

The Siamese twins of the Zeta function zeros.

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Abstract The Zeta function shares its zeroes with other functions. We propose the construction of the latter using a general model which allows studying a plethora of such objects. In addition, completing the list of nontrivial zeroes with a second infinite list, we can highlight the same effects, chiefly the generation of bundles of curves converging towards these zeroes, when we compare them at real abscissas $1/2$ and 1 , hence providing further insight on the Riemann's hypothesis. These results tip the scales on the famous conjecture side.

Les frères siamois des zéros de la fonction Zêta et les deux clefs de l'hypothèse de Riemann.

Résumé La fonction Zêta partage ses zéros avec d'autres fonctions. Nous proposons la construction de celles-ci à partir d'un modèle général ce qui permet d'en étudier une pléthore. De plus, complétant la liste des zéros non triviaux par une seconde liste infinie, nous pouvons mettre en évidence les mêmes effets, notamment la génération de faisceaux de courbes convergeant vers ces zéros, en comparant ceux-ci aux abscisses réelles $1/2$ et 1 , apportant ainsi un éclairage complémentaire sur l'hypothèse de Riemann. Ces résultats font pencher la balance du côté de la célèbre conjecture.

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1.Context.

The mathematical literature is abundant with evidence for the Riemann hypothesis [1]. One of the most important is the proof by André Weil in the 1940s of an analogue of this hypothesis for curves over a finite field [4]. We leave here also an analogy between two sets of mathematical objects, but it is closer still. We begin here also with an analogy, but somewhat closer, namely that of the sharing of the same equations by two collections of mathematical objects: the zeros of Riemann and some peculiar list of imaginary numbers

Indeed, we stage the Siamese twins of Riemann zeros. We named them Dirichlet zeroes for practical reasons (and symmetry to respond to a name by another name). The most convincing echo for the Riemann hypothesis for us is the mere existence of these imitators with their constant real value up to infinity.

2.Objectives.

The first objective of this article is to show that **any** non-trivial Riemann zero (or Dirichlet zero) is a solution of the set of equations

$$FG1_{\infty}(s) = \sum_{m=1}^{\infty} \sum_{n=1}^m (m.n)^{-a} \cdot (-1)^{m+n} \cdot \text{if}(m = n, 1, 2) \cdot ((m/n)^{i.b} \cdot F(m) + (m/n)^{-i.b} \cdot F(n)) = 0 \quad (1)$$

on one hand, and of the set of equations

$$FG2_{\infty}(s) = \sum_{m=1}^{\infty} \sum_{n=1}^m (-1)^{m+n} \cdot \text{if}(m = n, 1, 2) \cdot ((F(n)/m^{a+i.b}) + (F(m)/n^{a+i.b})) = 0 \quad (2)$$

on the other hand.

The second objective is then to show that any solution s of one or the other of these families admits no distinct symmetric solution to the real axis $1/2$.

The Riemann hypothesis is then true.

Note 1 : It is not necessary to get a proof for the second point for the two families of equations and only the first set is investigated.

Note 2: Some conditions, little restrictive for your purpose and targets, apply to the function F for the $FG1_{\infty}(s)$ or $FG2_{\infty}(s)$ sums to be actually null. They will be specified later on.

The second objective is achieved only summarily hereby. However, we will expose all the steps necessary to a proof. We are not in a position to judge whether what is said is enough or not.

The hereby method of investigation is simple. We seek the (Dirichlet) Siamese zeros properties and we hope the equivalent for the Riemann zeros. It is so and it will result in numerous illustrations.

3.The zeros of the Riemann Zeta function and of the Dirichlet Eta function.

Let us have thus $s = a + i.b$, a complex number. Subsequently when necessary, we index a and b by r (for Riemann) or s (for Siamese or Dirichlet). We will also use, whenever necessary, $s = s_0$ to designate a zero, that is a root of the given equation and $s \neq s_0$ to designate a different number than this zero. A point s in the neighbourhood of a zero s_0 is denoted by $s \approx s_0$. When this sign is used, the said neighbourhood is taken small enough so that the stated property applies, without be reduced to s_0 to the right or to the left. Signs may be cumulative. Thus $s \approx s_0$ and $s \neq s_0$ is a point in the neighbourhood of s_0 different of this point. In addition, for graphics represented versus axis b , we will call b the abscissa (although ordinate would possibly be more appropriate). Thus, at a zero, we talk about Riemann abscissa or Dirichlet abscissa.

Let us have also $\text{Ln}(x)$ the natural logarithm of x .

The Riemann Zeta function is defined for $\text{Re}(s) > 1$ by the series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} \quad (3)$$

For $\text{Re}(s) > 0$, it admits an analytic extension based on Dirichlet Eta series $\eta(s)$.

$$\eta(s) = (1-2^{1-s}) \cdot \zeta(s) \quad (4)$$

To find of the Riemann Zeta function zeros is therefore essentially to find of the Dirichlet Eta function zeros. The zeros of $1-2^{1-s}$ are the previously mentioned zeros, Siamese brothers of Riemann zeros.

Hence we have the sets of solutions :

$$\{\text{Dirichlet zeros}\} \equiv \{\text{Riemann zeros}\} \cup \{\text{Dirichlet zeroes}\} \quad (5)$$

This being done, we still have to identify the list of common equations. We will however take necessary time for this.

4.Fundamental theorems.

These are general results of the theory of entire functions, that we will not prove again here.

Theorem 1 (principle of isolated zeros)

Let us have f an analytic function in a field U , cancelling in a . Then, or f is identically zero, or there is a disk D of centre a , for which $f(s)$ is non-zero, in any s in D other than a . [6]

This theorem is inferred from the principle of the analytic continuation
It is also called the principle of isolated zeros.

Theorem 2

The none-constant function $f(a,b)$, represented according the a -axis, with b constant, is not constant on any interval
Inverting a and b , the same applies.

This is a simple corollary of theorem 1.

We express by this that the function is not constant if a varies alone and is not constant if b varies alone. Simultaneous variations allow, of course, by continuity, to find a contradictory path.

This theorem will be used constantly in this article, most of the time without mentioning it.

5.Study at the boundaries of the critical strip.

5.1.Upper boundary of the critical strip.

We are taking of $\text{Re}(s) = 1$.

Theorem 3

$\zeta(s)$ admits no zero such as $\text{Re}(s) = 1$.

This is a historic result that we will just set out without rewriting any proof. In 1896, Hadamard and De La Vallée-Poussin independently proved that no zero could lie on the line $\text{Re}(s) = 1$, and therefore that all non-trivial zeros should lie inside the critical strip $0 < \text{Re}(s) < 1$. This was to be a key result in the first full demonstration of the theorem of prime numbers [5].

The zeros of $\eta(s)$ are those of $\zeta(s)$, but also those of $1-2^{1-s}$. Further digital illustration shows moreover that these solutions are appropriate. The zeros of $1-2^{1-s}$ are equal to

$$s = 1 + i.2\pi.k/\text{Ln}(2) \quad (6)$$

where k is any relative integer. $\zeta(s)$ is not defined at $s = 1$, the zero corresponding to the value $k = 0$ should therefore be dismissed.

Hence, the $\eta(s)$ function has an infinity of zeros with real value exactly equal to 1 and imaginary values worth $2\pi.k/\text{Ln}(2)$, perfectly periodic. It also has an infinite number of other zeros, according to the general theory of entire functions [5][6], with the first billions all actually of real value equal to 1/2 in agreement with the Riemann hypothesis. What fly would have stung this function Eta, for suddenly, while indefinitely keeping custody to 1, choosing to waive custody to 1/2 ?

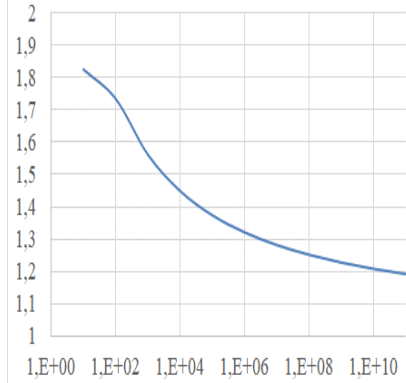
The similarity does not end there for the remainder zeroes. In fact, if $2\pi.k/\text{Ln}(2)$ is the imaginary value of Dirichlet's k -th zero, $2\pi.k/\text{Ln}(k)$ is the asymptotic imaginary value of the Riemann's k -th zero (see [7]), the “asymptotic supplement” being linked to the infinite number of terms in $\zeta(s)$ instead of the unique 2 in $1-2^{1-s}$.

Hence the continuation of the relationship (6) :

$$s \rightarrow 1/2 + i.2\pi.k/\text{Ln}(k) \quad (7)$$

The asymptotic convergence, as is often the case when the logarithm function is present, is extremely slow :

Rank k	$s_k =$ zeros at rank k	$a_k =$ $2\pi.k/\text{Ln}(k)$	s_k/a_k
10	49,77383248	27,28752708	1,82405069
100	236,5242297	136,4376354	1,73357028
1000	1419,422481	909,5842359	1,5605179
10000	9877,782654	6821,881769	1,44795571
100000	74920,8275	54575,05415	1,37280354
1000000	600269,677	454792,1179	1,31987705
10000000	4992381,014	3898218,154	1,28068282
100000000	42653549,76	34109408,85	1,2504922
1000000000	371870203,8	303194745,3	1,2265061
10000000000	3293531632	2728752708	1,20697329
100000000000	29538618432	24806842797	1,19074477



Let us have then the truncated function (indispensable to the implementation of the graphics) :

$$\eta_n(s) = \sum_{m=1}^n \frac{(-1)^{m-1}}{m^s} \quad (8)$$

One gets the Eta function for n tending towards $+\infty$ in $\eta_n(s)$.

We also have for a sum truncated at step n :

$$\eta_n(s) = \sum_{m=1}^n m^{-a} \cdot (-1)^{m-1} \cdot \cos(b \cdot \text{Ln}(m)) + i \cdot \sum_{m=1}^n m^{-a} \cdot (-1)^{m-1} \cdot \sin(b \cdot \text{Ln}(m)) \quad (9)$$

To get the zeros of $\eta_\infty(s)$ means to solve the two equations :

$$\sum_{m=1}^{\infty} m^{-a} \cdot (-1)^{m-1} \cdot \cos(b \cdot \text{Ln}(m)) = 0 \quad (10)$$

and

$$\sum_{m=1}^{\infty} m^{-a} \cdot (-1)^{m-1} \cdot \sin(b \cdot \text{Ln}(m)) = 0 \quad (11)$$

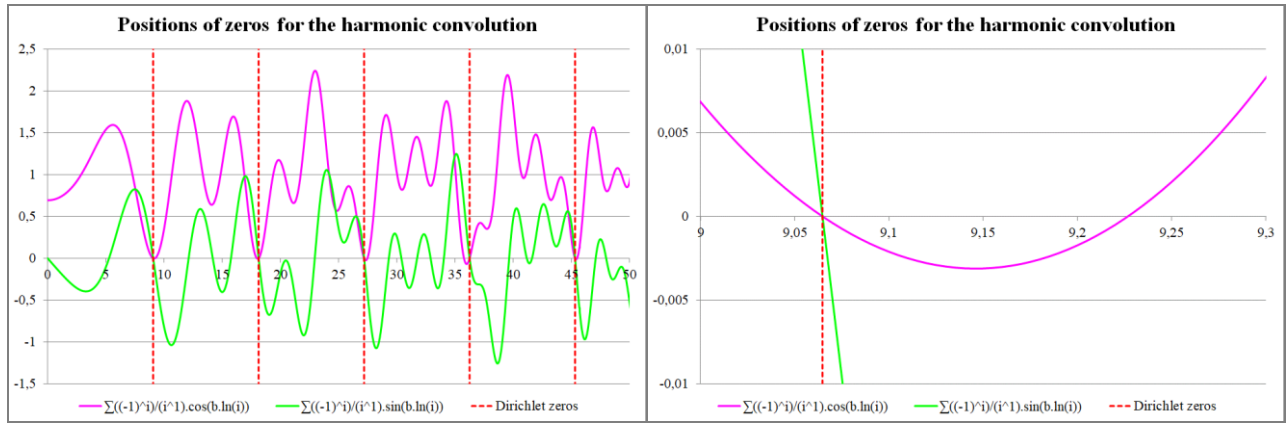
Ask for truncated sums to step n

$$\text{TC}_n(a,b) = \sum_{m=1}^n m^{-a} \cdot (-1)^{m-1} \cdot \cos(b \cdot \text{Ln}(m)) \quad (12)$$

and

$$\text{TS}_n(a,b) = \sum_{m=1}^n m^{-a} \cdot (-1)^{m-1} \cdot \sin(b \cdot \text{Ln}(m)) \quad (13)$$

Let us draw the two curves $\text{TC}_n(a,b)$ and $\text{TS}_n(a,b)$ for $a = 1$ according to b for n about 1 000 000 ($n = 2^{20}-1$ in fact)



We observe that the $TC_n(a,b)$ function basically oscillates in the half plane above the $y = 0$ axis, exceeding nevertheless this axis regularly. On the other hand, the $TS_n(a,b)$ function oscillates around the same $y = 0$ axis crossing the $y = 0$ axis regularly at the same time as $TC_n(a,b)$.

The first zero approximate value thus obtained is $bs_1 = 9,0647$ and corresponds indeed to :

$$bs_1 = 2\pi/Ln(2) \quad (14)$$

The regularity of the other solutions is obvious, what we proved with (3) and we write for other zeros :

$$bs_k = 2\pi.k/Ln(2) \quad (15)$$

We call these imaginary numbers the Dirichlet zeroes as announced earlier. They are solutions of $\eta(s)$ without being solutions of $\zeta(s)$ according to theorem 3.

The following formulas are the result of the previous basic arguments :

$\forall k \in \mathbb{Z}^*$

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^{1+i.2\pi.k/Ln(2)}} = 0 \quad (16)$$

or

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \cdot \cos(2\pi.k.Ln(m)/Ln(2)) = 0 \quad (17)$$

and

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \cdot \sin(2\pi.k.Ln(m)/Ln(2)) = 0 \quad (18)$$

From trigonometric identity $\cos(x+\varphi) = \cos(x).\cos(\varphi) - \sin(x).\sin(\varphi)$, we draw more generally, for any constant φ argument :

$$TC_{\infty}(1,b,\varphi) = \sum_{m=1}^{\infty} (-1)^{m-1} \cdot \cos(2\pi.k.Ln(m)/Ln(2)+\varphi)/m = 0 \quad (19)$$

For $k = 0$, we refer to the particular case of

$$Ln(1+x) = \sum_{m=1}^{\infty} (-1)^{m-1} \cdot x^m/m \quad (20)$$

taking $x = 1$, thus having immediately $TC_{\infty}(1,0) = Ln(2)$ and $TC_{\infty}(1,b,\varphi) = \cos(\varphi).Ln(2)$.

5.2. Lower boundary of the critical strip.

Because of the functional equation $\zeta(s) = 2^\pi \cdot \pi^{s-1} \cdot \sin(\pi s/2) \cdot \Gamma(1-s) \cdot \zeta(1-s)$ established by Riemann, which the reader will find for example in [2], we may expect a certain analogy between the cases $\text{Re}(s) = 1$ and $\text{Re}(s) = 0$, especially for zeros. It is not so, as it is the $1-2^{1-s}$ factor which has an impact on zeros at $\text{Re}(s) = 1$ and this term does not vanish for s a pure imaginary.

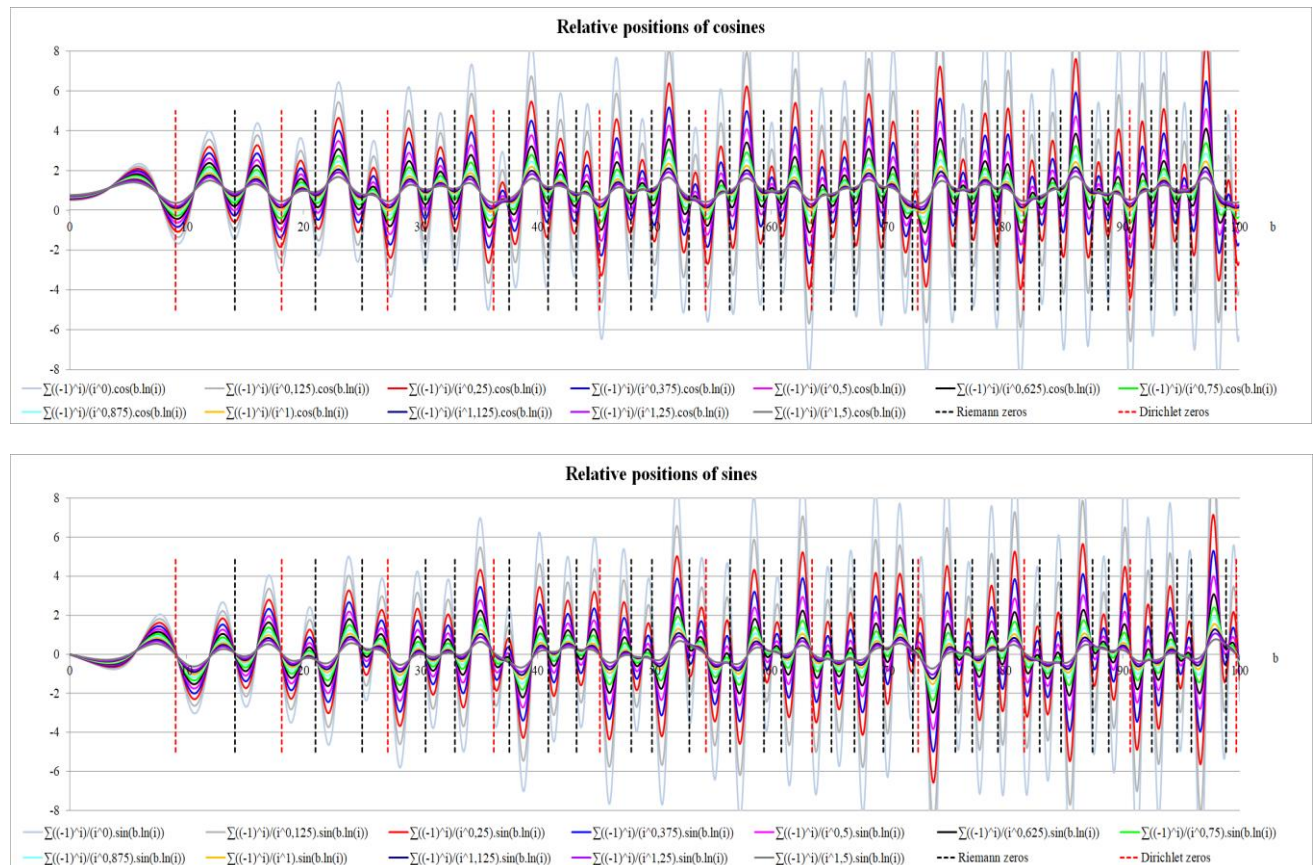
6. First steps among the non-trivial zeros.

6.1. The waves' separation.

We have given below the approximate curves representing the real and imagined values of the $\eta(s = a+ib)$ function, for different values of a , with abscissa the b parameter. More specifically, it is $\text{TC}_{1500}(a, b, 0)$ and $\text{TS}_{1500}(a, b, 0) = \text{TC}_{1500}(a, b, \pi/2)$ with successively $a = 0$ (in light blue), $a = 0.125$ (in grey), $a = 0.25$ (in red), $a = 0.375$ (in blue), $a = 0.5$ (in pink), $a = 0.625$ (in black), $a = 0.75$ (in green), $a = 0.875$ (in sky blue), $a = 1$ (in yellow ochre), $a = 1.125$ (in night blue), $a = 1.25$ (in purple) and $a = 1.5$ (in dark grey).

The Riemann abscissas are highlighted by a black dashed line and the Dirichlet abscissas by a red dashed line, below and throughout the whole article.

The drawings show the relative positions of these curves.



To indicate the positions of the Dirichlet zeroes in addition to the Riemann zeros allows to isolate a unique and systematic rising wave between two zeros, hardly visible when zeros are close.

6.2. Proximity of zeros.

Considering the imaginary parts only, there is a Riemann zero arbitrary close to a Dirichlet zero.

Argument

The gap between two zeros of Dirichlet is constant (and equal to $2\pi/\ln(2)$). It is a well-known fact [5] that the average difference between zeros of Riemann tends towards $2\pi/\ln(b)$, b being the imaginary value of the current zero. This average tends towards 0 when b increases and therefore the smallest gap between zeros of Riemann tends towards 0. For a "random" distribution, one will find imaginary values of the Riemann zeros, in average, closer and closer to a Dirichlet zero.

We will take the time to show that they cannot be the same (uniqueness of a for given b).

7.Synthesis of cosine and sine curves in a single equation.

7.1.Convergence and cancellation.

The cancellation equation (16) is written in a single equivalent equation using squares :

$$T_{\infty}(s) = T_{\infty}(a+i.b) = \left(\sum_{m=1}^{\infty} m^{-a} \cdot (-1)^{m-1} \cdot \cos(b \cdot \text{Ln}(m)) \right)^2 + \left(\sum_{m=1}^{\infty} m^{-a} \cdot (-1)^{m-1} \cdot \sin(b \cdot \text{Ln}(m)) \right)^2 = 0 \quad (21)$$

For the first square, one gets so one term in cosine brought to the square and two terms otherwise, which are $\cos(b \cdot \text{Ln}(r)) \cdot \cos(b \cdot \text{Ln}(s))$ and $\cos(b \cdot \text{Ln}(s)) \cdot \cos(b \cdot \text{Ln}(r))$. We can therefore sum up by choosing $r > s$ and adding a multiplicative factor of 2.

Using the remarkable identities $\cos(r-s) = \cos(r)\cos(s) + \sin(r)\sin(s)$ and $\cos^2(m) + \sin^2(m) = 1$, the truncated development at step n is :

$$T_n(s) = \sum_{i=1}^n \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot \cos(b \cdot (\text{Ln}(i) - \text{Ln}(j))) \cdot \text{if}(i=j, 1, 2) = \sum_{i=1}^n \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot \cos(b \cdot \text{Ln}(i/j)) \cdot \text{if}(i=j, 1, 2) \quad (22)$$

This is not exactly a double sum, the second depending of the first one, but we will use regularly this term later on. In the same time, we note that this let us free of problems of summability for double sums.

