Function $\cos(l.\ln(m))$ versus m. Function $\sin(l.\ln(m))$ versus m. $a = 1/2$ and $b = 1.55$.

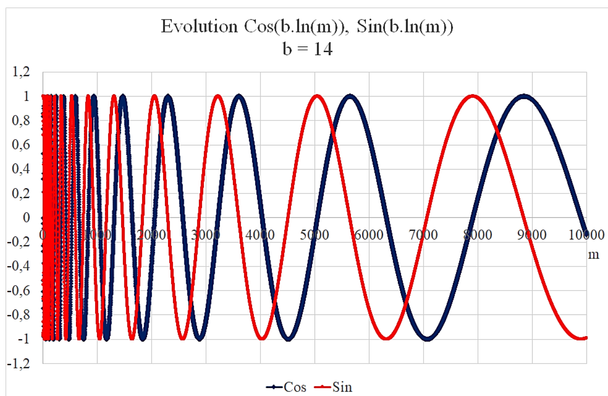

FIGURE 36. Function $\cos(l.\ln(m))$ versus m. Function $\sin(l.\ln(m))$ versus m. $a = 1/2$ and $b = 14$.

strict framework. We note that this frame is the existence of this sinusoidal distribution. Thus, the deviation from the minimum rule can be acceptable up to the somewhat quite approximate value $b \approx b_r1$ (≈ 14). Beyond this region, a unique distribution type $\cos(\pi.(m/mmax+1))$ settles permanently and will remain so up to b infinite. Of course, truncation cannot be limited to $mmax = 10000$ terms when b increases (as was the approximation choice here).

Of course also, except for $b = 0$, by taking a truncation with more terms (than 10000), we can find a sinusoidal profile for small values of b. But this takes place while the additional terms have only a negligible effect on the asymptotic value of $R2x$, the latter being essentially built on the first terms. The profile of the distribution must be "complete" in the useful truncation zone, where it has a real effect on the value of $R2x$ (that is $R2$), otherwise it is strictly speaking effectively "incomplete".

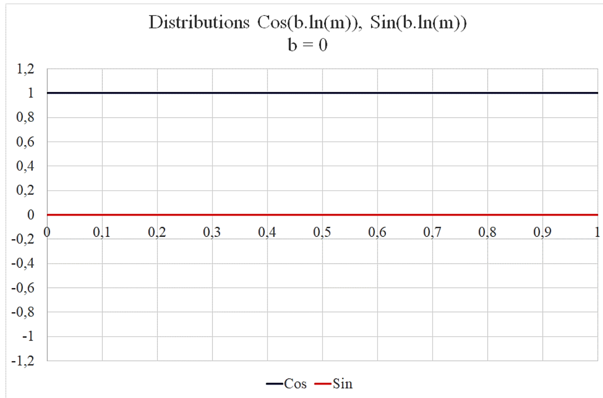

FIGURE 37. $a = 1/2$ and $b = 0$.

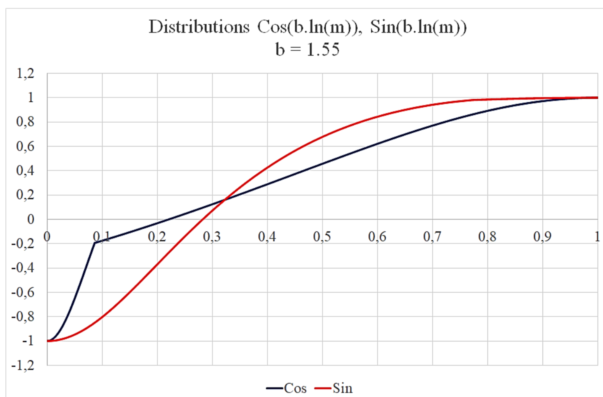

FIGURE 38. $a = 1/2$ and $b = 1.55$.

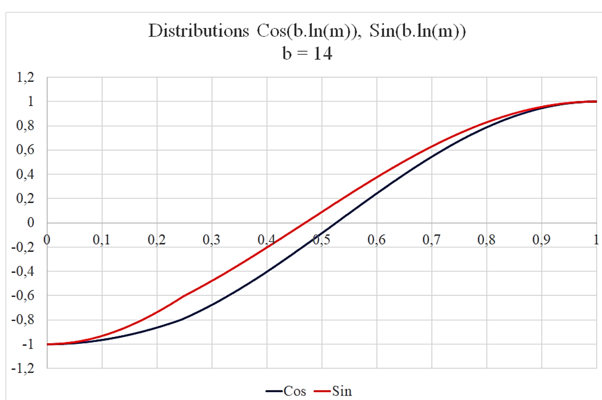

FIGURE 39. $a = 1/2$ and $b = 14$.