We will also use the shortcut of writing :

$$\text{or}(1,2) = \text{if}(i=j, 1, 2) \quad (23)$$

which means if $i = j$ then take 1, otherwise take 2

Then we have :

$$T_n(s) = \sum_{i=1}^n \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot \cos(b \cdot \text{Ln}(i/j)) \cdot \text{or}(1,2) \quad (24)$$

Alternatively, the other unambiguous expression is obtained by isolating the terms for which $i = j$:

$$T_n(s) = H_n(s) + A_n(s) = \sum_{i=1}^n i^{-2a} + 2 \sum_{i=1}^n \sum_{j=1}^{i-1} (i,j)^{-a} \cdot (-1)^{i+j} \cdot \cos(b \cdot \text{Ln}(i/j)) \quad (25)$$

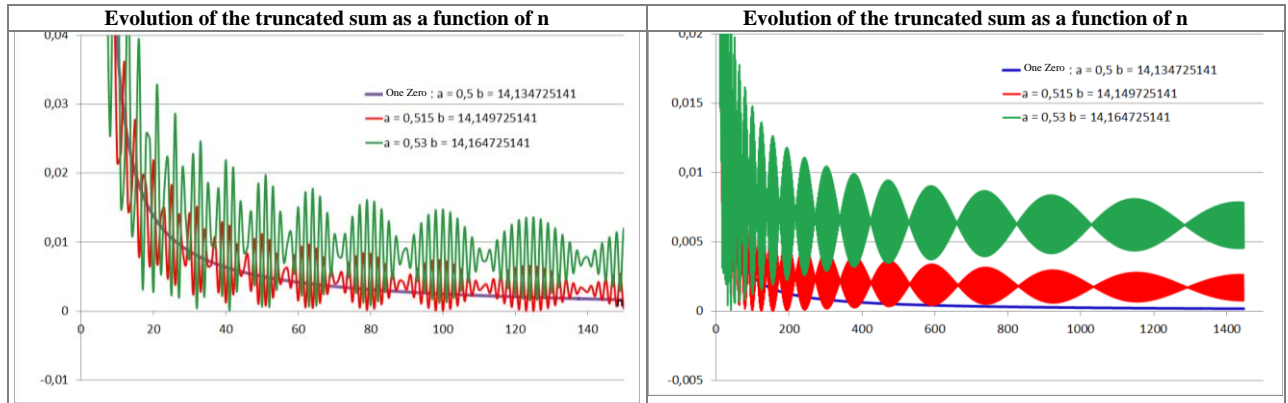
This last presentation of the formula highlights the special case $a = 1/2$ and $a = 1$.

Indeed, $a = 1/2$ is the radius of convergence of $H_{\infty}(s)$, which is the harmonic series when $a = 1/2$. To pass from $H_{\infty}(s)$ ' convergence to $H_{\infty}(s)$ ' divergence certainly has a particular impact on the whole of the term $T_n(s)$.

For $a = 1$, $H_{\infty}(s)$ converges, but the harmonic series is present to some extent in the second term by the $(i,j)^{-a}$ term when choosing $i = 1$ and $j = 2, 3, 4, \dots$, that is $1/2, 1/3, 1/4, 1/5, \dots$ by failing to see the other fractions. However, the possibility of a particular impact is more questionable.

Amazingly, even if the harmonic series is here much better hidden when $a = 1$ than when $a = 1/2$, in fact, we fully know the zeros of $T_{\infty}(s)$ in the first case (Dirichlet zeroes), while the second case remains the subject of speculations at this point (Riemann zeroes).

We represent graphically, for three different values of the pair (a,b) , the evolution of the term $T_n(s)$ when n increases. In the sample, we are located near the first (non-trivial) zero of the Eta function.



The first series of curves stopping at $n = 150$ allows to see figures of interference resulting from the trigonometric functions present in $T_n(s)$. The second series stopping at $n = 1500$ shows the same thing but detail is no more perceived with this backwards step.

We observe two types of behaviours on curves :

In the case of a curve taken at (a,b) corresponding to a zero (here $a = 0.5$ and $b \approx 14.134725141$), the curve representing $T_n(s)$ becomes smooth (without any interference pattern).

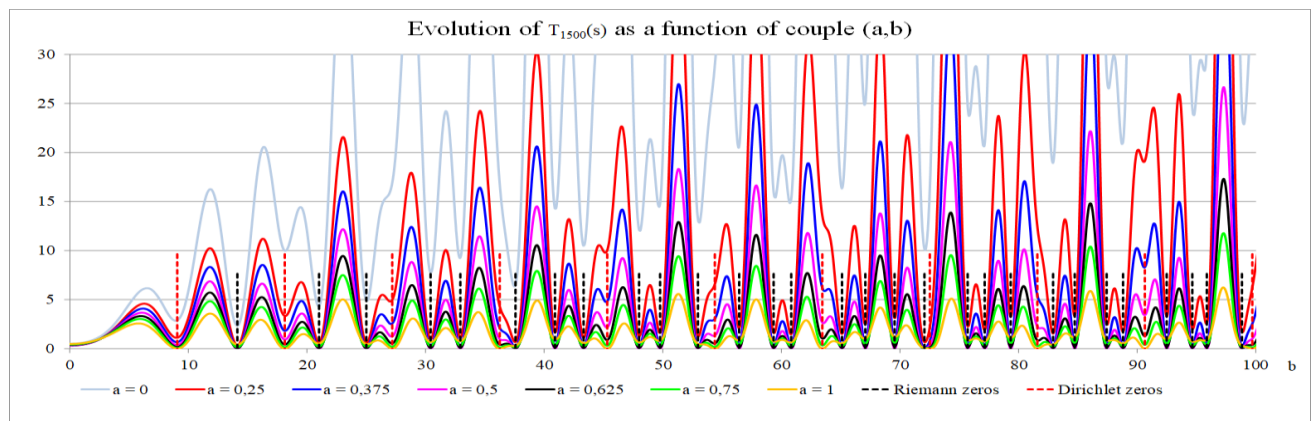
For the case of a curve (a,b) not corresponding to a zero, the curve fluctuates and form bellies and knots of interference. Then, we can consider two situations :

The first situation is that interferences occur to infinity without depreciate completely and in this case the $T_\infty(a,b)$ expression would oscillate indefinitely. There would be no convergence towards a constant number, so there would be no convergence towards zero either. This loophole would obviously confirm the Riemann guess.

The second situation is a progressive damping (even if clearly very slowly) of oscillations and convergence towards a given constant value. In this case, the tangent of $T_n(a,b)$ tends towards 0 when $n \rightarrow +\infty$. Even if it then no longer allows to conclude immediately to what concerns us here (that the Riemann hypothesis is true), this is certainly the usual situation since the general term of the series converges.

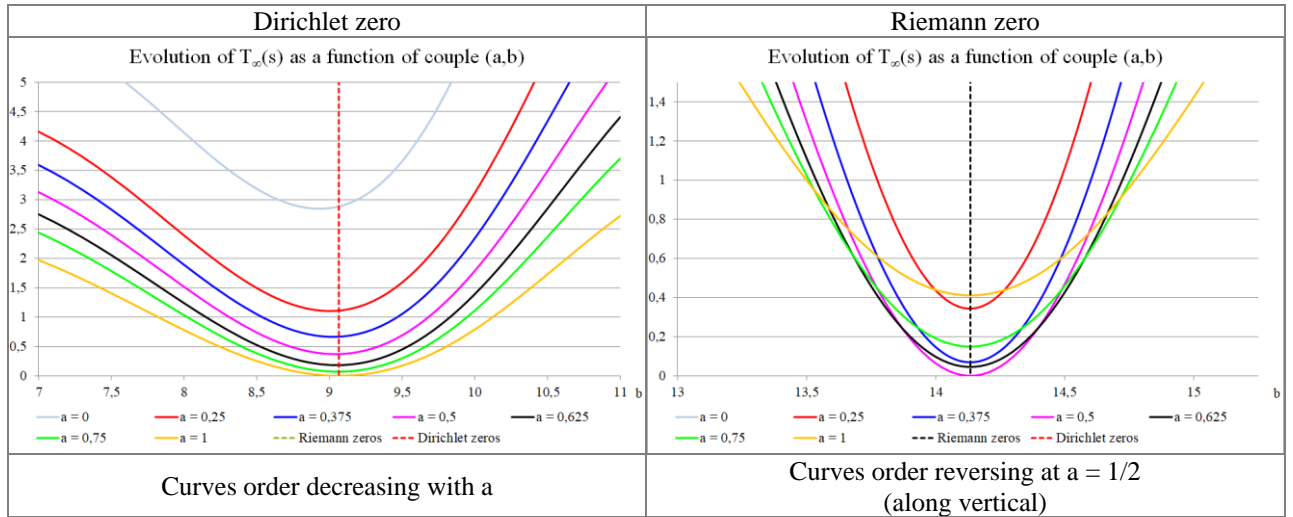
It is worth noting that these descriptive aspects do not interfere in the upcoming demonstrations.

We have charted the evolution of $T_{n=1500}(a, b)$ in the critical strip for $b < 100$ and a number of values a (with the same colour code as above, also systematically used afterwards), as follows :

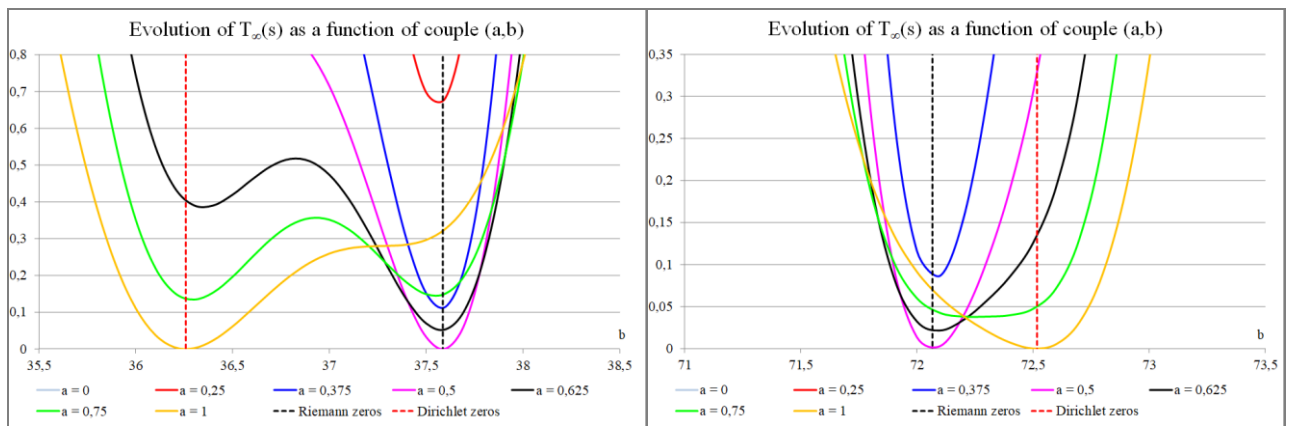


As $T_\infty(a, b)$ is a square, the whole set presents above the $y = 0$ axis, the axis being reached for the Riemann zeros (for $a = 1/2$ in the studied area) and for the Dirichlet zeroes (for $a = 1$).

The overall look is given by the two examples below :



Let us see the case, however, when zeros are relatively close, which requires a zoom in the neighbourhood of the zeros.



This highlights the reversal of the two configurations (Riemann zero and Dirichlet zero).

The inversion at $a = 1/2$ for the Riemann zeros is a simple scaling of what occurs in the same way for the Dirichlet zeroes at $a = 1$. The fact is that things are not as simple as that as we will see later on.

Note: The graphics are based on approximate calculations that explains sometimes less stringent alignments (last chart).

7.2.Successive derivations.

One can derivate $T_{\infty}(s)$ with respect to the parameters a or b several times since the function is holomorphic (at $a \neq 1$). One yields :

$$C_{\infty}(m,n,s) = \sum_{i=1}^{\infty} \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot (\text{Ln}(i,j))^m \cdot (\text{Ln}(i,j))^{2n} \cdot \cos(b \cdot \text{Ln}(i,j)) \cdot \text{or}(1,2) \quad (26)$$

and

$$S_{\infty}(m,n,s) = \sum_{i=1}^{\infty} \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot (\text{Ln}(i,j))^m \cdot (\text{Ln}(i,j))^{2n+1} \cdot \sin(b \cdot \text{Ln}(i,j)) \cdot \text{or}(1,2) \quad (27)$$

When one derives versus a , a new factor in $\text{Ln}(i,j)$ appears and therefore any full power in m exists. On the other hand, when one derives versus b , one gets a new factor in $\text{Ln}(i,j)$ at the same time as cosine becomes sine and vice versa, whence exponents $2n$ and $2n+1$ above. Of course, without recourse to the derivation, we can add a factor even or odd in $\text{Ln}(i,j)$ respectively in front of sine or cosine. This then gives half n values in $C_{\infty}(m,n,s)$ et $S_{\infty}(m,n,s)$.

7.3.Walk upon a.

Let us move stay then to a $T_\infty(s)$ zero ($s_0 = a+i.b$) and step out an epsilon versus a from this position.

We have :

$$T_\infty(s) = \sum_{i=1}^{\infty} \sum_{j=1}^i (i.j)^{-a-\varepsilon} \cdot (-1)^{i+j} \cdot \cos(b.Ln(i/j)).or(1,2) \quad (28)$$

As ε is small, we have

$$(i.j)^{-\varepsilon} = e^{-\varepsilon.Ln(i.j)} = 1 - \varepsilon.Ln(i.j) + \varepsilon^2.Ln^2(i.j)/2 + 0(\varepsilon^2)$$

Let us replace in the previous equation. It follows :

$$T_\infty(s) = \sum_{i=1}^{\infty} \sum_{j=1}^i (i.j)^{-a} \cdot (-1)^{i+j} \cdot \cos(b.Ln(i/j)).or(1,2) - \varepsilon \sum_{i=1}^{\infty} \sum_{j=1}^i (i.j)^{-a} \cdot (-1)^{i+j} \cdot \cos(b.Ln(i/j)).or(1,2) \cdot Ln(i.j) + 0(\varepsilon) \quad (29)$$

The first double sum is zero since we located at a zero.

We extracted ε from the second double sum since that term is the same for all elements of this sum.

The terms in $\varepsilon^2, \varepsilon^3 \dots$ are negligible compared to ε .

By construction (sum of two squares), the term $T_\infty(s)$ is positive. Developing up to second order, it comes

$$T_\infty(s) = -\varepsilon \sum_{i=1}^{\infty} \sum_{j=1}^i (i.j)^{-a} \cdot (-1)^{i+j} \cdot \cos(b.Ln(i/j)).or(1,2) \cdot Ln(i.j) + \varepsilon^2 \sum_{i=1}^{\infty} \sum_{j=1}^i (i.j)^{-a} \cdot (-1)^{i+j} \cdot \cos(b.Ln(i/j)).or(1,2) \cdot Ln^2(i.j)/2 + 0(\varepsilon^2) \quad (30)$$

We necessarily have, for the first double sum of the previous equation, a value equal to 0, otherwise we would have reversal of sign of $T_\infty(s)$, for ε an infinitesimal, when ε changes sign (which is impossible since $T_\infty(s)$ is positive by construction). Moreover, the second double sum must be positive or zero for the same reason. The second term is the curvature of the curve $T_\infty(s)$ at a zero. It is necessarily non-null at the immediate neighbourhood of that zero (isolated zero theorem) and so the so-called double sum is non-null.

Hence the two theorems :

Theorem 4

Let us have (a,b) corresponding to a Riemann or Dirichlet zero s_0 , then :

$$C_\infty(1,0,s = s_0) = \sum_{i=1}^{\infty} \sum_{j=1}^i (i.j)^{-a} \cdot (-1)^{i+j} \cdot Ln(i.j) \cdot \cos(b.Ln(i/j)).or(1,2) = 0 \quad (31)$$

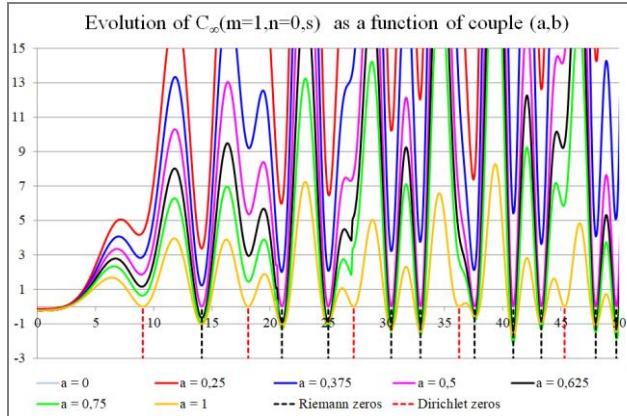
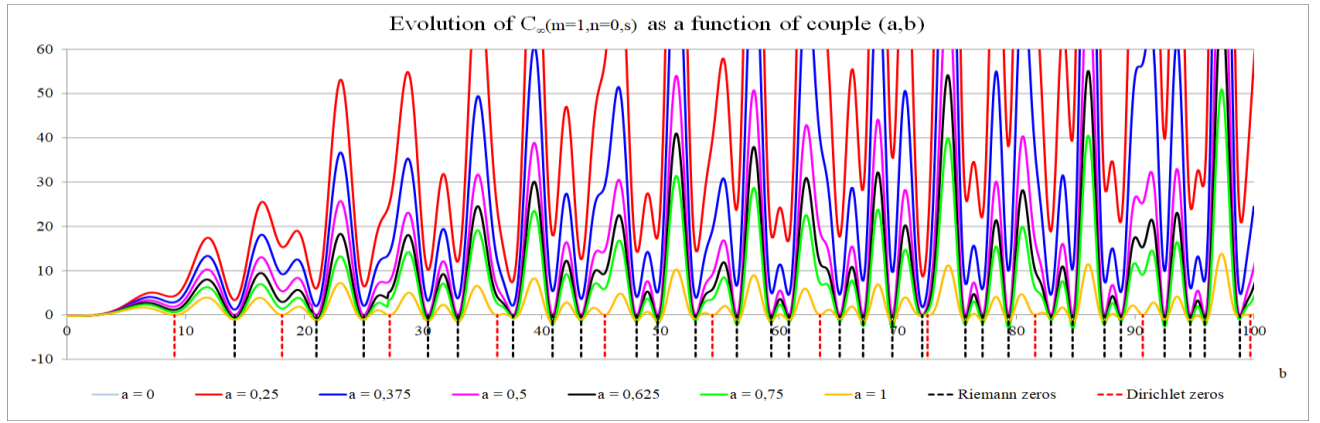
The converse is false, since $C_\infty(1,0,s)$ also cancels with each crossing from a Riemann zero to a Dirichlet zero and each crossing from a Dirichlet zero to a Riemann zero (see third graph below).

Theorem 5

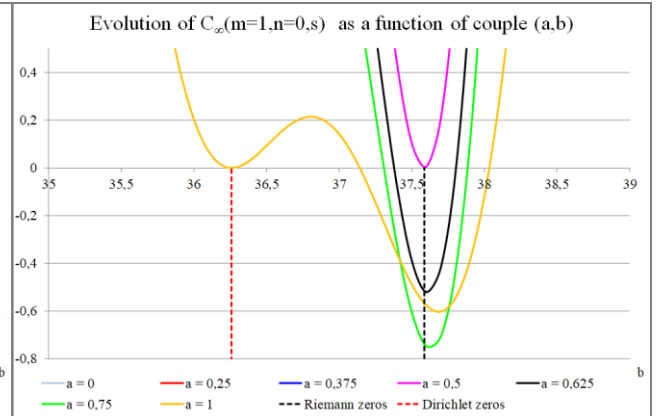
In the immediate neighbourhood of a Riemann or Dirichlet zero (a, b), we have (including for $s = s_0$) :

$$C_\infty(2,0,s \approx s_0) = \sum_{i=1}^{\infty} \sum_{j=1}^i (i.j)^{-a} \cdot (-1)^{i+j} \cdot (Ln(i.j))^2 \cdot \cos(b.Ln(i/j)).or(1,2) > 0 \quad (32)$$

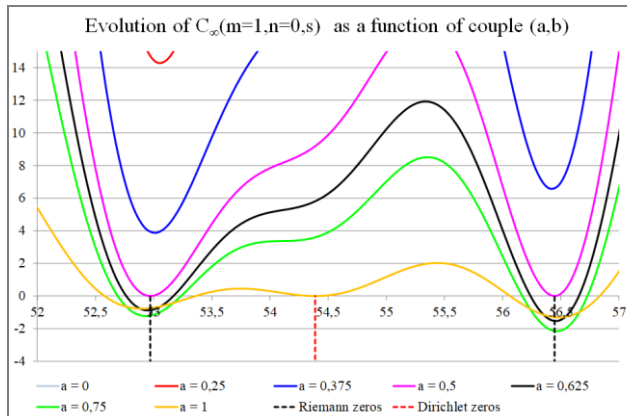
The curves below give the look of truncated $C_{1500}(1,0,s)$ and $C_{1500}(2,0,s)$ functions.



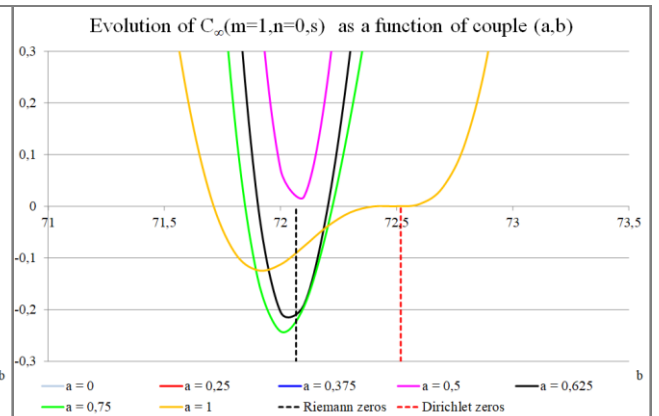
Formula (31) illustration



Curves order reversed at $a = 1/2$

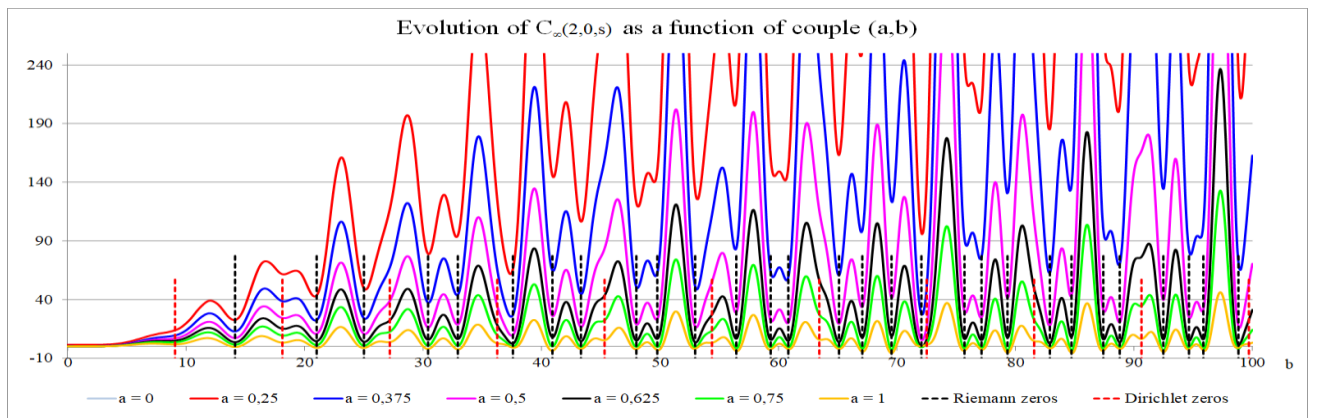


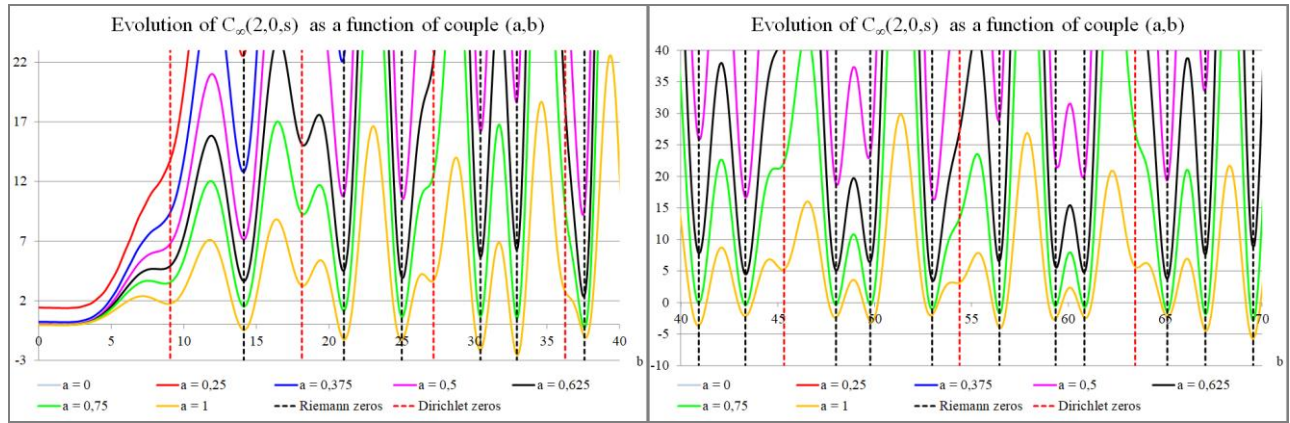
Formula (31) illustration



Curves order reversed at $a = 1/2$

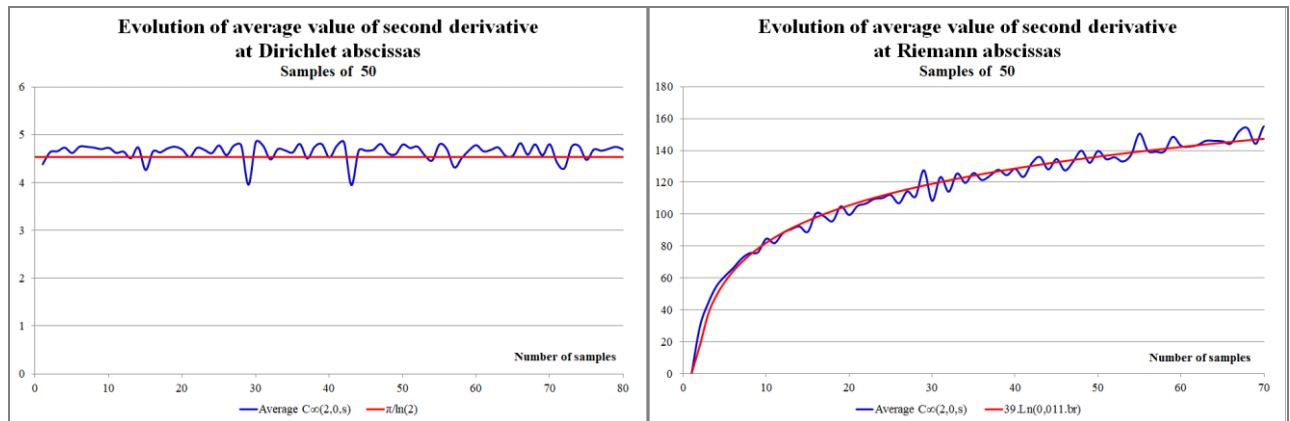
(the curve for $a = 1/2$ does not join the $y = 0$ axis on the chart due to numerical approximations)





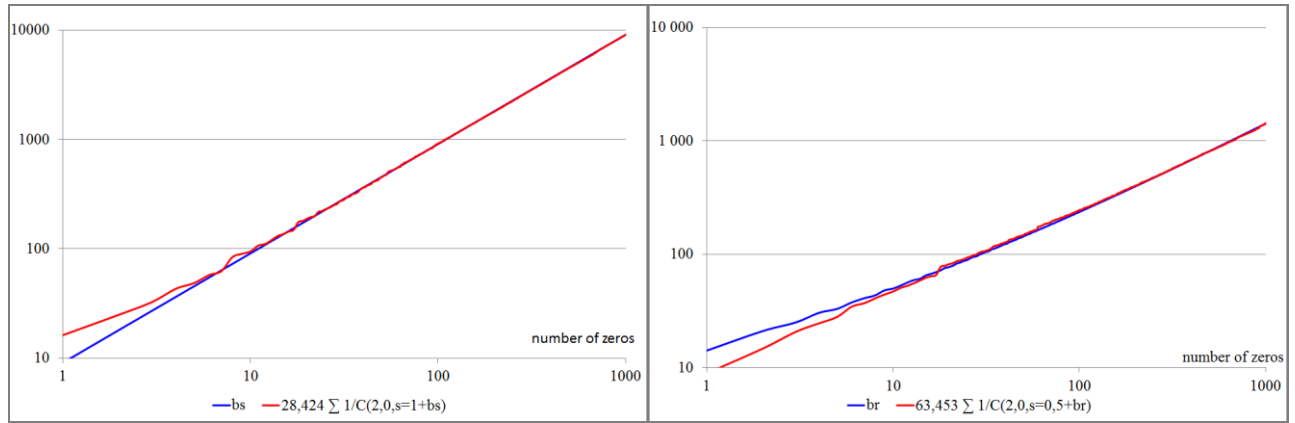
The last two graphs illustrate well theorem 5. The $C_{\infty}(2,0,s \approx s_0)$ expression is positive for $a = 1/2$ near Riemann abscissas and is positive for $a = 1$ near Dirichlet abscissas.

We give, in appendix 2, the approximate numeric values of $C_{\infty}(2,0,s)$ for the first 500 Riemann zeroes and the first 500 Dirichlet zeroes. The graphs below show in addition a few thousand of them. These numeric data are certainly not very accurate. What we have to retain here is that, from one zero to another, the values of $C_{\infty}(2,0,s)$ vary quite much. However, by grouping the results by samples of 50, the average values vary, with a multiplicative factor, as the reverse of the average gaps between zeros of each type. Everything happens as if the mean curvature increases linearly with the lack of space. In addition, the curves' curvatures at the Dirichlet abscissas express somehow their indifference to the Riemann zeros environment, since approximately constant like the gap between Dirichlet zeroes (this constant is close to half of the gap between two such zeros, that is $\pi/\ln(2)$). More numerous are the Riemann zeros, their relative amount gradually tending towards infinity in between Dirichlet zeroes. It is therefore somewhat unrealistic and unnecessary to check the indifference of the curvature at Riemann abscissas towards Dirichlet zeroes. We note simply that the values of the curvatures are inverse of the logarithm of br , the imaginary value of Riemann zeros, which can be compared to the gaps between the so-called zeroes.



In the previous graph relating the Dirichlet zeroes of (first chart), the lower values correspond certainly to numeric errors caused by sums' truncation.

We can also compare the evolution of br or bs , imaginary values of one or the other type's zero, compared with $r \cdot \sum 1/C_{\infty}(2,0,s)$ by adjusting with a multiplicative coefficient r .



The graphics are shown from the first 1000 zeros.

Only the values near the origin do not fit. The coordinates are logarithmic to better view that. In linear coordinates, the red and blue curves would differ little.

Theorem 6

The function $C_{\infty}(1,0,s)$ below is strictly positive in the immediate neighbourhood of a Riemann or Dirichlet zero (it is null at this zero according to theorem 4).

$$C_{\infty}(1,0,s \approx s_0 \text{ et } s \neq s_0) = \sum_{i=1}^{\infty} \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot \text{Ln}(i,j) \cdot \cos(b \cdot \text{Ln}(i/j)) \cdot \text{or}(1,2) > 0 \quad (33)$$

Proof

We have $T_{\infty}(s) = 0$ at a zero and $T_{\infty}(s) > 0$ (strictly) in the immediate neighbourhood of a zero by construction. The derivative of $T_{\infty}(s)$, with respect to the variable a , is $C_{\infty}(1,0,s)$. We have, according to the relation (30), in the immediate neighbourhood of a zero

$$T_{\infty}(s) = -\varepsilon \cdot C_{\infty}(1,0,s) + 0(\varepsilon)$$

where ε changes sign at the crossing of the said zero (referring to the earlier construction of this expression). In the immediate neighbourhood of a zero, $C_{\infty}(1,0,s)$ is therefore of same sign before and after the said zero. As $C_{\infty}(1,0,s)$ is holomorphic (and thus continuous and differentiable), after verification of the sign with one zero, the other neighbourhoods of zeros bearing same sign by the same relation, we conclude that $C_{\infty}(1,0,s)$ is positive in the neighbourhood of a zero and null at this zero.

Let us place then on $C_{\infty}(1,0,s)$ which is extremum in a zero. Its derivative, versus b , is thus null at this zero.

Let us write this derivative

$$-S_{\infty}(1,0,s) = - \sum_{i=1}^{\infty} \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot \text{Ln}(i,j) \cdot \text{Ln}(i/j) \cdot \sin(b \cdot \text{Ln}(i/j)) \cdot \text{or}(1,2) \quad (34)$$

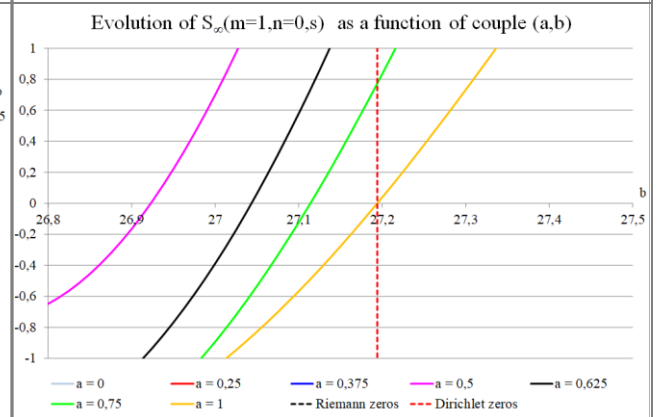
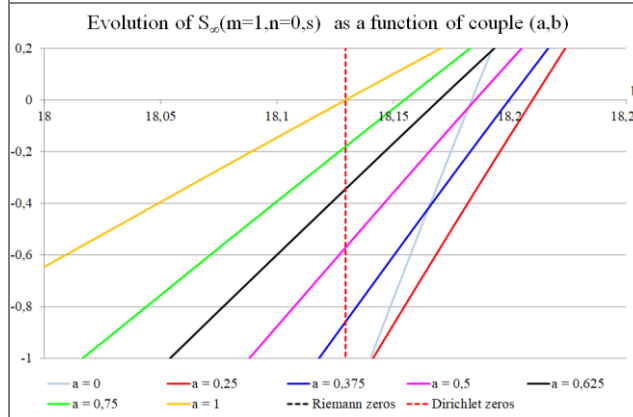
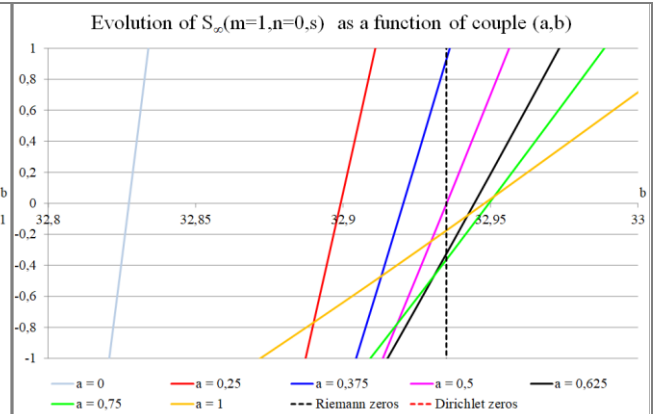
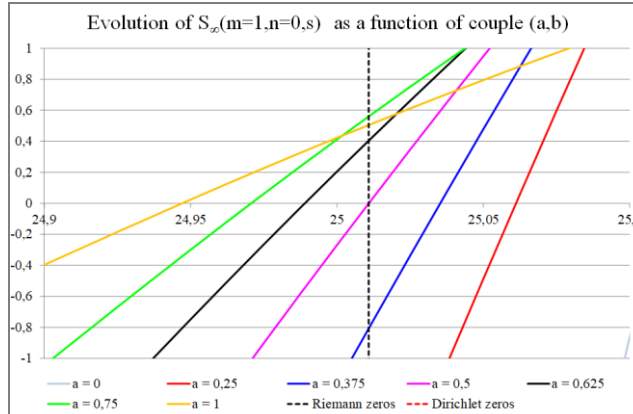
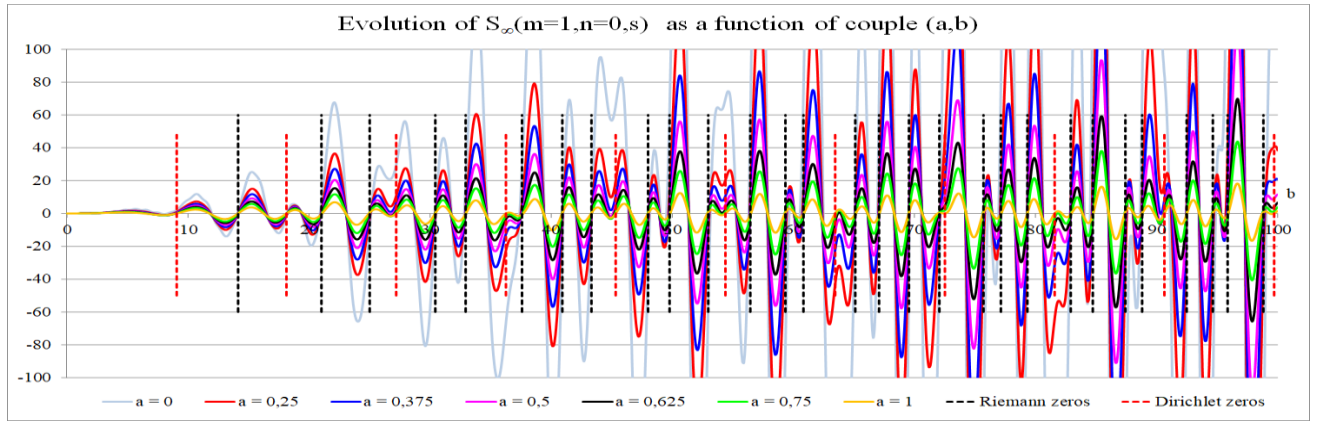
Hence the theorem :

Theorem 7

Upon a Riemann or Dirichlet zero, we have :

$$S_{\infty}(1,0,s = s_0) = \sum_{i=1}^{\infty} \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot \sin(b \cdot \text{Ln}(i/j)) \cdot \text{or}(1,2) \cdot ((\text{Ln}(i))^2 - (\text{Ln}(j))^2) = 0 \quad (35)$$

The converse is false, since $S_{\infty}(1,0,s)$ cancels also for the maxima of $T_{\infty}(s)$ giving at least an intruder among two. We illustrate these results below.



We note effectively the intersections with the horizontal $y = 0$ axis to the Riemann abscissas for $a = 1/2$ and Dirichlet abscissas for $a = 1$.

7.4.Walk upon b .

Let us move again, at a non-trivial Riemann zero ($s_0 = a+ib$) and step out an epsilon versus b of this position.
We have :

$$T_\infty(s) = \sum_{i=1}^{\infty} \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot \cos((b+\epsilon) \cdot (\ln(i/j))) \cdot \text{or}(1,2) \quad (36)$$

Using trigonometric identity, we get :

$$\cos((b+\epsilon) \cdot \ln(i/j)) = \cos(b \cdot \ln(i/j)) \cdot \cos(\epsilon \cdot \ln(i/j)) - \sin(b \cdot \ln(i/j)) \cdot \sin(\epsilon \cdot \ln(i/j))$$

As ϵ is an infinitesimal, we have :

$$\begin{aligned} \cos((b+\epsilon) \cdot \ln(i/j)) &= \cos(b \cdot \ln(i/j)) \cdot (1 - (\epsilon \cdot \ln(i/j))^2/2) - \sin(b \cdot \ln(i/j)) \cdot (\epsilon \cdot \ln(i/j)) + 0(\epsilon^2) \\ &= \cos(b \cdot \ln(i/j)) - \epsilon \cdot \ln(i/j) \cdot \sin(b \cdot \ln(i/j)) - (1/2) \cdot \epsilon^2 \cdot (\ln(i/j))^2 \cdot \cos(b \cdot \ln(i/j)) + 0(\epsilon^2) \end{aligned} \quad (37)$$

Replace in equation (36), it follows :

$$T_{\infty}(s) = \sum_{i=1}^{\infty} \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot \cos(b \cdot \text{Ln}(i/j)) \cdot \text{or}(1,2) - \varepsilon \cdot \sum_{i=1}^{\infty} \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot \sin(b \cdot \text{Ln}(i/j)) \cdot \text{or}(1,2) \cdot \text{Ln}(i/j) + 0(\varepsilon) \quad (38)$$

By the same argument on the positivity of $T_{\infty}(s)$, with the first term of the equation right member being null, it immediately induces the theorem :

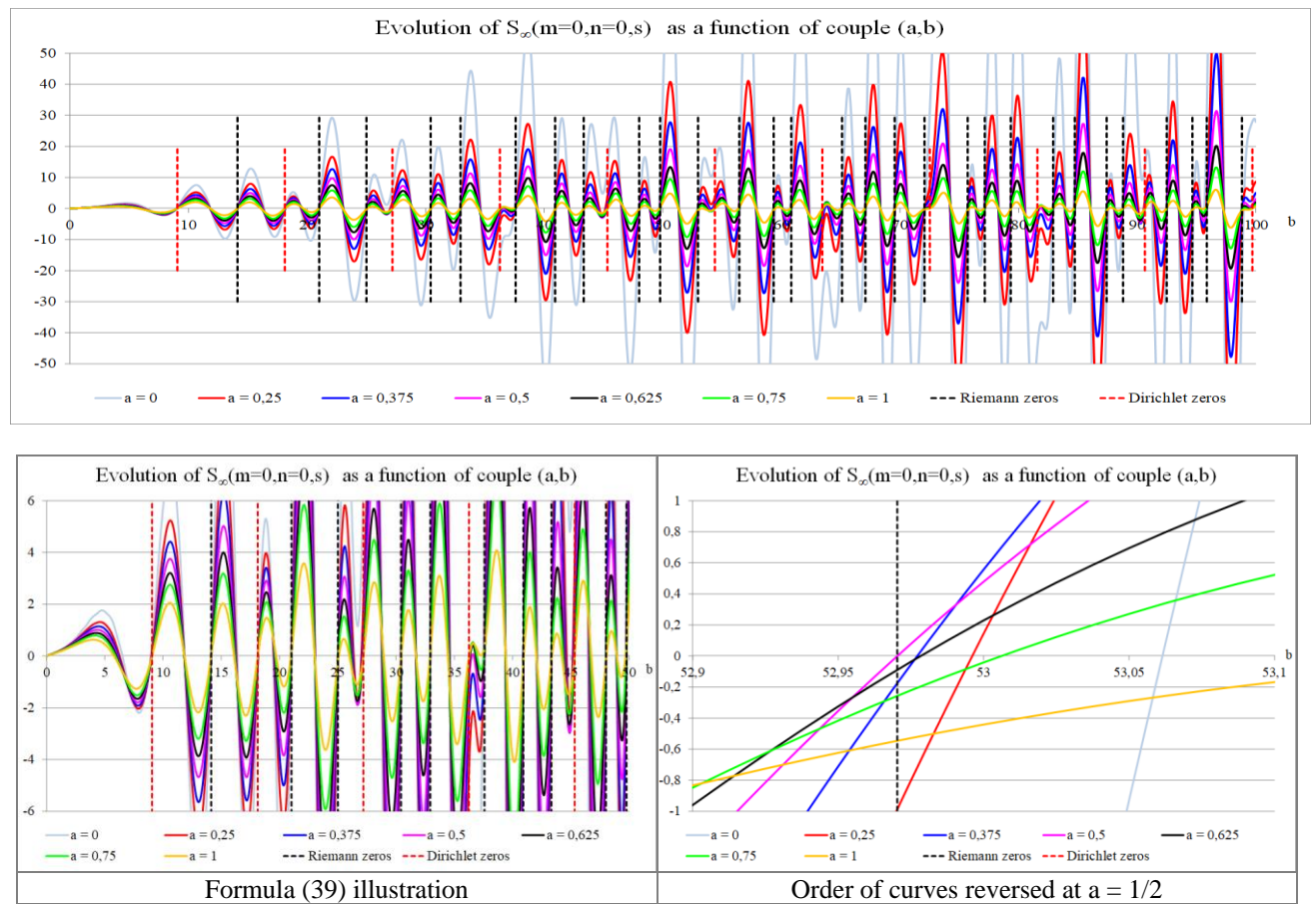
Theorem 8

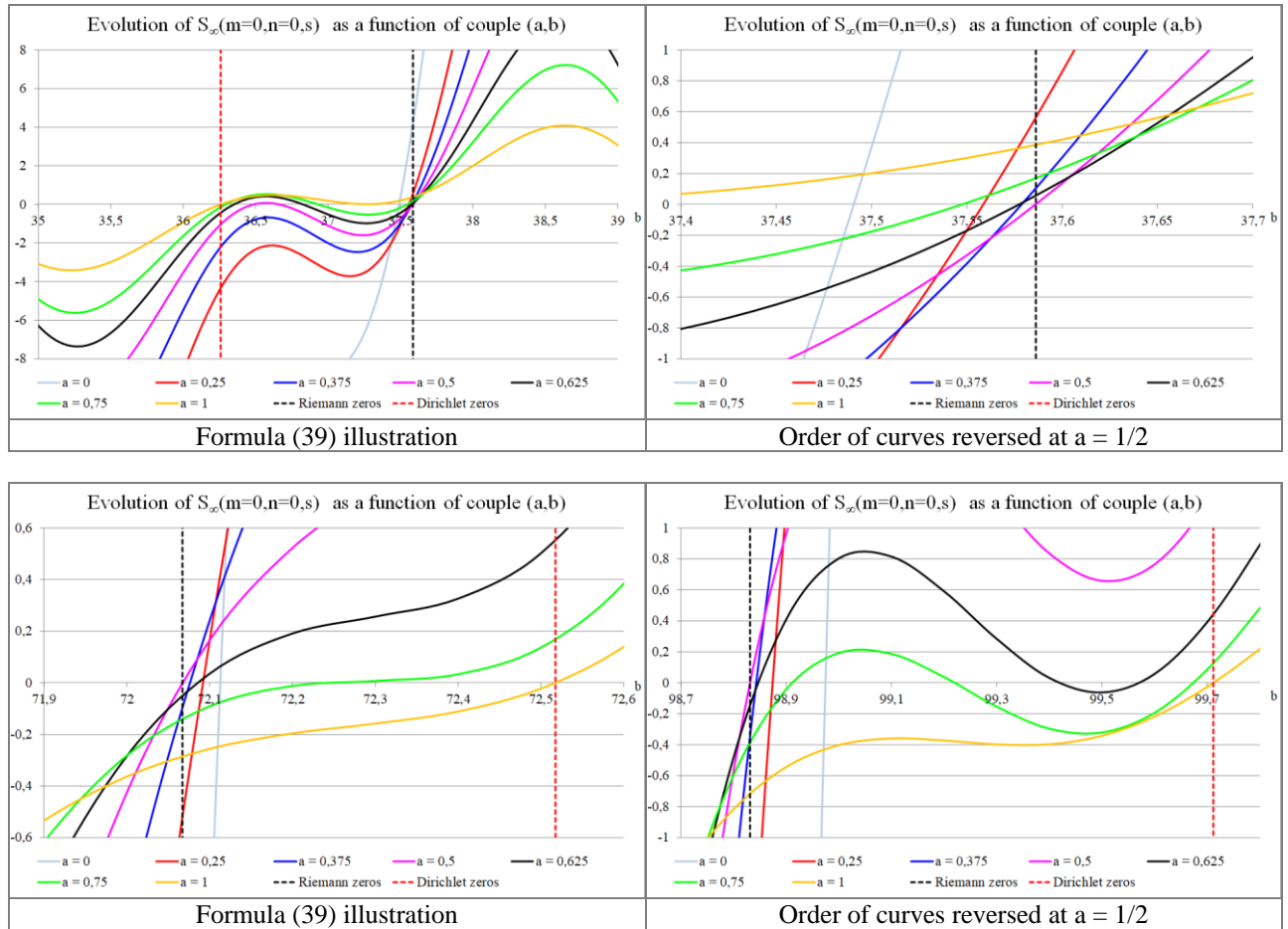
Let us have (a,b) corresponding to a Riemann or Dirichlet zero, then :

$$S_{\infty}(0,0,s) = \sum_{i=1}^{\infty} \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot \text{Ln}(i/j) \cdot \sin(b \cdot \text{Ln}(i/j)) \cdot \text{or}(1,2) = 0 \quad (39)$$

The converse is false since the function $S_{\infty}(0,0,s)$ cancels also for the $T_{\infty}(s)$ maxima causing the appearance of an intruder every two cases (at least).