7.5.3. *Is the unique asymptotic distribution sufficient to imply $R2x$ greater than -1 for $b > b_r 1$?* In other words, how many b-values need to be checked

FIGURE 40. $a = 1/2$ and $b = 100$.

before concluding that the minimum value cited is legitimate each time (and that we are in the presence of a theorem)?
Well, to whom will object that this is only a few calculations on a tiny part of the values that can take b, we recall that the parameter b is encapsulated in the cosine and sinus functions that can only take values between -1 to 1. The neighbourhoods of all values within this interval are reached thousands of times (for $b < 20000$ for example and we went a hundred times further) and the functions are continuous. Of course, not all possibilities are covered, but the sample is quite representative of the whole system of equations. In addition, if the examples are necessarily specific, the relationships and thus conclusions are general.

Note 18. We do not say, however, that the sinusoidal distribution is sufficient to get $R2$ greater than -1. If we affect random values to $\cos(b.\ln(m))$ between -1 and 1 (with corresponding values deduced for $\sin(b.\ln(m)))$ in the way that we still get a correct distribution, we still can get counter-examples to the $R2$ result (we checked the failure). It is absolutely necessary to use both $(\cos(b.\ln(m)), \sin(b.\ln(m)))$ and $m = 0$, 1, 2, etc. in this order in the equations for everything to work according to expectation.

Proposition 2. *The Riemann hypothesis is true.*

Argument. Going back to the proposition 1, we mentioned that we had to establish the particular event $a = 1/2$ as some limit case which clearly happens in the study of $R2(a, b)$. The fact that $D1(a, b) \neq 0$ and $D2(a, b) > 0$ on the left side of the critical band allows then to conclude.

8. FINAL NOTE

We studied the local extrema of a function, labelled $R2(a, b)$, over the lower half of the Riemann critical band. The maximums of this function may have any positive high value without incidence on our goal, but the minimums are expected to stay within the $]-1, 0[$ negative interval in order to prove the Riemann hypothesis. The way the proof was implemented here enables to calibrate the "distance" to a possible denial and shows a quite wide gap to such a mishap. Indeed, numerically, the lowest value one can find in the $(a, b) \in ([0, 1/2], [3, 2000000])$ range is $R2(a, b) \approx -0,51209$ which event happens for the coordinates $(a, b) = (1/2, 78974.87502)$. Moreover it is easy to prove that the asymptotic values of the local negative extrema tend towards $-0,5$. Most of the theoretical work was therefore here intended to get proof in the intermediate range (a range being within the very lower range of coordinates (a, b) already known void of any Riemann hypothesis exception, thus even more reassuring in our conclusion).

There are more numerical examples and graphics in reference [7] for a reader who may wish to get more of this kind of supporting evidences. One has to highlight the difficulty to get rapid evaluation of $R2(a, b)$ for large b. It would be convenient to be able to use a method of acceleration of convergence reducing the number of terms m, which order of magnitude is proportional to b, to a number of terms for example of size proportional to $b^{1/2}$. Unfortunately, referring to appendix A and therefore the constraint reminded in relation 9, the jumps in values of $\cos(b.\ln(m))$ and $\sin(b.\ln(m))$ makes such alleviation impossible.

Several formulas, such as relationships 11, 12 or 14 have been established empirically. Therefore, even though not essential to establish our proof, it would be still interesting to find some alternative mathematical method to determine the exact relationships. Those are necessarily quite complex, as according to further investigations again provided at reference [7] (within Appendix 8), there are effectively better approximations which consist in including not only the imaginary part of the nearby Riemann's zeroes but also, at least, the imaginary part of the nearby Dirichlet's zeroes.

If such an exact alternative is reached, in a thousand years, when another eminent reader that the one who reads us here, will wake up, his wish will be all the more satisfied looking at these additional relationships. If not, we will tell him: "Young man, in mathematics, you don't understand things, you just get used to them". (By John Von Neumann).