The curves below give the look of truncated functions.





Theorem 9

The $C_{\infty}(0, 1/2, s)$ function is negative (or null) in the immediate neighbourhood of a Riemann or Dirichlet zero (as well as at this zero).

$$C_{\infty}(0, 1/2, s \approx s_0) = \sum_{i=1}^{\infty} \sum_{j=1}^i (i \cdot j)^{-a} \cdot (-1)^{i+j} \cdot (\ln(i/j))^2 \cdot \cos(b \cdot \ln(i/j)) \cdot \text{or}(1, 2) \leq 0 \quad (40)$$

Proof

Using the relation (37), we get in the neighbourhood of a zero

$$T_{\infty}(s) = C_{\infty}(0, 0, s_0) - \epsilon \cdot S_{\infty}(0, 0, s_0) - \epsilon^2 \cdot C_{\infty}(0, 1/2, s_0) + O(\epsilon^2) \quad (41)$$

As $T_{\infty}(s)$ is a square, the first two terms $C_{\infty}(0, 0, s_0)$ and $S_{\infty}(0, 0, s_0)$ being null (by construction for the first term, by theorem 8 for the second term), the third term $C_{\infty}(0, 1/2, s_0)$ is necessarily of negative sign, possibly zero, because of the ϵ^2 square. Equality is certainly strict but this point is not proven here.

8. Global order of curves.

This paragraph has nothing essential but allows understanding the evolutions of the curves. For the moment, we are interested in curves near the zeros (Riemann or Dirichlet). We now focus on the evolution of these away from these positions.

8.1. Order of $T_{\infty}(s)$ curves.

Specifically, we seek to assess the order of the curves $T_{\infty}(s)$ on all of the critical strip, and beyond to be complete, remaining at constant $b = b_0$.

He had previously

$$T_{\infty}(s = a + i.b) = \sum_{i=1}^{\infty} \sum_{j=1}^i (i.j)^{-a} \cdot (-1)^{i+j} \cdot \cos(b \cdot \ln(i/j)) \cdot \text{or}(1,2) \quad (42)$$

Consider two coordinates (a_1, b_0) and (a_2, b_0) . The ratio of the general terms $(i.j)^{-a} \cdot (-1)^{i+j} \cdot \cos(b \cdot \ln(i/j)) \cdot \text{or}(1,2)$ of (42), is equal to $(i.j)^{-(a_1 - a_2)}$, which is thus **independent of b**. This multiplicative function, independent of b, **acts exponentially** on each of the terms in the evolution of $T_{\infty}(s)$ with a.

$T_{\infty}(s)$ is a holomorphic function. Thus it is infinitely differentiable versus variable a or b (or s). Its derivative with respect to a, which is $-S_{\infty}(0,0,s)$, is null at a zero as we have demonstrated at theorem 8.

Moving away from a zero, the slope becomes steeper and that in an exponential manner. The zero acts as a **centre of a “homothety”**, in the geometric sense, this homothety being of a particular type (exponential and not linear). The term “homothety” reflects the involved phenomenon. It is an illustration and should not be taken in its flat literal sense.

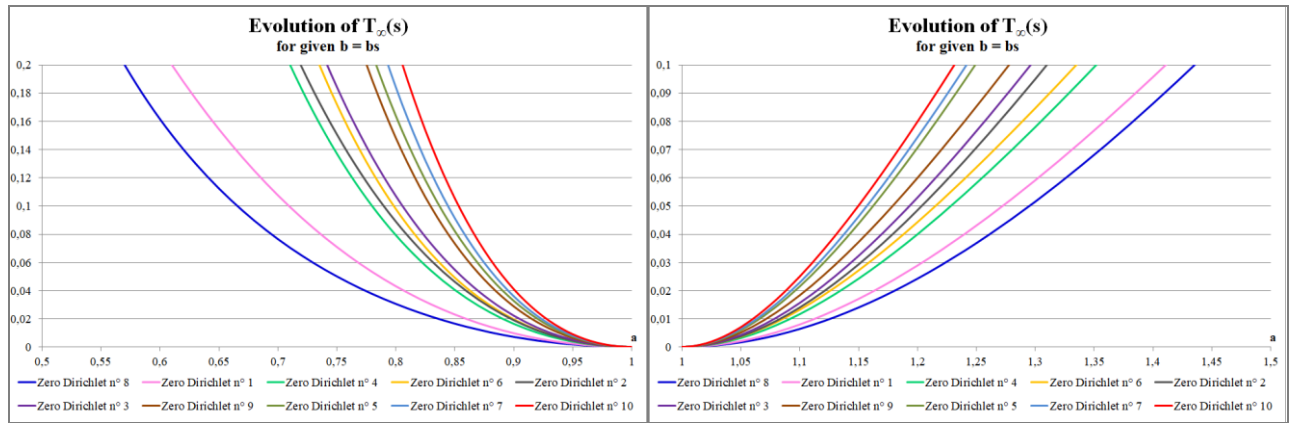
At a zero, by continuity, there is necessarily a range on which the order of curves is that of a (which reverses at the said point) with an exponential evolution. The **evolution** along the axis above a zero is **necessarily alike this origin off perturbations**. We call these perturbations the **effects**.

The graphs below illustrate this introduction.

8.1.1. Far away zeros. Separated effects.

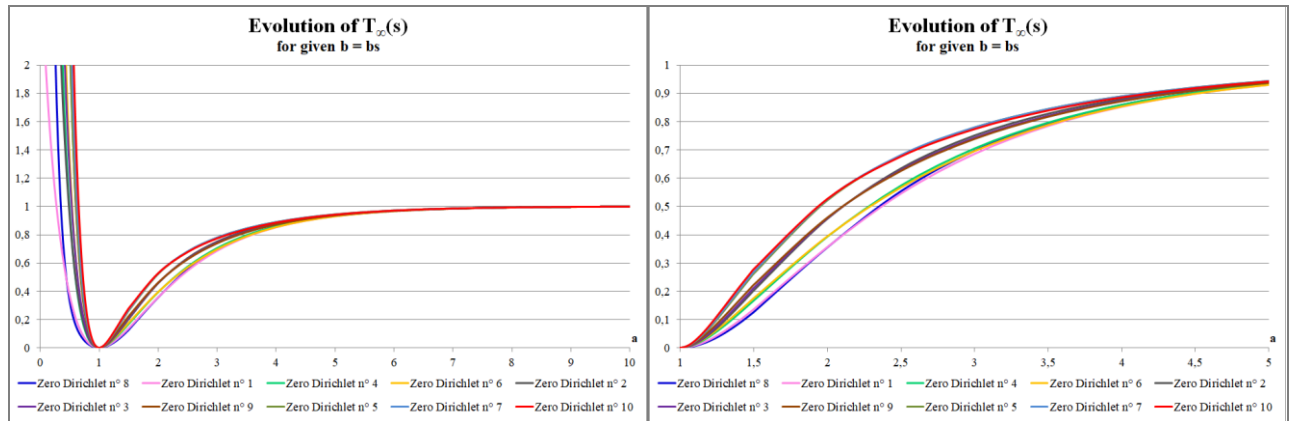
We first investigate the first ten Riemann zeros and the first ten of Dirichlet zeroes.

Dirichlet abscissas

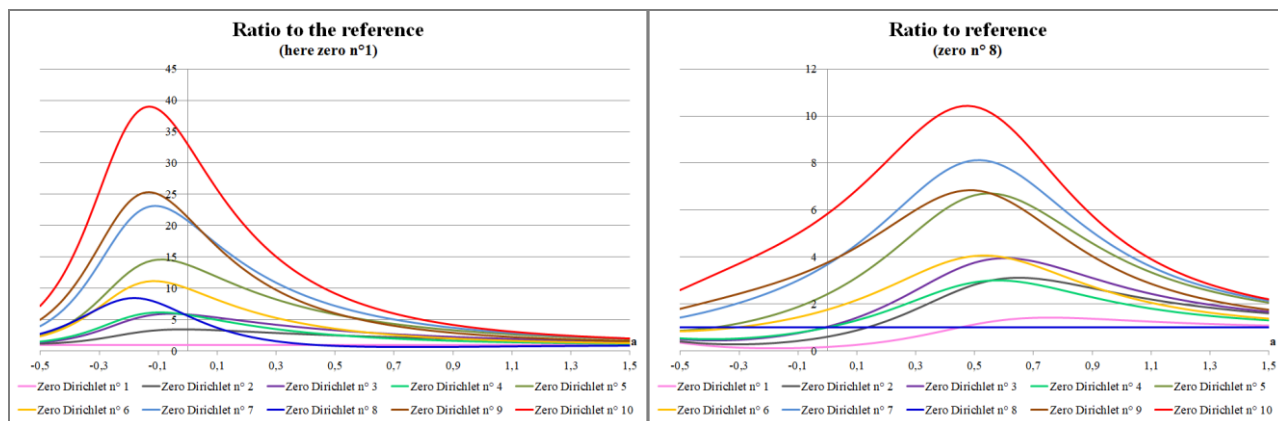


We note that the curves on both sides of $a = 1$ are in the same order. This is because $C_{\infty}(2,0,s)$ is the same when approaching from the right or left of $a = 1$ and the ratio is then held by homothety. However, this similarity is not eternal. Thus, the curves of zero n°2 and zero n°6 do switch in the range of graphics.

The curves on the left tend to infinity. Curves on right tend underneath towards 1 asymptotically.

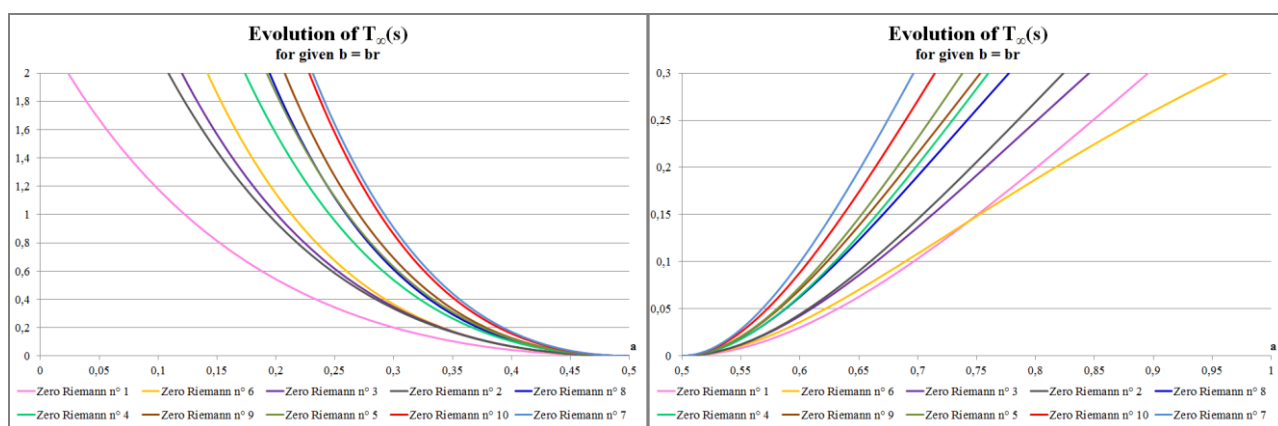


Drawing the ratios $T_{\infty}(s = a + i.bs) / T_{\infty}(s = a + i.bs_{\text{ref}})$, where bs_{ref} is a chosen reference for bs (here the first zero, then the eighth zero, we get the following paces :



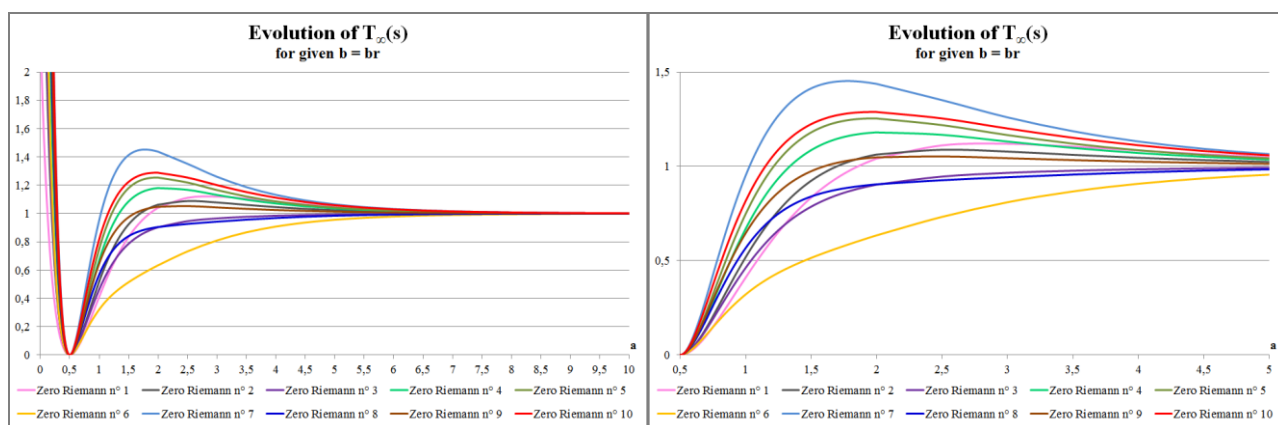
The peak, somewhat more on left that slightly distinguishes the curve corresponding to zero n°8 from the other zeros on the first chart, comes from the proximity with the 18th Riemann zero. When this zero is taken in reference, the other curves more or less have their maxima in the same region (with a shift to the right in this case).

Riemann abscissas

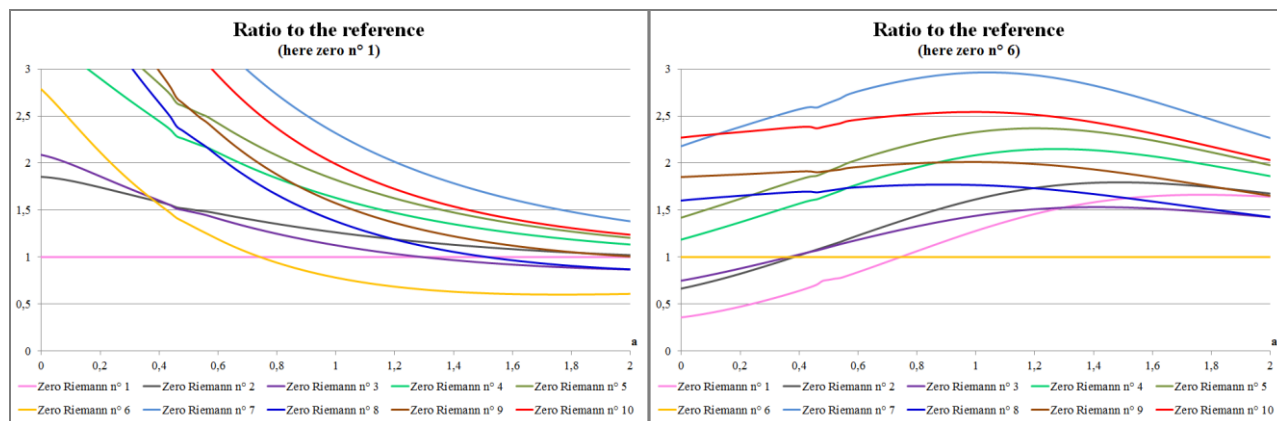


The look is similar to the Dirichlet zeroes curves with both sides' order more or less respected at remote distance than previously.

The curves on the left tend to infinity. Curves right tend towards 1 asymptotically downwards or upwards.



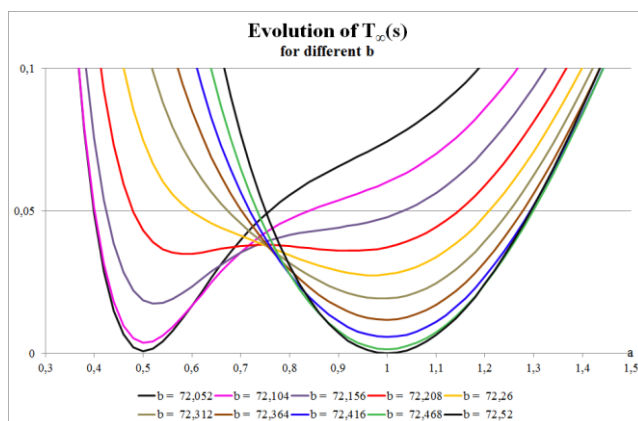
The look is similar to curves for Dirichlet zeroes with both sides' order, less respected at remote distance as previously.



Here, it is zero n°6 that offsets the maxima of curves to the right. It is close (relatively) of the 4th Dirichlet zero. The ratios at abscissas near $a = 0.5$ and $a = 1$ are obtained by unrefined smoothing.

Intermediary abscissas between Riemann and Dirichlet zeroes

In this case, we have two centres of homothety. The result is an additive effect on the look of the curves and the distortion may result in two minima de $T_{\infty}(s)$ between these two zeros (curve in red here). The abscissas between the Riemann zero n°18 ($br \approx 72,0671576744819$) and the Dirichlet zero n°8 ($bs \approx 72,5177622692351$) perfectly illustrates this point.



Intermediate summary

The first ten examples at Riemann and Dirichlet abscissas show curves with similar looks with the expected minimum at $a = 0.5$ for the first of them and $a = 1$ for the latter.

We call these "potential well" the **attractive effect of zeros**.

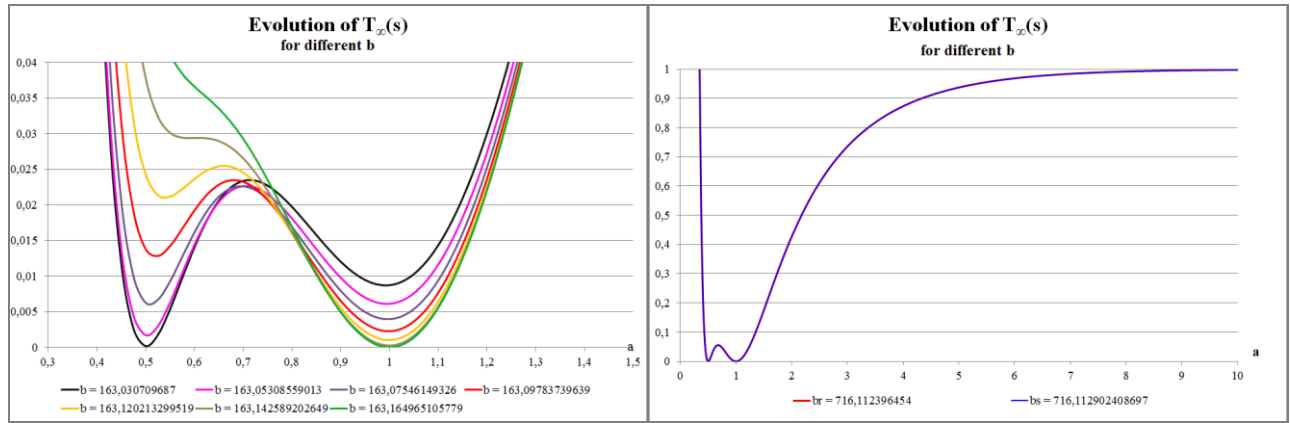
8.1.2. Nearby zeros. Conjugated effects.

We observe that, when two abscissas of distinct type are close, the two effects combine. The median curve at $b = 72,208$ shows this at best.

This combination of effects exists at any abscissa, intermediate or not. It is only a matter of degree of intensity. The examples below are eloquent.

The first example shows the evolution of effects for a range of values of b between $br \approx 163,030709687$ and $bs \approx 163,164965105779$. The combined effect is more pronounced here above $a = 1$ than above $a = 0,5$.

For the second example, the Riemann abscissas $br \approx 716,112396454$ and Dirichlet abscissa $bs \approx 716,112902408697$ are so close that curves are not distinct at the drawing's scale. Magnification at $a = 0.5$ and $a = 1$ would however show the expected order at these abscissas near the $y = 0$ axis. Under the conditions of the numeric application, the curve corresponding to $bs \approx 716,112902408697$ is located under the curve corresponding to $br \approx 716,112396454$ everywhere on the range $[0,10]$, except a small interval around $a = 0.5$ (see underneath data, remembering approximations due to truncations used for compilations).



a	$T_x(s=a+i.br)$	$T_x(s=a+i.bs)$	ΔT	a	$T_x(s=a+i.br)$	$T_x(s=a+i.bs)$	ΔT
0,4	0,27003244	0,26992652	0,00010592	0,5	0,0001912	0,00019382	-2,6212E-06
0,41	0,1997609	0,19968361	7,7292E-05	0,51	0,00113853	0,0011405	-1,9736E-06
0,42	0,14413682	0,14408217	5,4653E-05	0,52	0,00367473	0,0036754	-6,7821E-07
0,43	0,1007764	0,10073936	3,7043E-05	0,53	0,00733796	0,00733689	1,0716E-06
0,44	0,06762549	0,06760185	2,3631E-05	0,54	0,01174731	0,0117442	3,1162E-06
0,45	0,04291837	0,04290468	1,3697E-05	0,55	0,01659144	0,01658611	5,3248E-06
0,46	0,02514142	0,0251348	6,6227E-06	0,56	0,02161857	0,02161098	7,5919E-06
0,47	0,01300096	0,01299909	1,8753E-06	0,57	0,02662787	0,02661804	9,8334E-06
0,48	0,00539504	0,00539604	-9,9943E-07	0,58	0,03146183	0,03144984	1,1983E-05
0,49	0,00138852	0,00139091	-2,3888E-06	0,59	0,03599965	0,03598566	1,3992E-05
0,5	0,0001912	0,00019382	-2,6212E-06	0,6	0,04015157	0,04013575	1,5822E-05

Note : $\Delta T = T_x(s=a+i.br) - T_x(s=a+i.bs)$

General summary

We complete the previous remarks.

The look of $T_\infty(s)$ curves, function of a , is characterized by effects of two kinds :

- the effects related to the poles of the equation, that we can also appoint asymptotic effects,
- the effects related to the zeros of the equation, also called zero attractive effect.

The effects are all the more accentuated that actors are close (all zeros and poles interact). The effects are locally exponential (from the fact that a is an exponent) so that trends, once begun, are strong.

The result of this is, following a from $-\infty$ to $+\infty$:

- a brutal decrease from infinity before abscissa $a_r = 0.5$ (assuming the Riemann hypothesis)
- a potential well around this abscissa a_r , if one has $s = a_r + i.b$ with b sufficiently close to a_{br} (imaginary value of a Riemann zero), the axis $y = 0$ being reached if $s = s_r = a_r + i.br$ where br is one zero Riemann,
- a potential well around $a_s = 1$, if $s = 1 + i.b$ with b sufficiently close to $a_{bs} = 2k.\pi/\ln(2)$ (a Dirichlet zero imaginary value), axis $y = 0$ being reached in case of equality
- a growth towards y -ordinate 1, possibly exceeded to return back to this axis if the last centre of homothety is strong enough to cause this temporary overflow,
- an asymptotic branch $y = 1$, quickly reached with great precision.

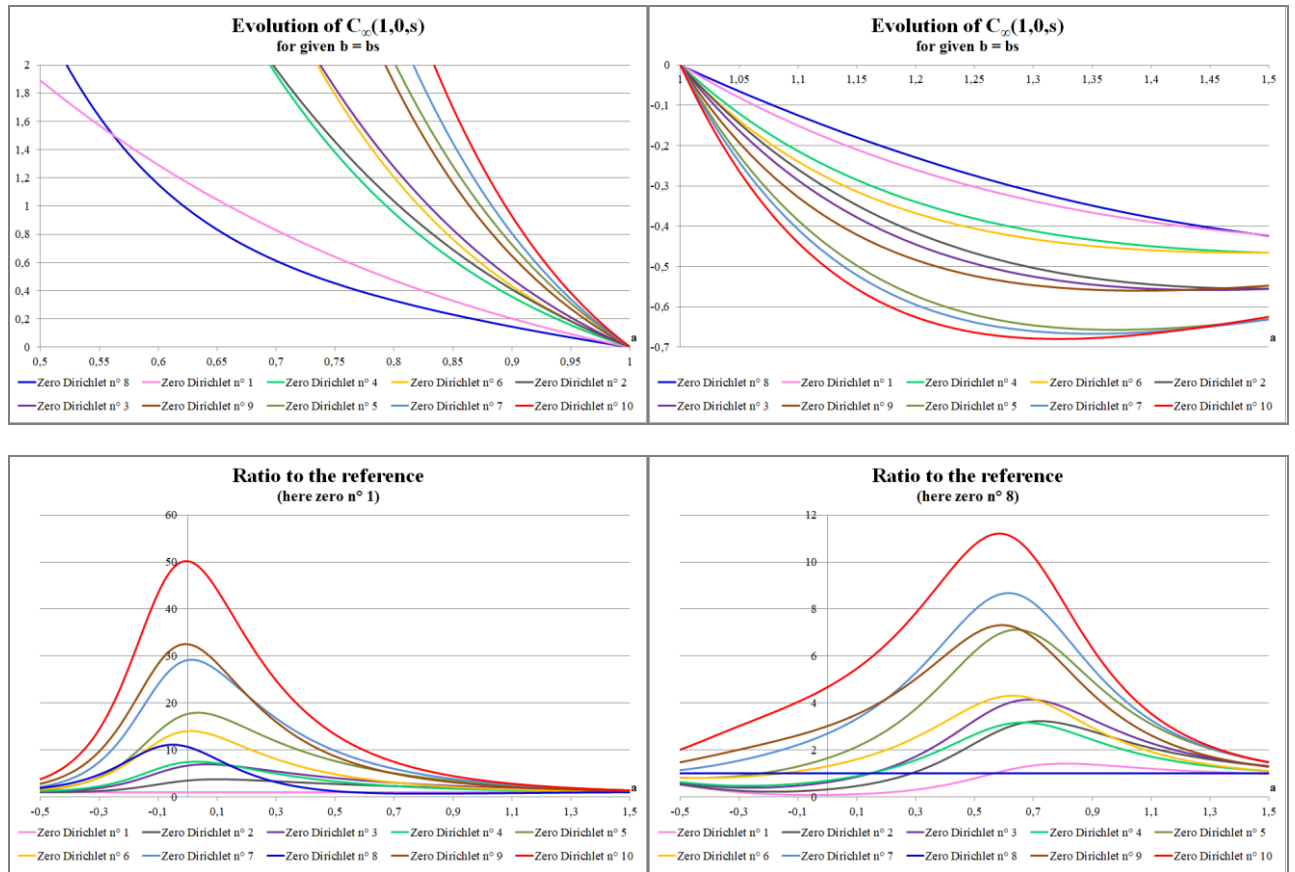
The purpose of all this numerical research is to find the worst scenarios within the a and b choices. The range of values $b \approx 716,112$ to $716,113$ turns out be such a case and deserves to be examined closely later on.

8.2.Order of curves $C_\infty(1,0,s)$.

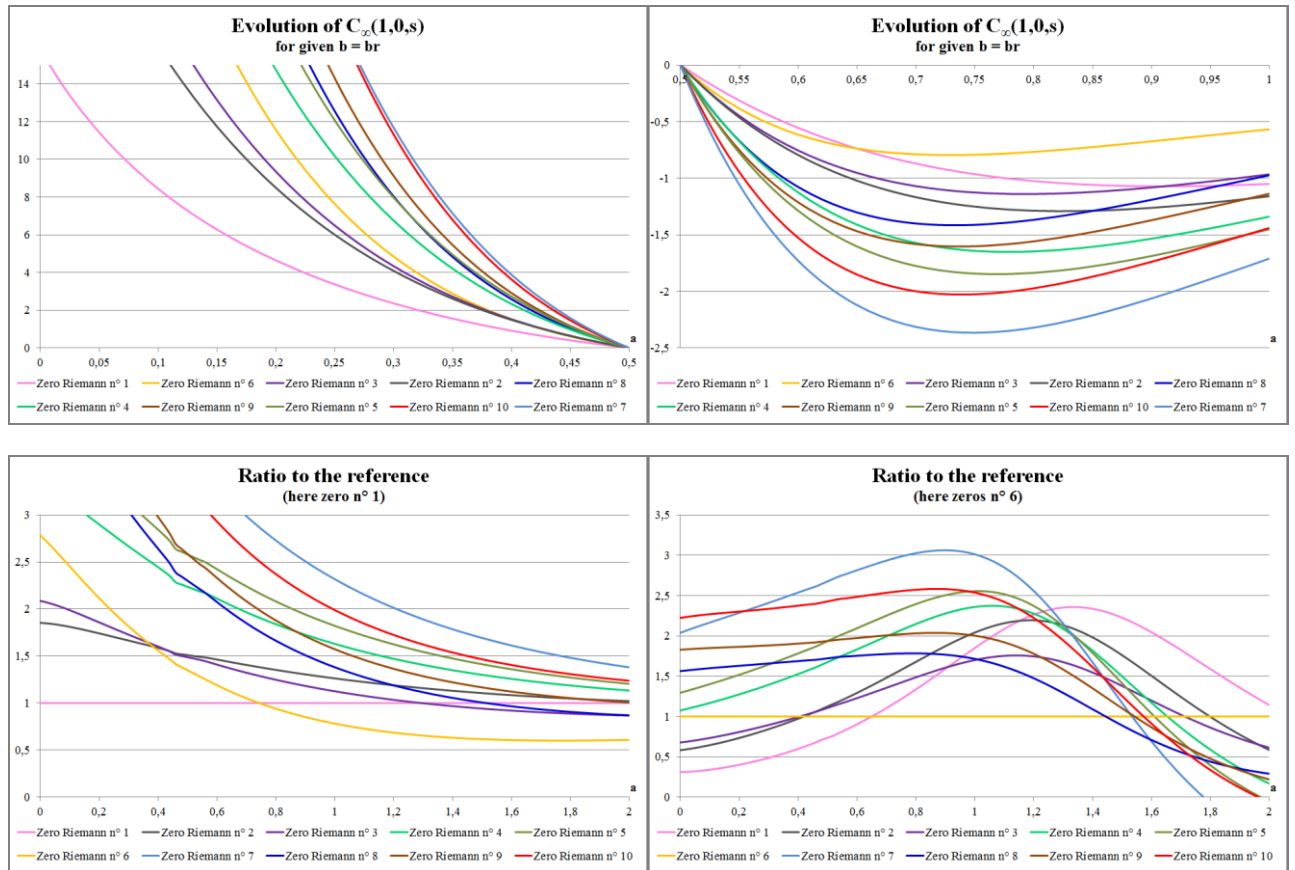
We proceed as previously.

The look of the curves is familiar after the previous paragraph.

Dirichlet abscissas



Riemann abscissas



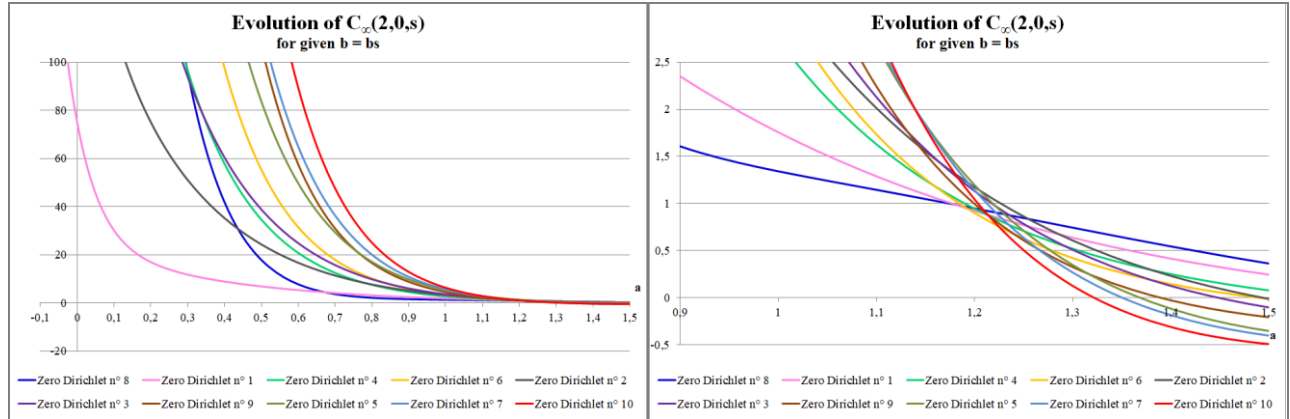
As $C_\infty(1,0,s)$ is not a square as was $T_\infty(s)$, the curves cross the $y = 0$ axis at $a = 0.5$ and $a = 1$ at the zeroes' abscissas.

8.3.Order of curves $C_{\infty}(2,0,s)$.

The series is the following

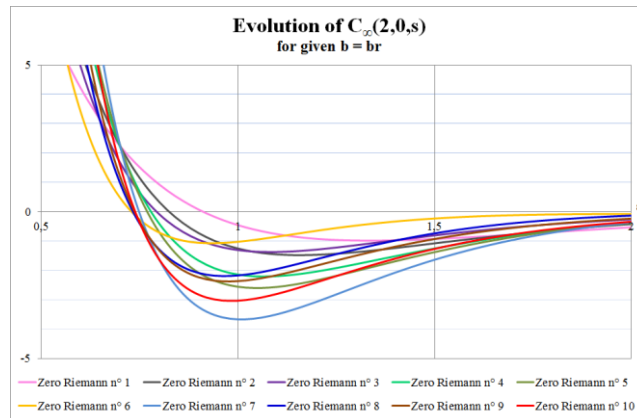
$$C_{\infty}(2,0,s) = \sum_{i=1}^{\infty} \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot \text{or}(1,2) \cdot \cos(b \cdot \text{Ln}(i/j)) \cdot (\text{Ln}(i,j))^2 > 0 \quad (43)$$

Dirichlet abscissas



The curves do not through the same point, the multiplicative ratio presented above has no more meaning here.

Riemann abscissas



Now, curves do no more cross axis $y = 0$ together.

Is there another equation extending, beyond the first derivative, such a clustering ?

9.The wall-through

This paragraph prevails over everything else (besides the Riemann hypothesis).

9.1.Remarkable infinite sums.

Let us note first that when the cosine is involved in our infinite sums, we have $\text{Ln}(i,j) = \ln(i) + \ln(j)$ factor, and that when the sinus occurs, we have $\text{Ln}(i/j) = \ln(i) - \ln(j)$ factor.

Let us summarize some of our results using the previous specified logarithm development.

Let us have (a,b) a Riemann or Dirichlet zero.

The referee equation for these zeros is

$$\sum_{i=1}^{\infty} \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot \cos(b \cdot \text{Ln}(i/j)) \cdot \text{or}(1,2) \cdot (\text{Ln}(i))^0 + (\text{Ln}(j))^0 = 0$$

and we have trivially

$$\sum_{i=1}^{\infty} \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot \sin(b \cdot \text{Ln}(i/j)) \cdot \text{or}(1,2) \cdot (\text{Ln}(i))^0 - (\text{Ln}(j))^0 = 0$$

From theorem 4

$$\sum_{i=1}^{\infty} \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot \cos(b \cdot \text{Ln}(i/j)) \cdot \text{or}(1,2) \cdot (\text{Ln}(i))^1 + (\text{Ln}(j))^1 = 0$$

From theorem 8

$$\sum_{i=1}^{\infty} \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot \sin(b \cdot \text{Ln}(i/j)) \cdot \text{or}(1,2) \cdot (\text{Ln}(i))^1 - (\text{Ln}(j))^1 = 0$$

From theorem 7

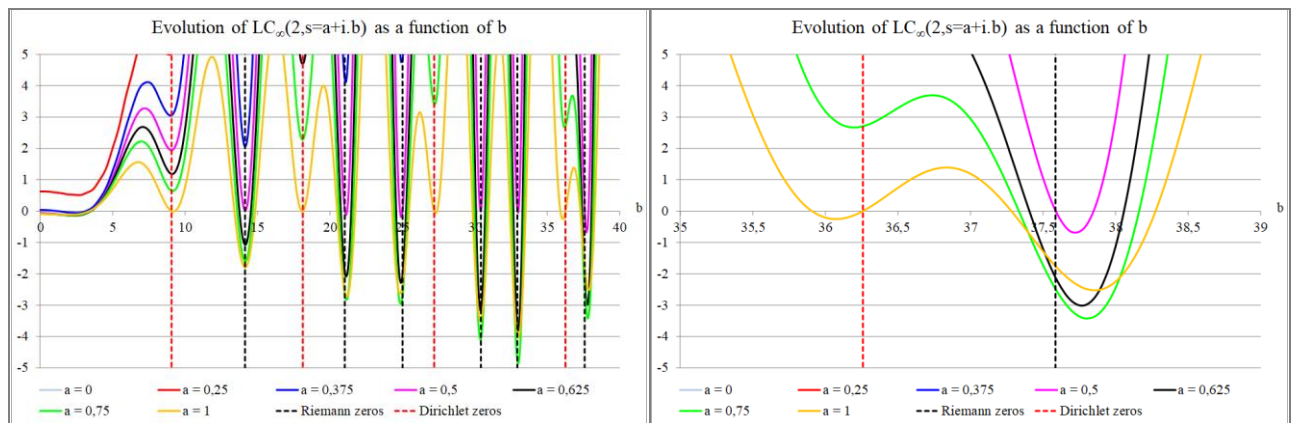
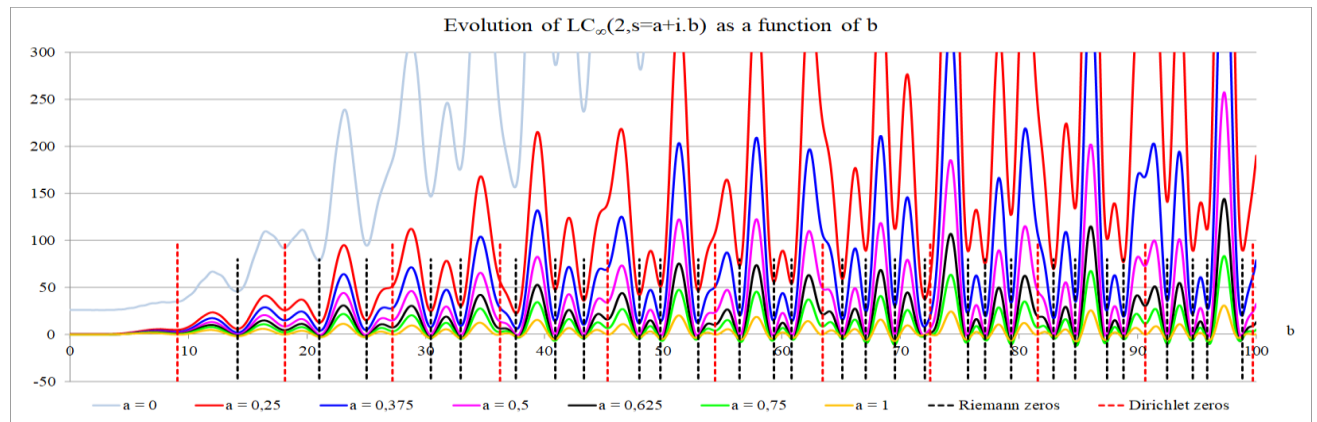
$$\sum_{i=1}^{\infty} \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot \sin(b \cdot \text{Ln}(i/j)) \cdot \text{or}(1,2) \cdot (\text{Ln}(i))^2 - (\text{Ln}(j))^2 = 0 \quad (44)$$

We propose to succeed :

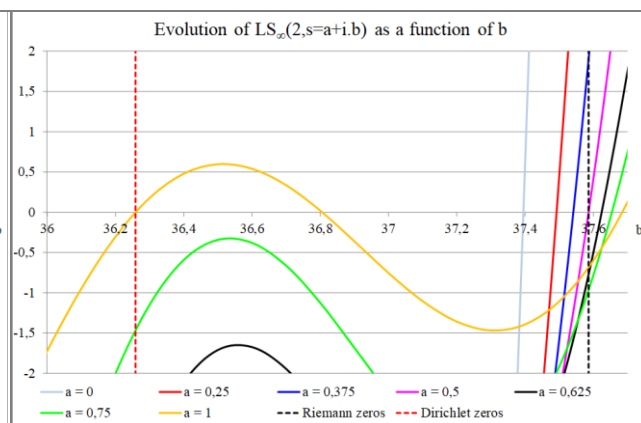
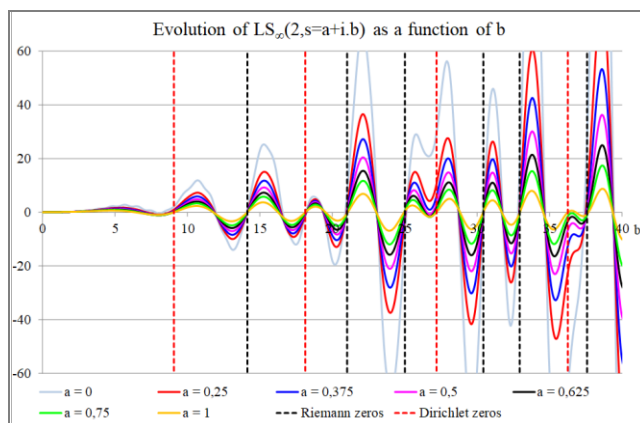
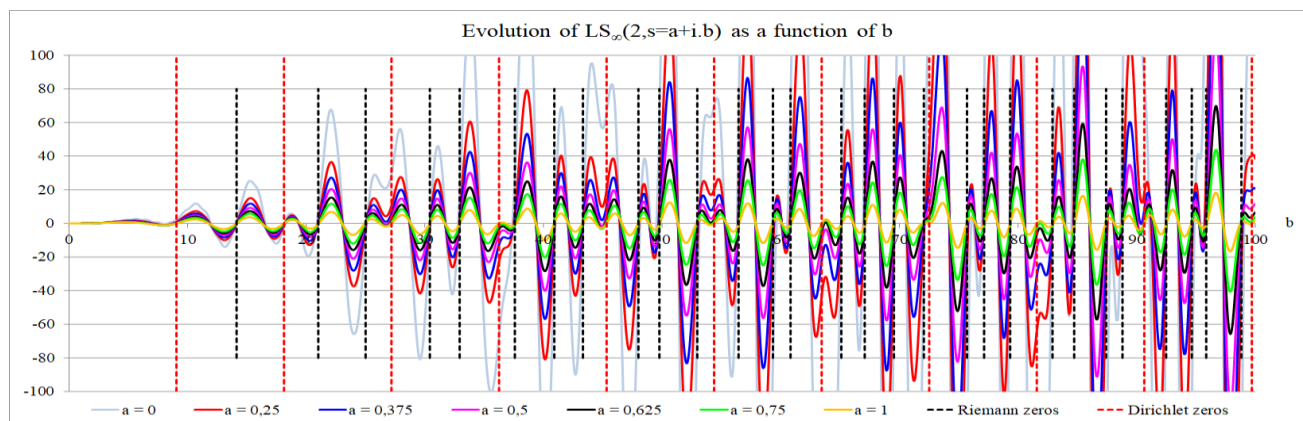
$$\sum_{i=1}^{\infty} \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot \cos(b \cdot \text{Ln}(i/j)) \cdot \text{or}(1,2) \cdot ((\text{Ln}(i))^2 + (\text{Ln}(j))^2) = 0 \quad (45)$$

We isolate this relation by circumstantial necessity. Indeed, it appears not as a natural derivative as its sister formula of theorem 7.

Let us first illustrate these two sisters' functions noted respectively $LS_{\infty}(2,s)$ and $LC_{\infty}(2,s)$.

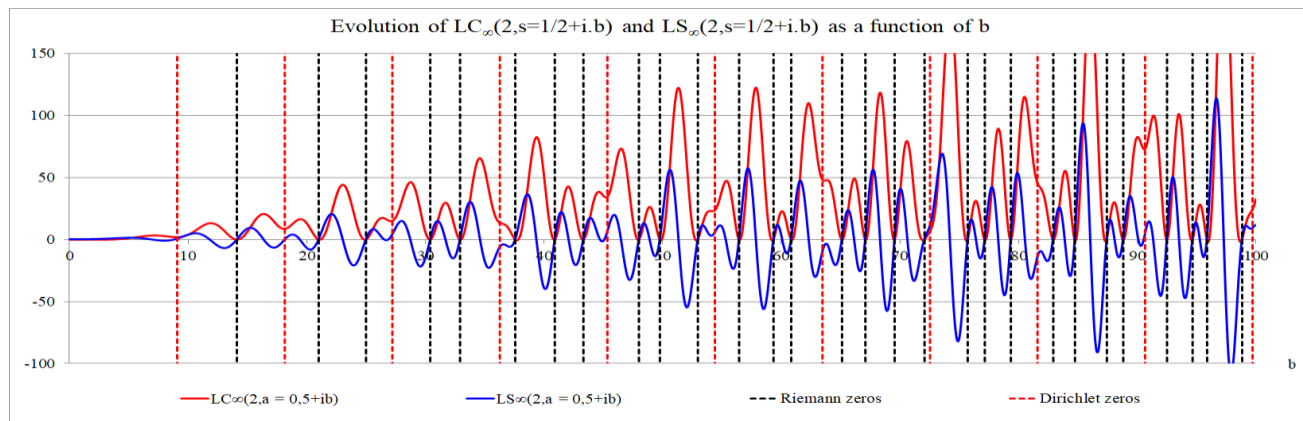


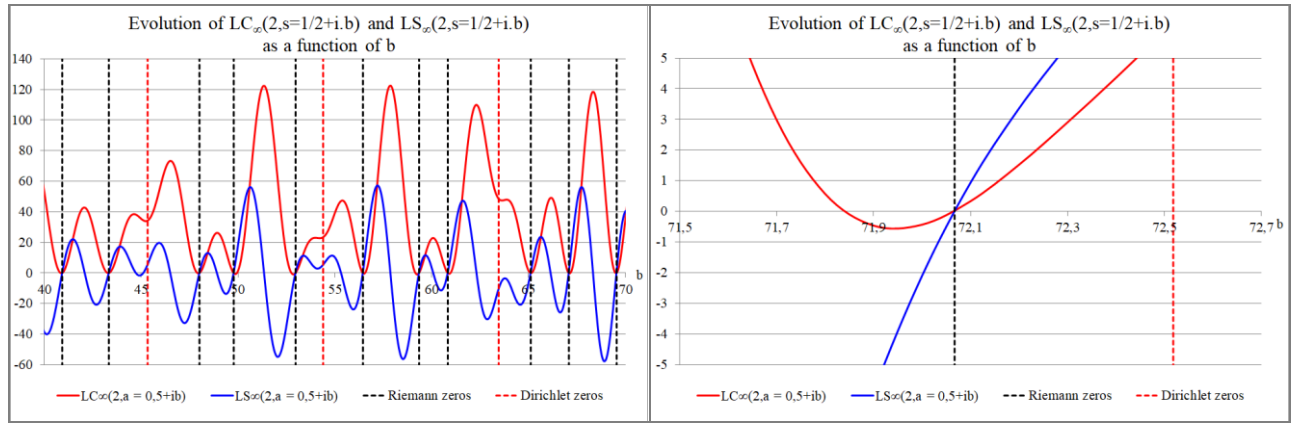
As hoped, intersections at Riemann and Dirichlet abscissas take place on $y = 0$ axis for $a = 1/2$ and $a = 1$ respectively. It is worth noting however that these are not extrema at these points.