Appendix A. Imperatives linked to the truncations

The numerical evaluations within of the body of text are drawn from expressions with infinite numbers of terms. They are based on approximations by truncation at a certain rank n. Expressions are the combinations of

$$C_k(a, b) = \sum_{m=1}^{\infty} (-1)^{m-1+k} (\ln(m))^k \cdot m^{-a} \cdot \cos(b.\ln(m))$$

and

$$S_k(a, b) = \sum_{m=1}^{\infty} (-1)^{m-1+k} (\ln(m))^k \cdot m^{-a} \cdot \sin(b.\ln(m))$$

which shapes as a function of n is typically those of figure 41.

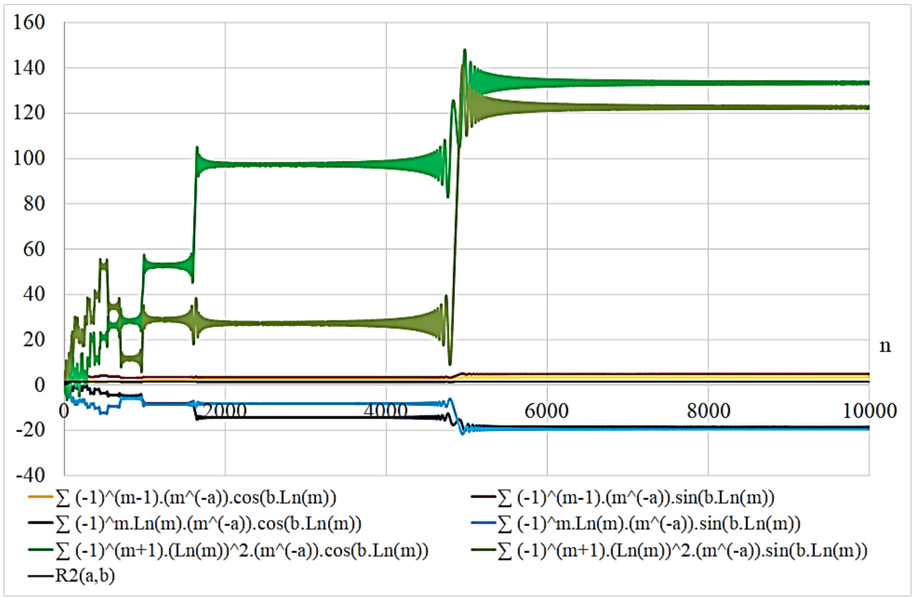

FIGURE 41. $a = 1/2$, $b = 15300$

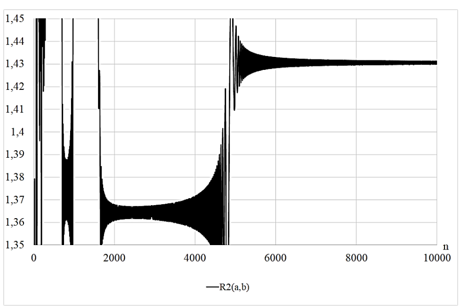

FIGURE 42.
Zoom on $R2(a, b)$
figure 41

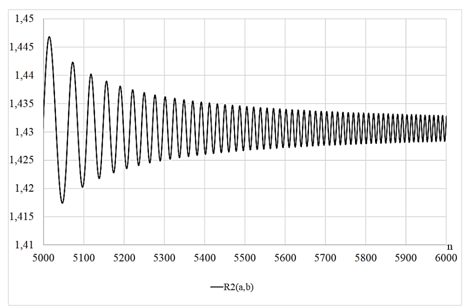

FIGURE 43.
Zoom on $R2(a, b)$
figure 41

The particular look of these graphics can give the reader the impression that it is impossible to assess the value of expression to infinity. Indeed, leaps in values appear at abscissas that may seem random. What guarantee do we have here that a new jump will not arise somewhere asymptotically? To find out what is happening, it is necessary to trace the origin of these jumps. The sums we are talking about here are partially alternating sums. A jump comes from the fact that, at some stage, a given term is followed by a term of the same sign and this "many" times. So let us consider what produces the sign of two terms that follow each other. Within $(-1)^{m-1+k}.(Ln(m))^k.m^{-a}.cos(b.Ln(m))$, neither $Ln(m)$ nor m^{-a} have any effect on the change of sign. It remains therefore $(-1)^m.cos(b.Ln(m))$, k

being a constant term that can be eliminated in the list. For two successive terms to be the same sign, it is sufficient asymptotically that $(-1)^m.cos(b.Ln(m)) \approx (-1)^{m+1}.cos(b.Ln(m+1))$ since $Ln(m)$ and $Ln(m+1)$ are of close values. From that, we deduce $cos(b.Ln(m+1)) \approx -cos(b.Ln(m))$, or $cos(b.Ln(m+1)) \approx cos(\pi + b.Ln(m))$, and then $b.Ln(m+1) \approx (1+2k).\pi + b.Ln(m)$, or finally:

$$b.Ln(1+1/m))/\pi \approx 1+2k$$

where $k \in Z$.
For $b > 0$ and $m > 0$, k is necessarily in N.
When $m \to +\infty$, and b has some given value, the product $b.Ln(1+1/m))/\pi \to 0$, so that the values of m for which $b.Ln(1+1/m))/\pi \approx 1$ are the last ones for which a jump occurs. The initial expression will converge after this last leap which intervenes at abscissa:

$$m \approx 1/(exp(\pi/b) - 1)$$

In the case of the graphics 41 to 43 examples, $m \approx 1/(exp(\pi/15300) - 1) \approx 4870$. The other jumps occur around m abscissas such as:

$$m \approx 1/(exp((1+2k).\pi/b) - 1)$$

This gives Table 1. This table explains the "chaos" near the origin of the abscissas.

Table 1

k	m
...	...
10	231
9	256
8	286
7	324
6	374
5	442
4	541
3	695
2	974
1	1623
0	4869

Table 2

k	m
...	...
10	243
9	270
8	304
7	347
6	405
5	487
4	608
3	811
2	1217
1	2435
0	$+\infty$

The resulting expression also allows to give approximately the rank n sufficient, for some b, to have a good asymptotic evaluation despite the truncation. Typically, one can choose twice the abscissa of the last jump:

$$n \approx 2/(exp(\pi/b) - 1)$$

or approximately when b is large enough in front of π (which is the case in general):

$$n \approx 2b/\pi \approx 0.63662\ b$$

This provides Table 3.

TABLE 3

Parameter b	Rank n
100	63
250	158
500	317
1000	636
2500	1591
5000	3182
10000	6365
25000	15914
50000	31830
100000	63661

Roughly speaking, the accuracy of the asymptotic evaluation therefore depends on a linear variation in the number of terms of the truncation with respect to b. The 2−times factor is adequate to get current $R2(a, b)$ value. But at some peak r_{peak}, the evaluation often necessitates a higher number of terms.

The graphic below gives the example of $b = 100$. The reader will therefore note, that the previous twice factor also does not apply to "small" b values ($b < 50$) due to the presence of quite significant oscillations. There is no use however to discuss further this particular case here.

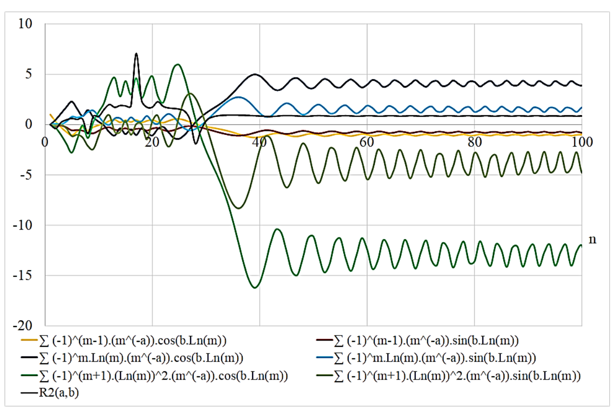

FIGURE 44. a = 1/2, b = 100

For the sake of accuracy, most of the calculations were conducted with 10000 terms even when not necessary. In the case of $b > 100000$ we used at least 100000 terms, the effective number being specified then most of the time.

Now, when, on the contrary, we are interested in areas where the function studied is not subject to a jump but is rather close to a zero slope, the equation to be solved is $(-1)^m.cos(b.Ln(m)) \approx -(-1)^{m+1}.cos(b.Ln(m+1))$ and therefore:

$$b.Ln(1 + 1/m))/\pi \approx 2k$$

The corresponding abscissas are:

$$m \approx 1/(exp(2.\pi.k/b) - 1)$$

So that, for our example, we get table 2.

Let us note that for the sinus, the expressions of the sought abscissas result in exactly the same.

Finally, in view of figure 43 for example, and directly related to the fact of having an alternating sum, the accuracy of the evaluation is subject to oscillations. Thus, the evaluation of a current $\sum$ is done by adding half of opposite the last term (or equivalently the average of the sum at ranks $n-1$ and n is made). Eventually, when necessary, the average of several terms in even number is made (up to 100 terms when $b > 100000$).

Literature and sources

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