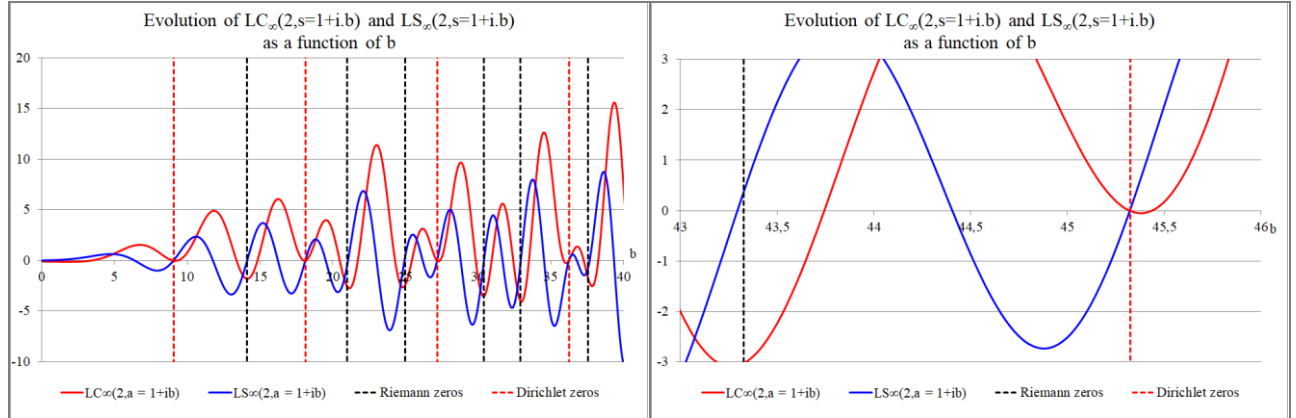
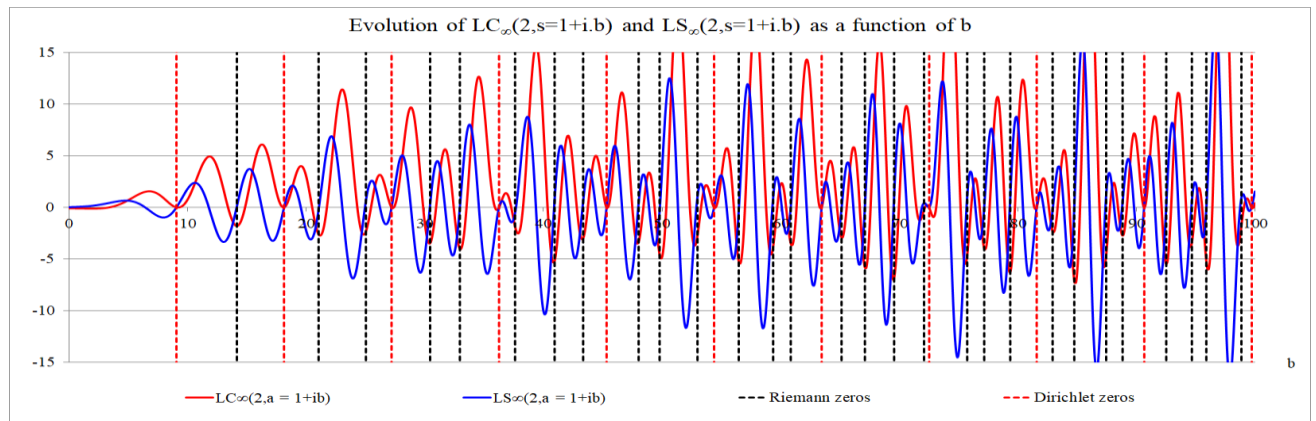
Again, the intersections at Riemann and Dirichlet abscissas take place on $y = 0$ axis without another remarkable fact.

We take interest below specifically to the values $a = 1/2$ and $a = 1$ by placing the two curves $LC_{\infty}(2,s)$ and $LS_{\infty}(2,s)$ on the same graphics. These views are reminiscent of the graphs on page (4).





The intersection with the $y = 0$ axis takes place at the Riemann abscissas for the two curves without so at Dirichlet abscissas.



The intersection with the $y = 0$ axis takes place at the Dirichlet abscissas for the two curves without so at Riemann abscissas.

Let us go back to the relation (45) and to

$$C_{\infty}(2,0,s) = \sum_{i=1}^{\infty} \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot \text{or}(1,2) \cdot \cos(b \cdot \ln(i/j)) \cdot (\ln(i) + \ln(j))^2 \quad (46)$$

This expression is reminiscent of $C_{\infty}(0,1/2,s)$ that can be found in (40) which was negative (or null) for of Riemann or Dirichlet zeroes.

$$C_{\infty}(0,1/2,s) = \sum_{i=1}^{\infty} \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot \text{or}(1,2) \cdot \cos(b \cdot \ln(i/j)) \cdot (\ln(i) - \ln(j))^2 \quad (47)$$

Then let us start from expression

$$LN2_{\infty}(s) = \left(\sum_{m=1}^{\infty} m^{-a} \cdot (-1)^{m-1} \cdot \cos(b \cdot \ln(m)) \cdot \ln(m) \right)^2 + \left(\sum_{m=1}^{\infty} m^{-a} \cdot (-1)^{m-1} \cdot \sin(b \cdot \ln(m)) \cdot \ln(m) \right)^2 \quad (48)$$

As the sum of two squares, it is necessarily positive or null. Developing and grouping the terms as we did in (22), we get :

$$LN2_{\infty}(s) = \sum_{i=1}^{\infty} \sum_{j=1}^i (i \cdot j)^{-a} \cdot (-1)^{i+j} \cdot \text{or}(1,2) \cdot \cos(b \cdot \ln(i/j)) \cdot \ln(i) \cdot \ln(j) \geq 0 \quad (49)$$

But $(\ln(i \cdot j))^2 = \ln(i)^2 + 2\ln(i) \cdot \ln(j) + \ln(j)^2$ and $(\ln(i/j))^2 = \ln(i)^2 - 2\ln(i) \cdot \ln(j) + \ln(j)^2$.
Thus

$$C_{\infty}(2,0,s) = 2 \cdot LN2_{\infty}(s) + \sum_{i=1}^{\infty} \sum_{j=1}^i (i \cdot j)^{-a} \cdot (-1)^{i+j} \cdot \text{or}(1,2) \cdot \cos(b \cdot \ln(i/j)) \cdot ((\ln(i))^2 + (\ln(j))^2) \quad (50)$$

Let us write

$$LC_{\infty}(2,s) = \sum_{i=1}^{\infty} \sum_{j=1}^i (i \cdot j)^{-a} \cdot (-1)^{i+j} \cdot \text{or}(1,2) \cdot \cos(b \cdot \ln(i/j)) \cdot ((\ln(i))^2 + (\ln(j))^2) \quad (51)$$

We then have to summarize

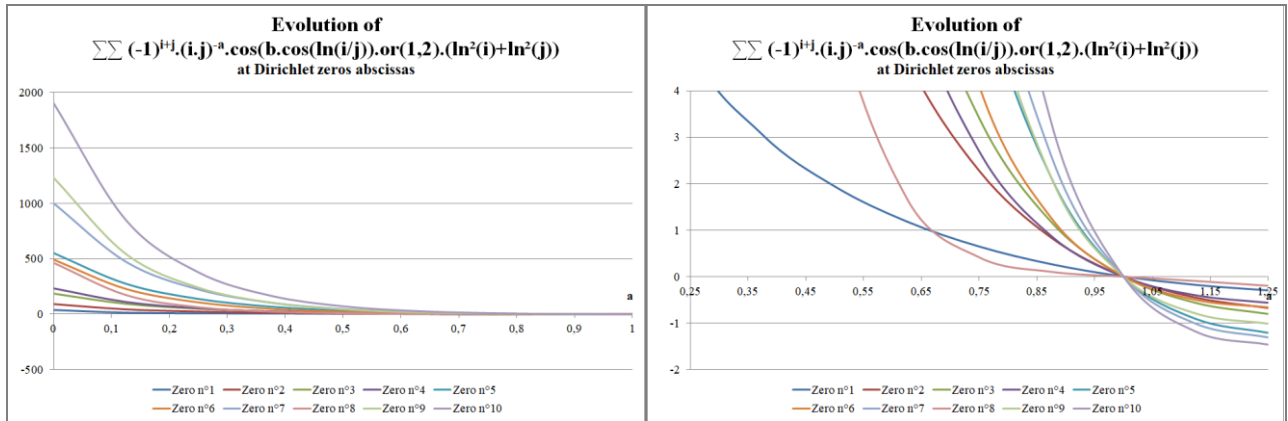
$$C_{\infty}(2,0,s) = 2 \cdot LN2_{\infty}(s) + LC_{\infty}(2,s) \quad (52)$$

$$-C_{\infty}(0,1/2,s) = 2 \cdot LN2_{\infty}(s) - LC_{\infty}(2,s) \quad (53)$$

$LN_{\infty}(2,s)$ is a square by construction and the positive or negative walks of $LC2_{\infty}(s)$ do not interfere with the positivity of $C_{\infty}(2,0,s)$ or $-C_{\infty}(0,1/2,s)$ at Riemann or Dirichlet zeroes.

Dirichlet abscissas

The chart below is a summary of the evolution of $LC_{\infty}(2,s)$ as a function of parameter a in the critical strip and beyond $a = 1$ (only really useful point here). The second chart is a simple zoom along the y axis of the first chart aimed particularly at the area around $a = 1$.

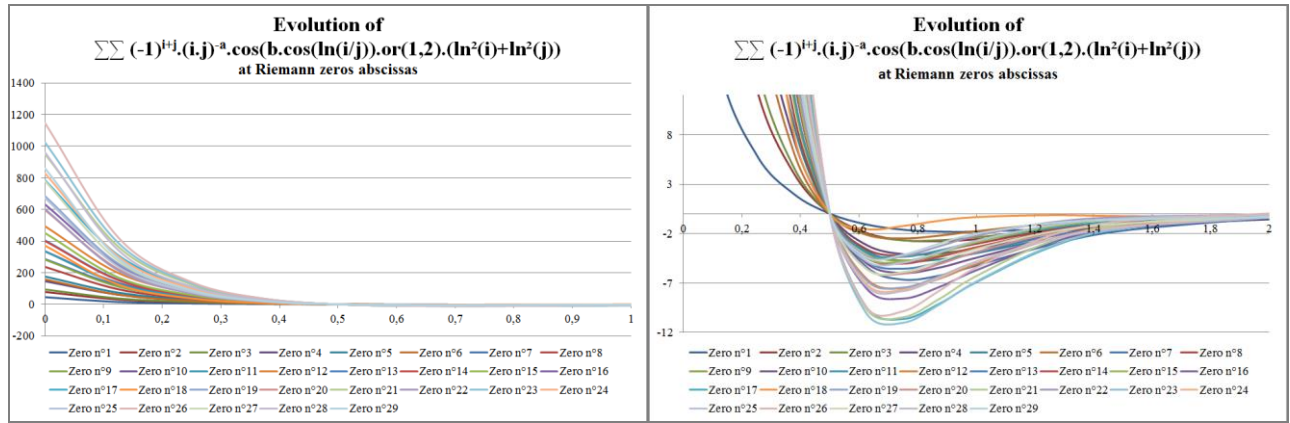


We observe that the function $LC_{\infty}(2,s)$ decreases in any interval $a = 0$ to $1,25$ represented here and crosses the $y = 0$ axis at abscissa $a = 1$ as announced. Our argument for the Riemann zeros is similar in all respects to that above.

Riemann abscissas

The chart below is a summary of the evolution of $LC_{\infty}(2,s = a + i \cdot b)$ as a function of a , around $a = 1/2$ (only really useful point here) for different b values corresponding to the imaginary values of the Riemann and Dirichlet zeroes. Again, the second chart is a simple zoom along the y axis of the first chart going beyond $a = 1/2$.

As previously, numerical applications show that the function crosses through the $y = 0$ axis at the Riemann abscissas, here for $a = 1/2$. It decreases in the interval $[0, 1/2]$, continues to decrease beyond, but increases then again.



9.2. The wall-through equations.

It is now time to find equations as general as possible with Riemann and Dirichlet zeroes as common solutions. One thinks immediately to the L functions (of all types) and in particular those associated with Dirichlet characters. We did not take this axis of research preferring a simpler way which provides us with a range of functions with much smaller requirements.

As a first step, we do only some observations from numerical examples, the theoretical part being postponed to the general case.

According to our investigations, there are at least two types of general equations.

9.2.1. The first type of general equations.

First generalisation

Let us come back then to our series of expressions. It is natural, the reader will agree, to generalize the relations to the powers 3, 4, etc. and then to intermediate powers.

Thus let us write :

$$LC_{\infty}(r,s) = \sum_{i=1}^{\infty} \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot \cos(b \cdot \ln(i/j)) \cdot \text{or}(1,2) \cdot ((\ln(i))^r + (\ln(j))^r) \quad (54)$$

$$LS_{\infty}(r,s) = \sum_{i=1}^{\infty} \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot \sin(b \cdot \ln(i/j)) \cdot \text{or}(1,2) \cdot ((\ln(i))^r - (\ln(j))^r) \quad (55)$$

$$LM_{\infty}(r,s,\varphi,\theta) = \sum_{i=1}^{\infty} \sum_{j=1}^i (i,j)^{-a} \cdot (-1)^{i+j} \cdot \cos(b \cdot \ln(i/j) + \varphi) \cdot \text{or}(1,2) \cdot ((\ln(i))^r + \theta \cdot (\ln(j))^r) \quad (56)$$

The function $LC_{\infty}(r,s)$ converges for $r = 0$ and $r = 1$ when $s = a + i \cdot b$ is in the critical strip. As, for all r and for all $\varepsilon > 0$, $\ln^r(x)/x^{\varepsilon} \rightarrow 0^+$ when $x \rightarrow +\infty$, we have still the convergence of $LC_{\infty}(r,s)$ for all r in the critical strip (having removed $\ln^r(1)$ which as a null contribution). The same holds for $LS_{\infty}(r,s)$.

Theorem 10

Let $s = (a, b)$ be a Riemann or Dirichlet zero.

For any real positive or null real number r

$$LC_{\infty}(r,s) = 0 \quad (57)$$

Theorem 11

Let $s = (a, b)$ be a Riemann or Dirichlet zero.

For any real positive or null real number r

$$LS_{\infty}(r,s) = 0 \quad (58)$$

Everything looks as if logarithms crossed the double sum thus still producing null products when $T_{\infty}(s)$ is null. One can see the same type of phenomenon with indefinite integrals (instead of infinite sums), what we called wall-through in

other articles, term that we have reused here. This was involving Diophantine equations with asymptotic branches where logarithms 'crossed' the integral symbol. In the same way here, the logarithms cross twice somehow the (double) sum sign.

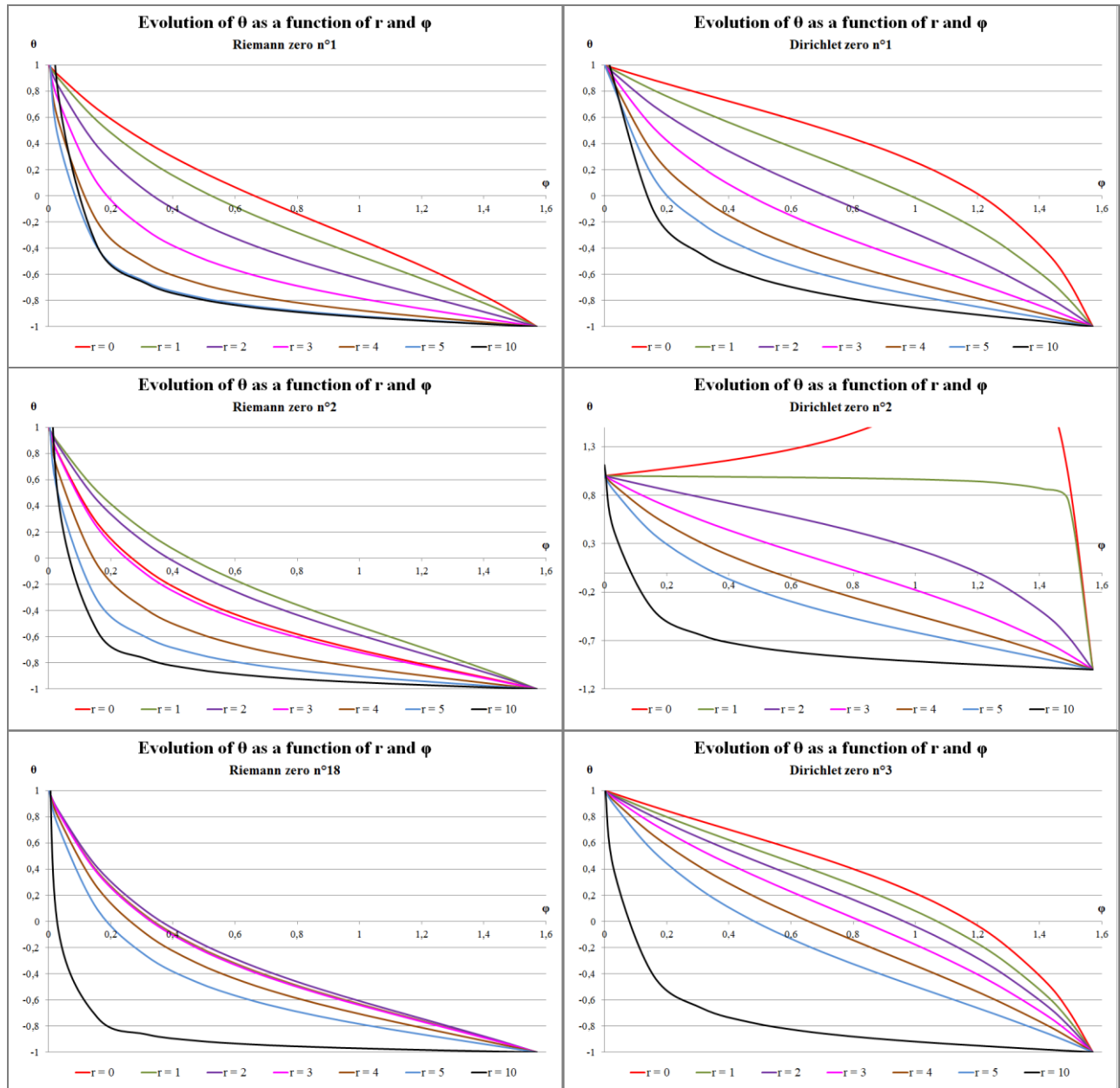
However, his wall-through is quite different as it applies not only to logarithms as we will see later on.

Theorem 12

For all real numbers $-\pi/2 \leq \varphi \leq 0$ (enabling crossing from cosine to sine) and $0 \leq r$, there exists θ such as :

$$LM_{\infty}(r,s,\varphi,\theta) = 0 \quad (59)$$

Being an intermediate equation between the previous two, by virtue of the continuity of functions, this relation is obvious and there is therefore nothing to prove more than the theorems (10) et (11). The study of variations of θ versus r and φ deserves certainly a longer look. However, numerical applications show a rather difficult to understand behaviour.



We see with some astonishment the possibility of excursion of θ outside the interval $[-1,1]$ (here for Dirichlet zero n°2), even if it is unusual.

The order of the curves seems to be respected in the case of Dirichlet zeroes (curve above the other for lower r and vice versa). Order seems also respected for Riemann zeros but with a round-trip (analogue to round-trips evoked to solve the Riemann hypothesis). The highest curve change from one zero to another ($r \approx 0$ for zero n°1, $r \approx 1$ for zero n°2, $r \approx 2$ for zero n°18).

It is to be noted that the connection at $(\varphi, \theta) = (0, 1)$ is inaccurate, when r is large (here $r = 10$). This follows again from the truncation of the functions.

Rewriting with complex numbers

Let us bring together the equations by writing

$$L_{\infty}(r, s) = LC_{\infty}(r, s) + i.LS_{\infty}(r, s) \quad (60)$$

Using the imaginary number i , we switch the indices i and j in the double sums for m and n . We get :

$$L_{\infty}(r, s) = \sum_{m=1}^{\infty} \sum_{n=1}^m (m.n)^{-a} . (-1)^{m+n} . \text{or}(1, 2) . (\exp(i.b.Ln(m/n)).ln^r(m)) + \exp(-i.b.Ln(m/n)).ln^r(n)) \quad (61)$$

This is also :

$$L_{\infty}(r, s) = \sum_{m=1}^{\infty} \sum_{n=1}^m (m.n)^{-a} . (-1)^{m+n} . \text{or}(1, 2) . ((m/n)^{i.b}.ln^r(m) + (m/n)^{-i.b}.ln^r(n)) \quad (62)$$

Let us note besides that the result remains true for negative r .

Generalization

The substitution $Ln^r(x) \rightarrow F(x)$ give a more general turn to the previous equation.

We have then :

$$FG1_{\infty}(s) = \sum_{m=1}^{\infty} \sum_{n=1}^m (m.n)^{-a} . (-1)^{m+n} . \text{or}(1, 2) . ((m/n)^{i.b}.F(m) + (m/n)^{-i.b}.F(n)) \quad (63)$$

Theorem 13

Let us have s a Riemann or Dirichlet zero. If $FG1_{\infty}(s)$ converge, then $FG1_{\infty}(s) = 0$.

Proof

Let us make the list of the terms including $F(r)$, r being an integer given in advance, when we develop the expression $FG1_{\infty}(s)$. This gives :

$$r^{-2a} . (-1)^{2r} . (r/r)^{i.b} . F(r) + r^{-2a} . (-1)^{2r} . (r/r)^{-i.b} . F(r) + 2 \sum_{\substack{n=1 \\ n \neq r}}^{\infty} (r.n)^{-a} . (-1)^{r+n} . (r/n)^{i.b} . F(r)$$

We distinguished the case $n = r$, but it is easy to reintroduce the term into the sum, so that :

$$2.F(r) . \sum_{n=1}^{\infty} (r.n)^{-a} . (-1)^{r+n} . (r/n)^{i.b}$$

which is also

$$2.(-1)^r . (1/r)^{a-ib} . F(r) . \sum_{n=1}^{\infty} (-1)^n . (1/n)^{a+i.b}$$

Getting all terms together, we have then :

$$FG1_{\infty}(s) = 2 \sum_{r=1}^{\infty} (-1)^r . (1/r)^{a-ib} . F(r) . \sum_{n=1}^{\infty} (-1)^n . (1/n)^{a+i.b} \quad (64)$$

The second sum is precisely the Dirichlet Eta function, which cancels at the Riemann and Dirichlet zeroes. One can then expect the same for $FG1_{\infty}(s)$. However, it is necessary here to consider the respective evolutions of the first and second

sums when r and n grow towards infinity. Our product was written somewhat rapidly without taking account of any relationship between r and n when we develop the initial double sum $FG1_{\infty}(s)$. It is better to write here :

$$FG1_{\infty}(s) = \lim_{r \rightarrow \infty} \sum_{m=1}^r (-1)^m \cdot (1/m)^{a+ib} \cdot F(m) \cdot \sum_{n=1}^r (-1)^n \cdot (1/n)^{a+ib} \quad (65)$$

Thus $FG1_{\infty}(s)$ will cancel at the Eta function zeros if and only if the first sum does not diverge too fast while the second sum converge. The divergence phenomena due to the first sum is instantaneous when $F(x)$ get too large as $F(x)$ is factor of an exponential term (that is $(1/m)^{a+ib}$). Hence the product $FG1_{\infty}(s)$ either cancels, either diverges. Thus, when we chose to say $FG1_{\infty}(s)$ cancels if $FG1_{\infty}(s)$ converges, we get free of an explicit determination of $F(x)$ to effectively realize this annulation.

We can nevertheless try to get such a determination. For that, let us consider the dominant terms of each of the sums and let us ignore the factors without effect on the module, that is r^{ib} and its opposite. This simplification gives however a speculative turn to what follows. We then have to compare $\sum (-1)^n \cdot (1/n)^a \cdot F(n)$ and $\sum (-1)^n \cdot (1/n)^a$. Thus, the divergence of the first sum will be faster than the convergence du second as soon as $F(x) \geq x^{2a}$ asymptotically. In particular, all the terms like $F(x) = \ln^r(x)$ will be illegible for convergence (and annulation) of the product.

Splitting of real and imaginary parts

Using $\cos(x) = (\exp(i.x) + \exp(-i.x))/2$ et $\sin(x) = (\exp(i.x) - \exp(-i.x))/2$, we get the corresponding real and imaginary parts :

$$FC1_{\infty}(s) = \sum_{m=1}^{\infty} \sum_{n=1}^m (m.n)^{-a} \cdot (-1)^{m+n} \cdot \text{or}(1,2) \cdot \cos(b \cdot \ln(m/n)) \cdot (F(m) + F(n)) \quad (66)$$

and

$$FS1_{\infty}(s) = \sum_{m=1}^{\infty} \sum_{n=1}^m (m.n)^{-a} \cdot (-1)^{m+n} \cdot \text{or}(1,2) \cdot \sin(b \cdot \ln(m/n)) \cdot (F(m) - F(n)) \quad (67)$$

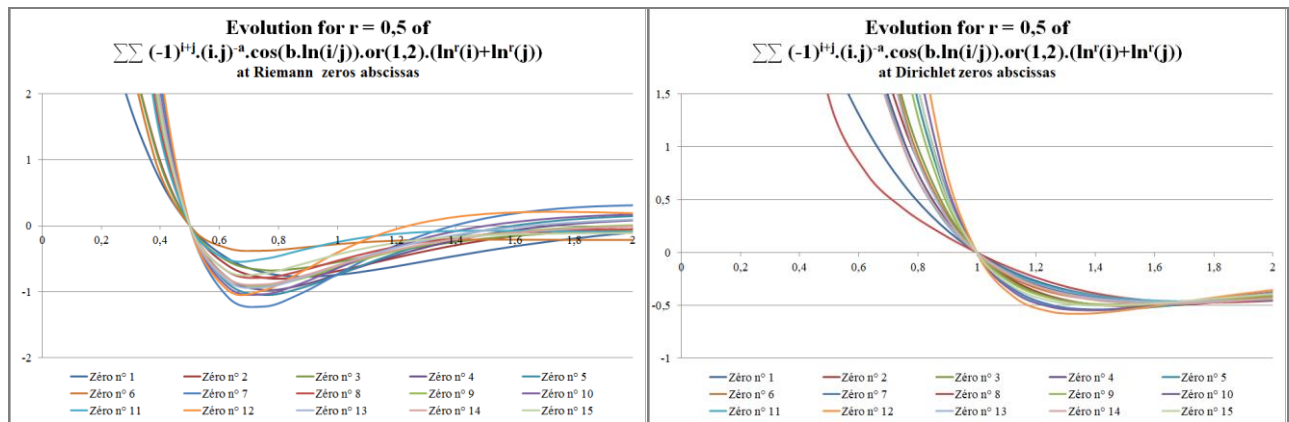
We get :

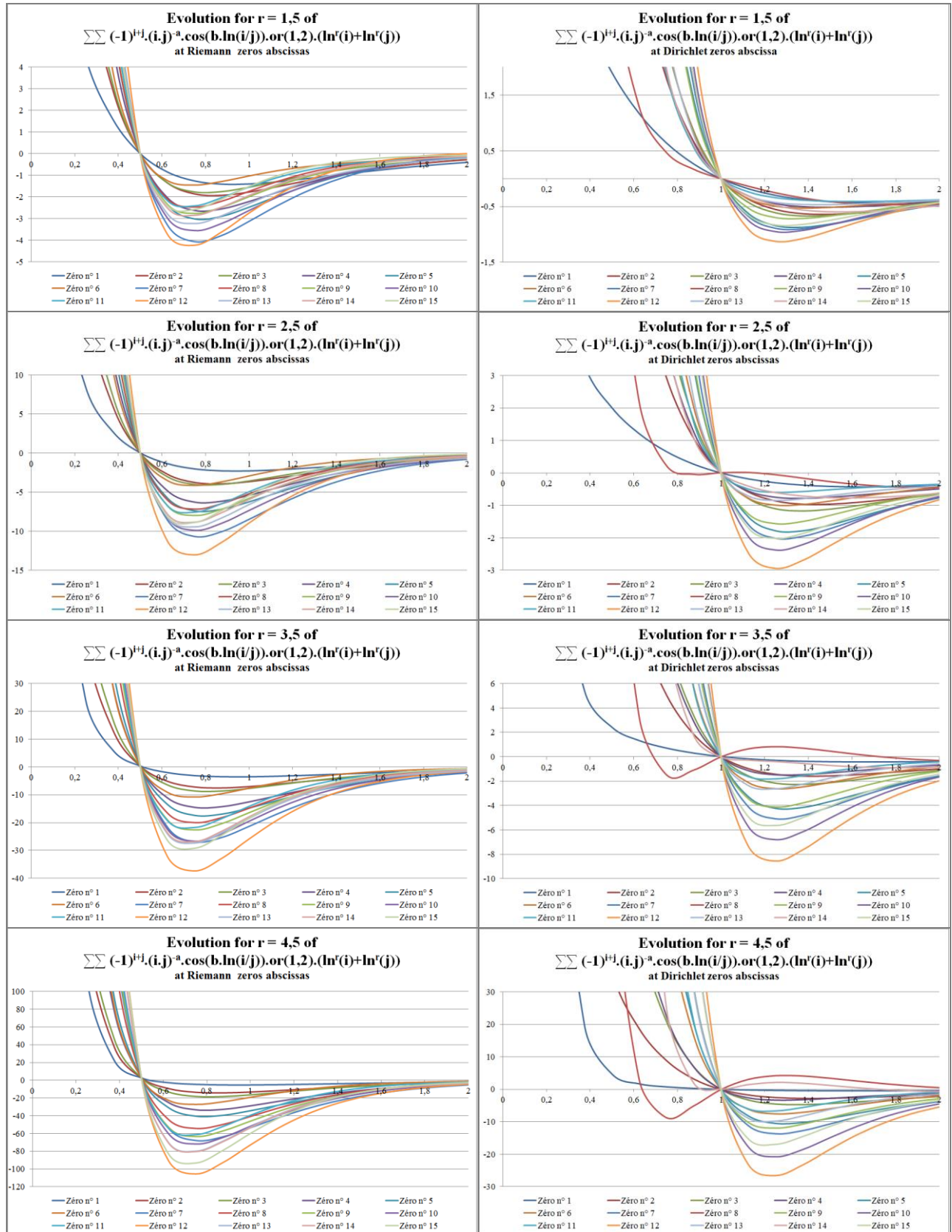
Theorem 14

Let us have s a Riemann or Dirichlet zero. If $FC_{\infty}(s)$ and $FS_{\infty}(s)$ converge, then $FC1_{\infty}(s) = 0$ and $FS1_{\infty}(s) = 0$ simultaneously.

Illustration of $LC_{\infty}(r,s)$ and $LS_{\infty}(r,s)$

We give below a sample of the variations of $LC_{\infty}(r,s)$ and $LS_{\infty}(r,s)$ as a function of a when this parameter varies in the interval $[0,2]$ for r values between 0.5 and 5 and for different b values corresponding to Riemann and Dirichlet zeroes imaginary values (thus the terminology 'Riemann or Dirichlet abscissas' and 'zeros n°').

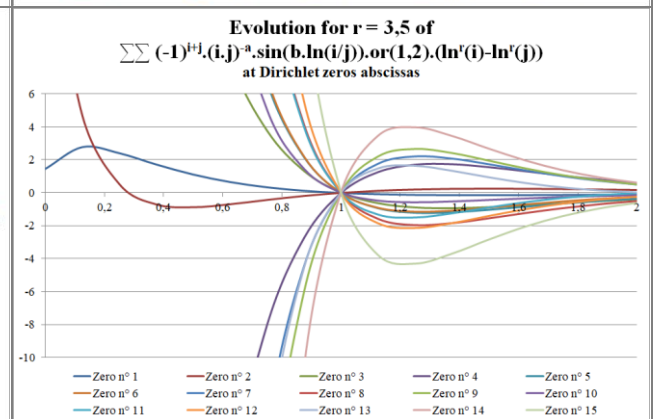
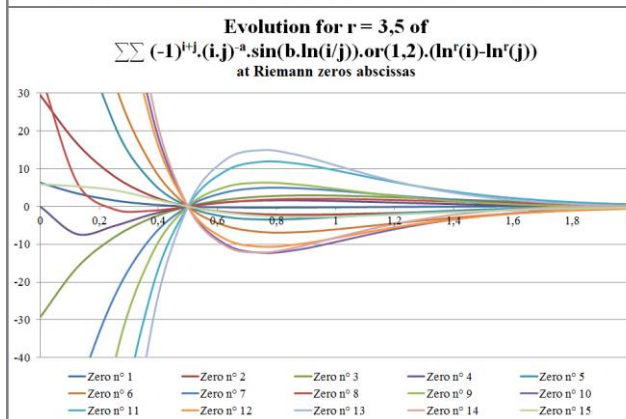
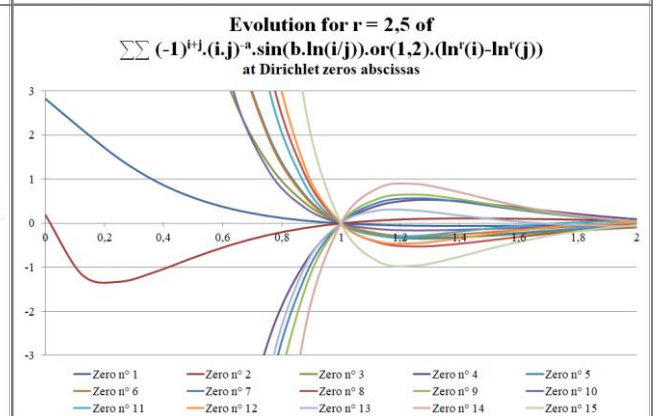
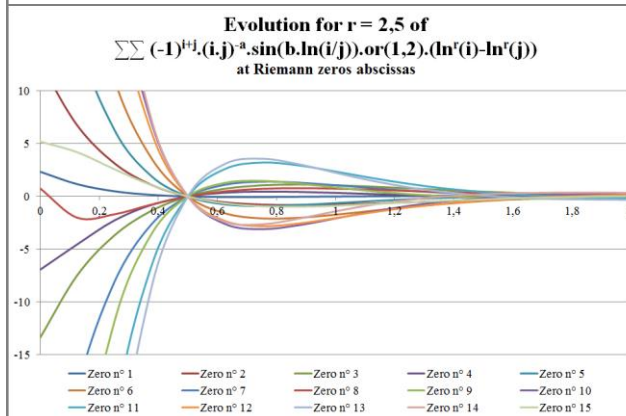
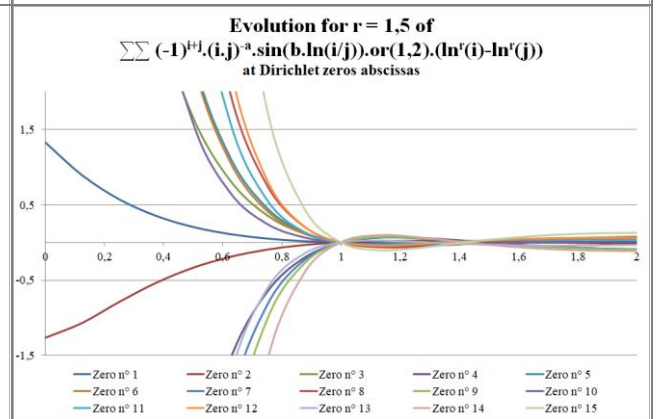
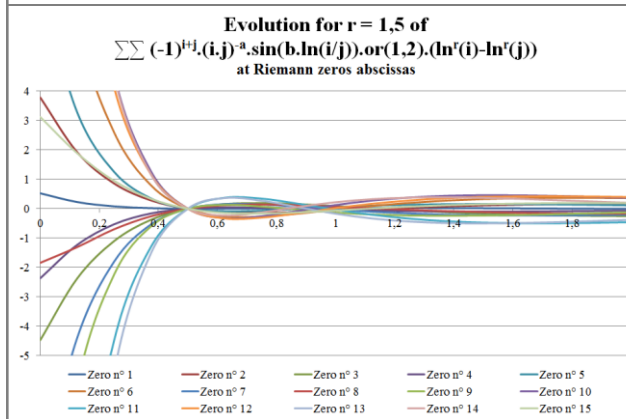
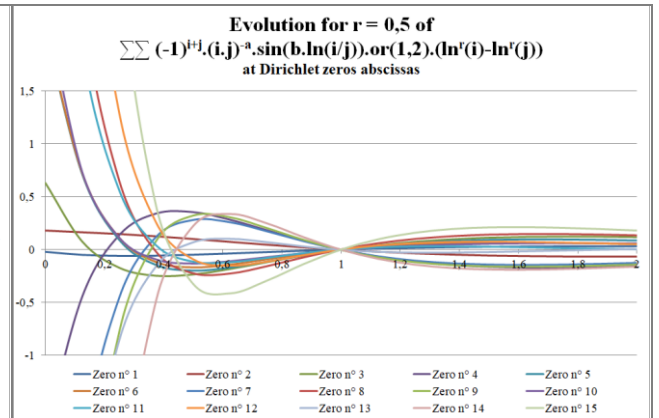
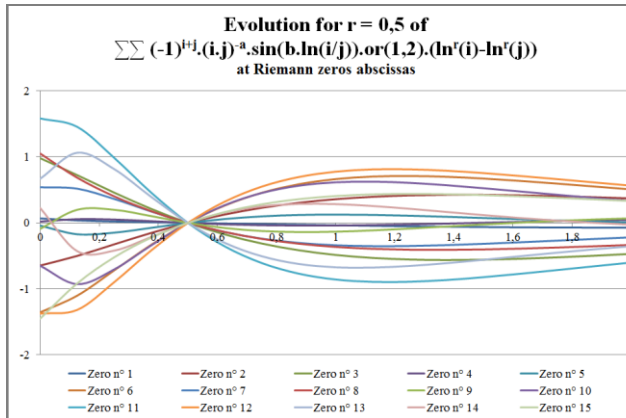


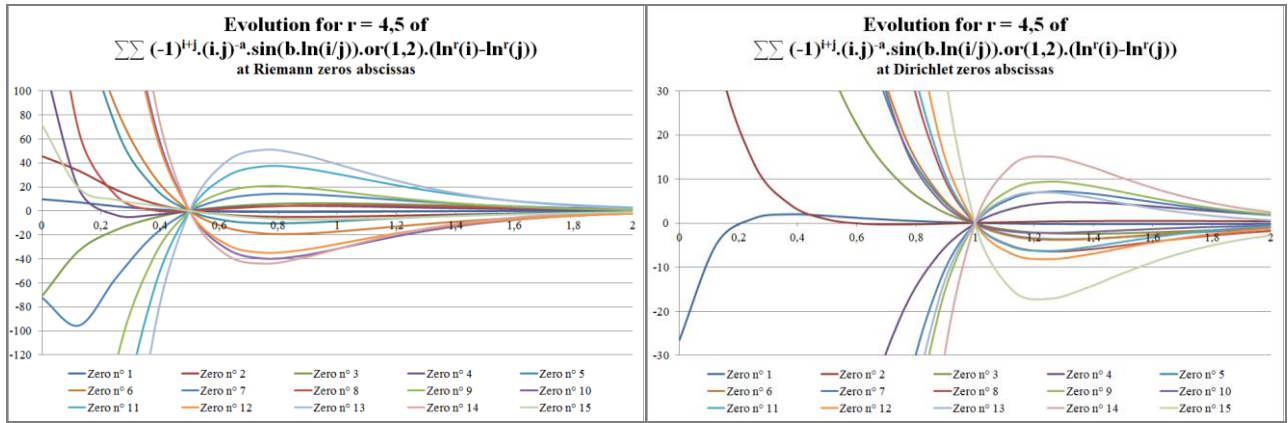


We can observe behaviour on the same pattern but with significant variations from one case to another. It seems difficult to give an a priori estimate of $LC_{\infty}(r,s)$ apart from those of Riemann or Dirichlet abscissas.

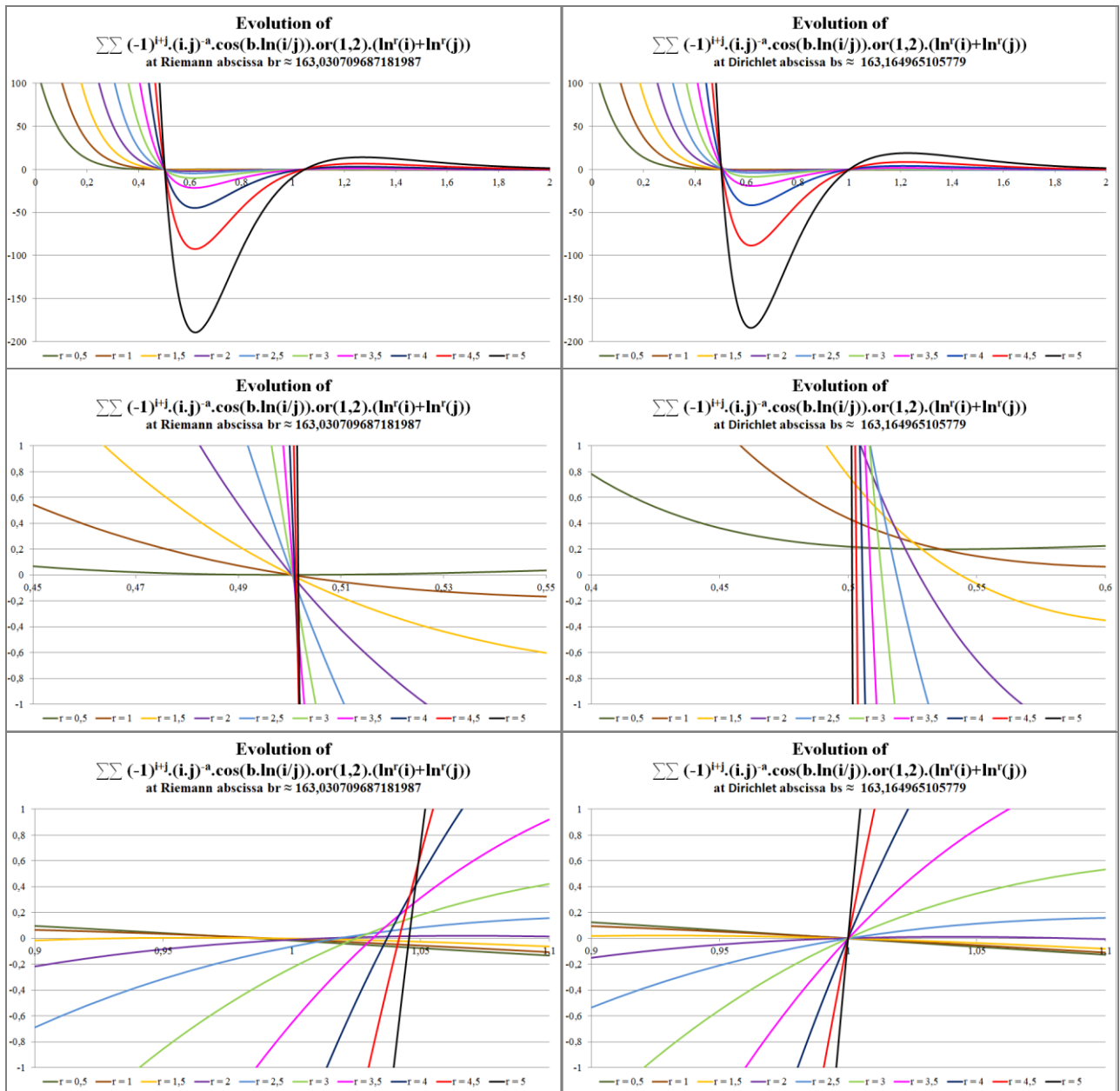
As graphics from truncated functions, the reader will not be surprised of vagueness at the intersection with the x-axis.

We note the attraction of the centre of homothety at $a = 0.5$ for the curve corresponding to the Dirichlet zero n°8 which is somewhat bullied by the pole $a = -\infty$, with more pronounced effect as r increases.

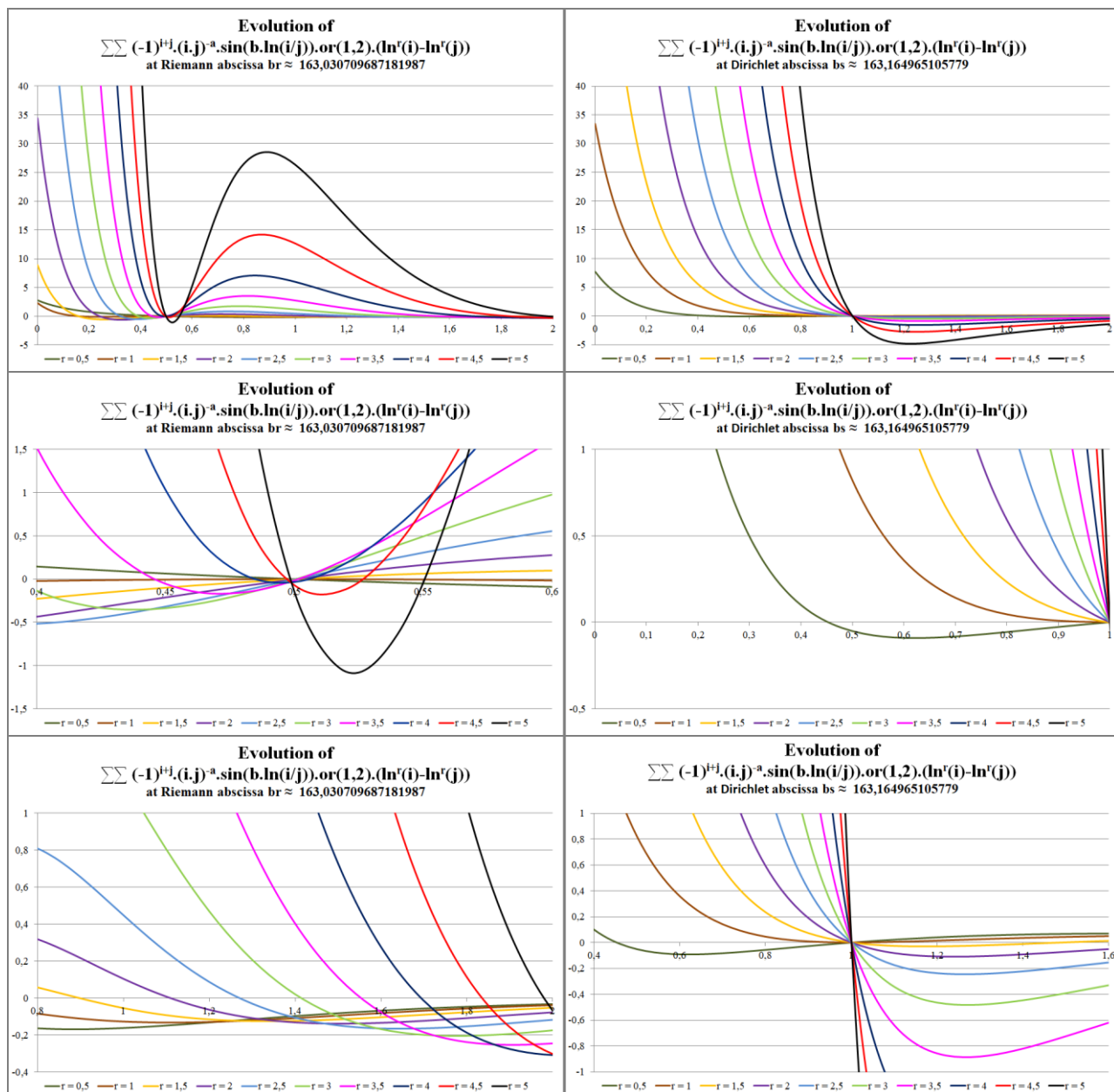




Let us check the curves' look for a case of relatively marked conjugated effects.

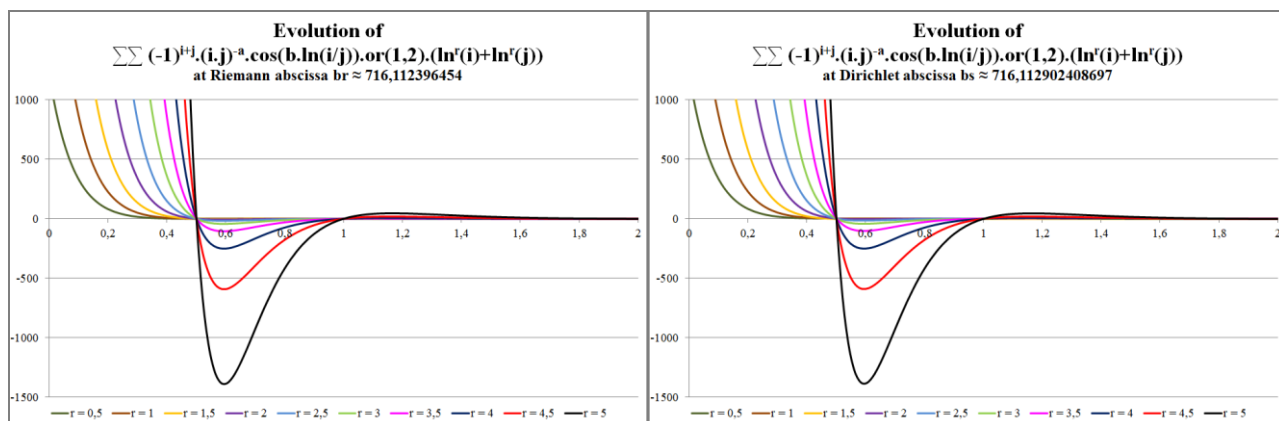


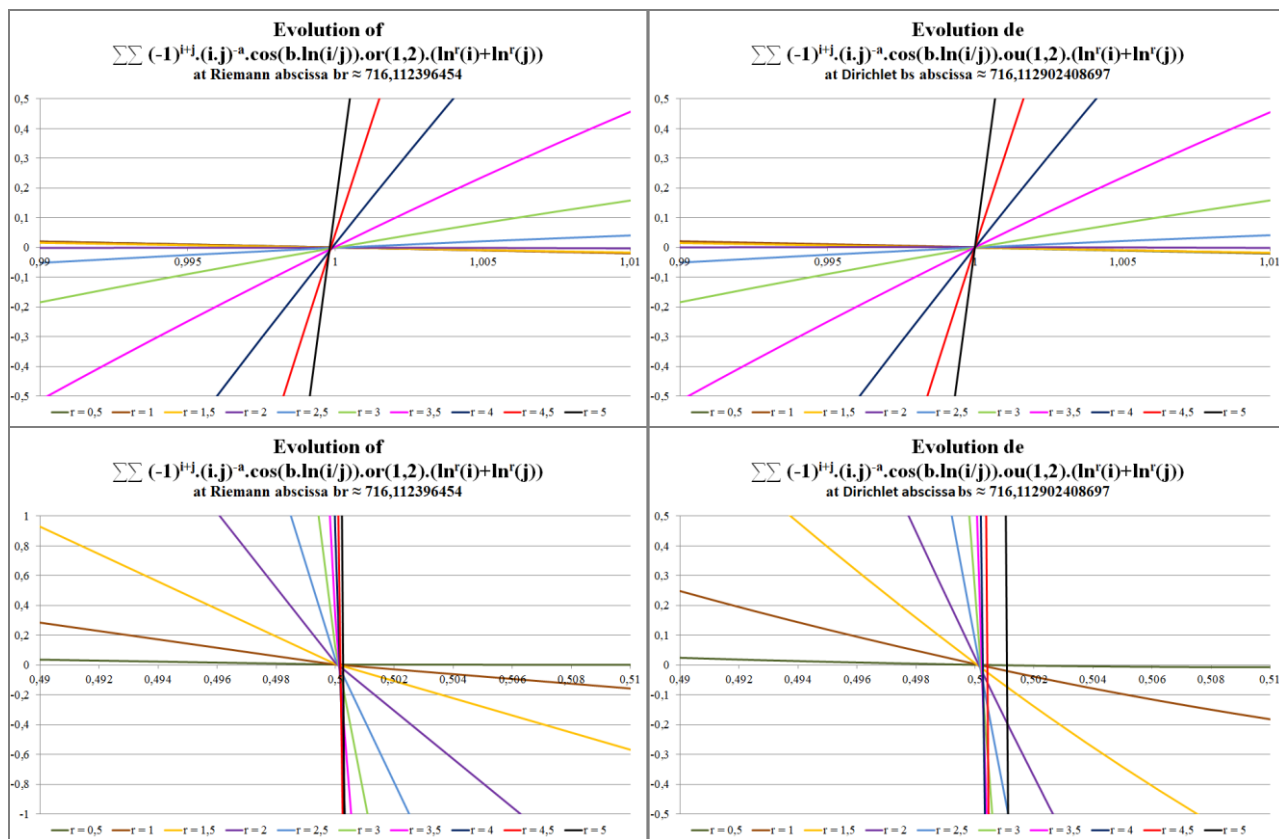
At remote distance, the cosine curves are very close.



Strangely enough here, the sine curves have differences between them much more marked than previous cosine curves, even remotely. This feature is certainly not a generality.

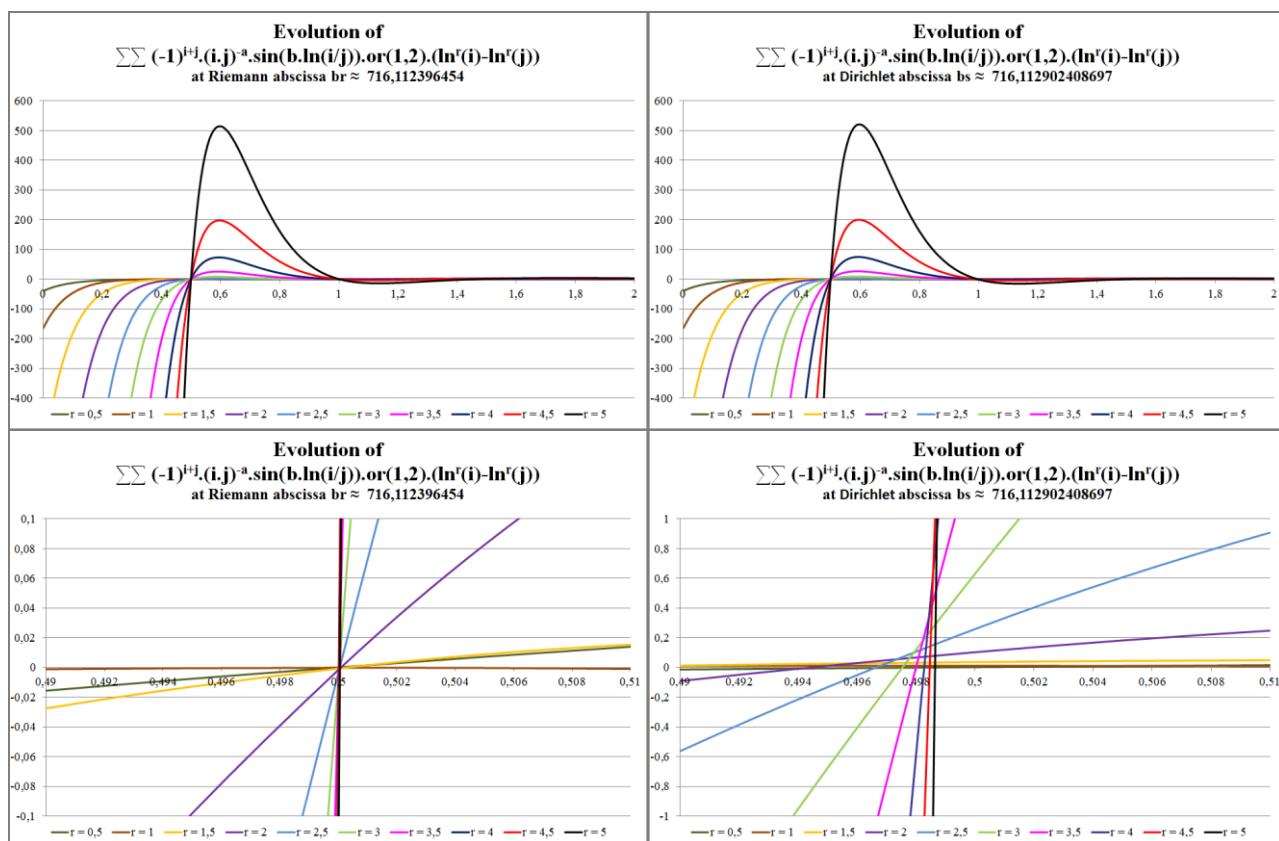
To finish with, let us have a look on curves for very marked conjugated effects. Note that this type of curves' look is not uncommon. It is even a general pattern around a Dirichlet zero of high number (anticipating later remarks).

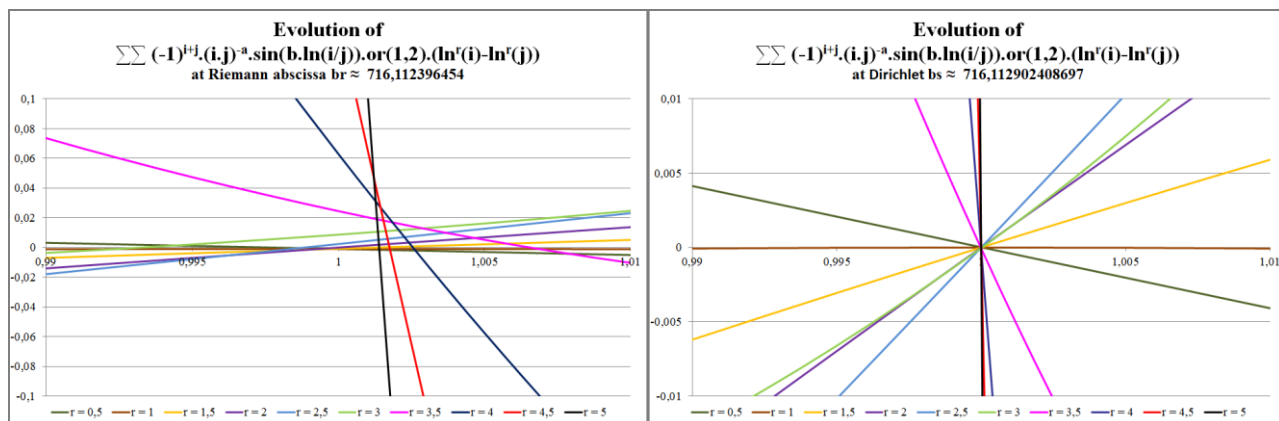




The curves are identical to a small shift in a . At the Riemann abscissa, the curves intersect (badly) before $a = 1$ and the Dirichlet abscissa, the curves intersect (badly) beyond 0.5.

It is easier to have a good accuracy of graphics, despite truncations, near $a = 1$ than near $a = 0.5$ (problem of parallax versus verticals with step $\Delta a = 0.02$).



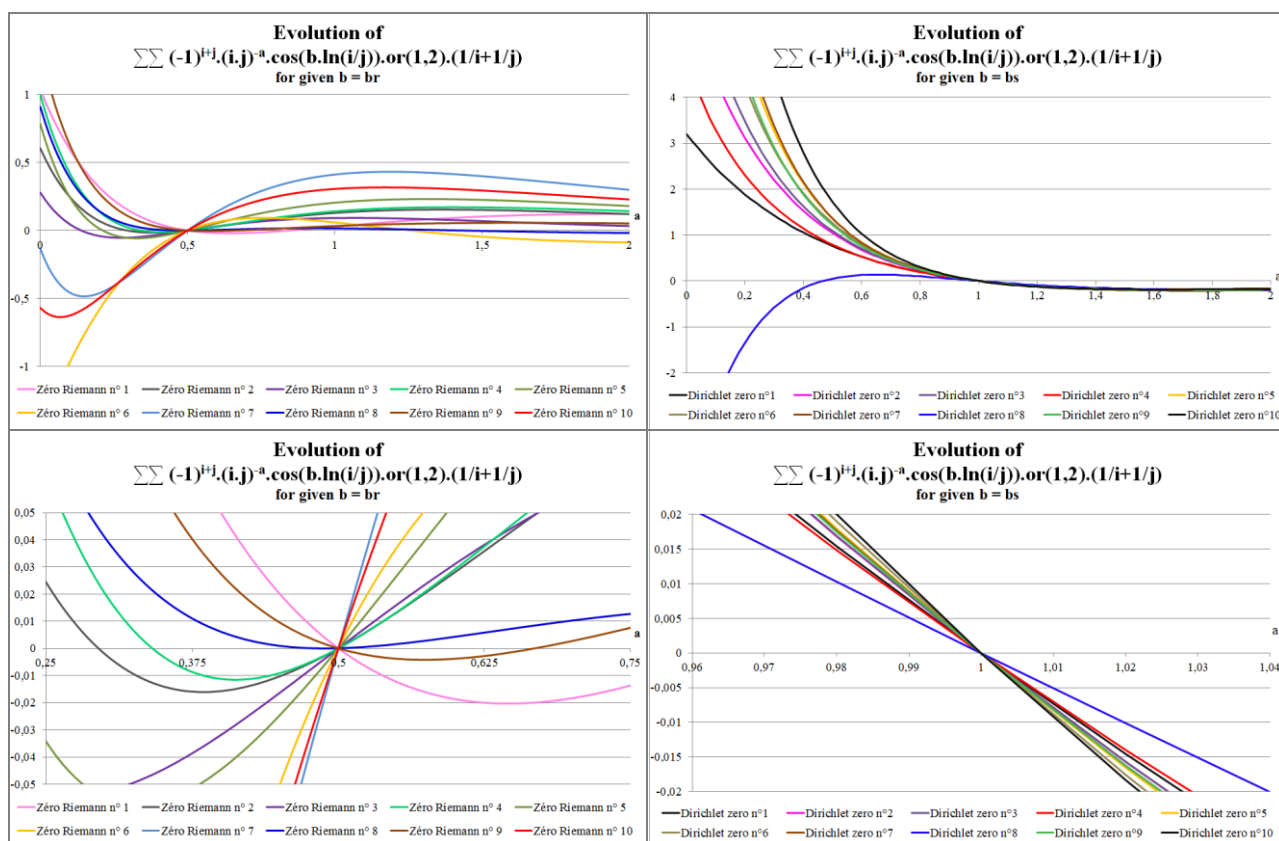


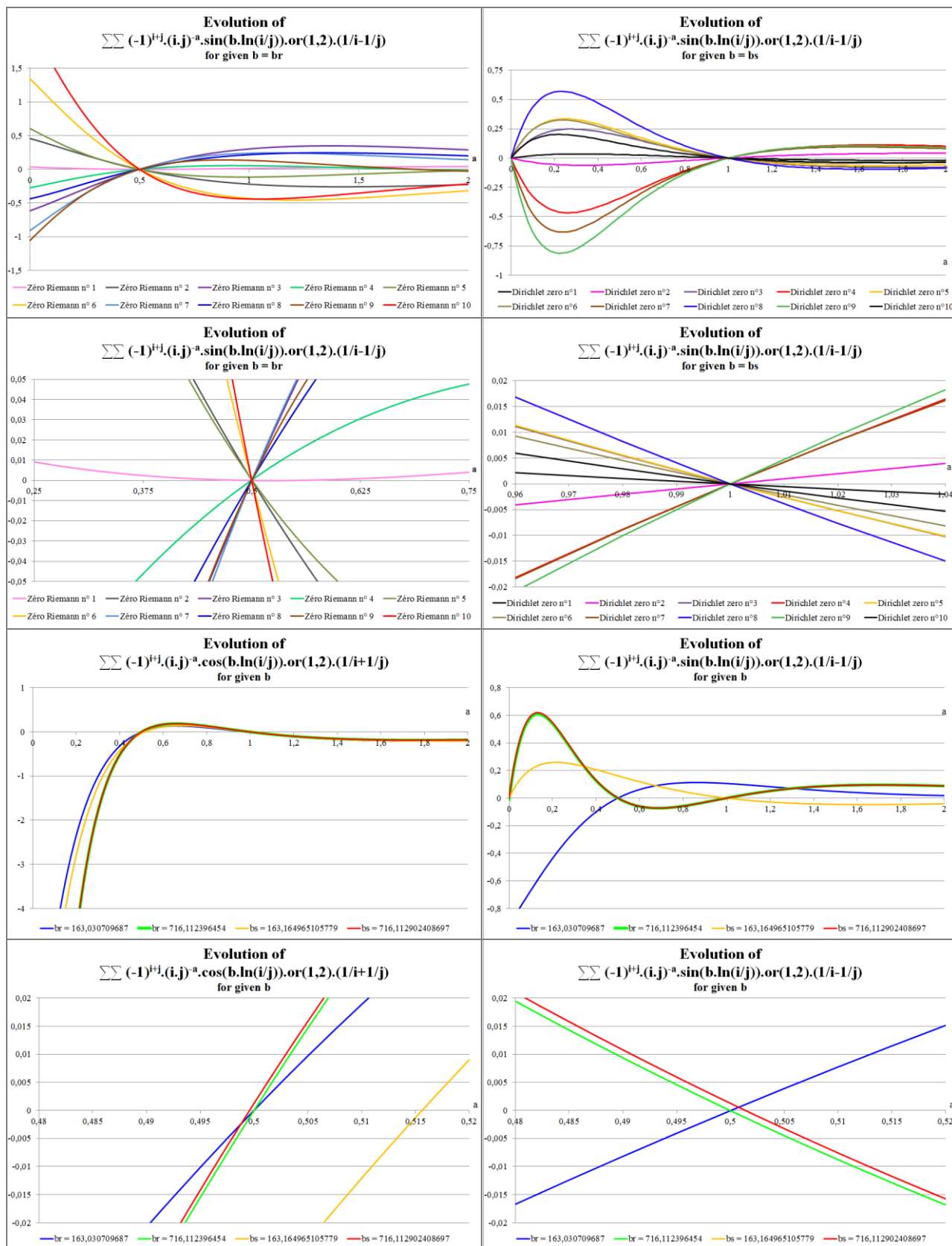
With hindsight, sine curves appear symmetrical to cosine curves versus the x-axis. This symmetry is only a semblance of symmetry. The curves obtained by summing cosine and sine still give curves similar to the previous ones, just being an example of an intermediate curve $LM_{\infty}(r,s,\phi,\theta)$ that we will expose underneath after the three illustrative examples.

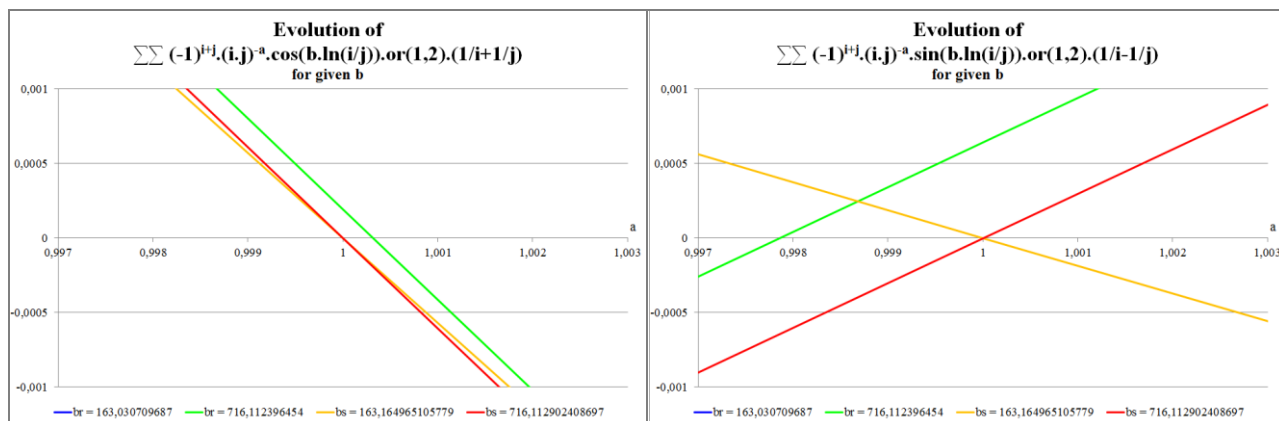
Illustration of $FC1_{\infty}(s)$ and $FS1_{\infty}(s)$

We give underneath a sample of the variations of $FC1_{\infty}(s)$ and $FS1_{\infty}(s)$, function of a , when this parameter varies in interval $[0,2]$.

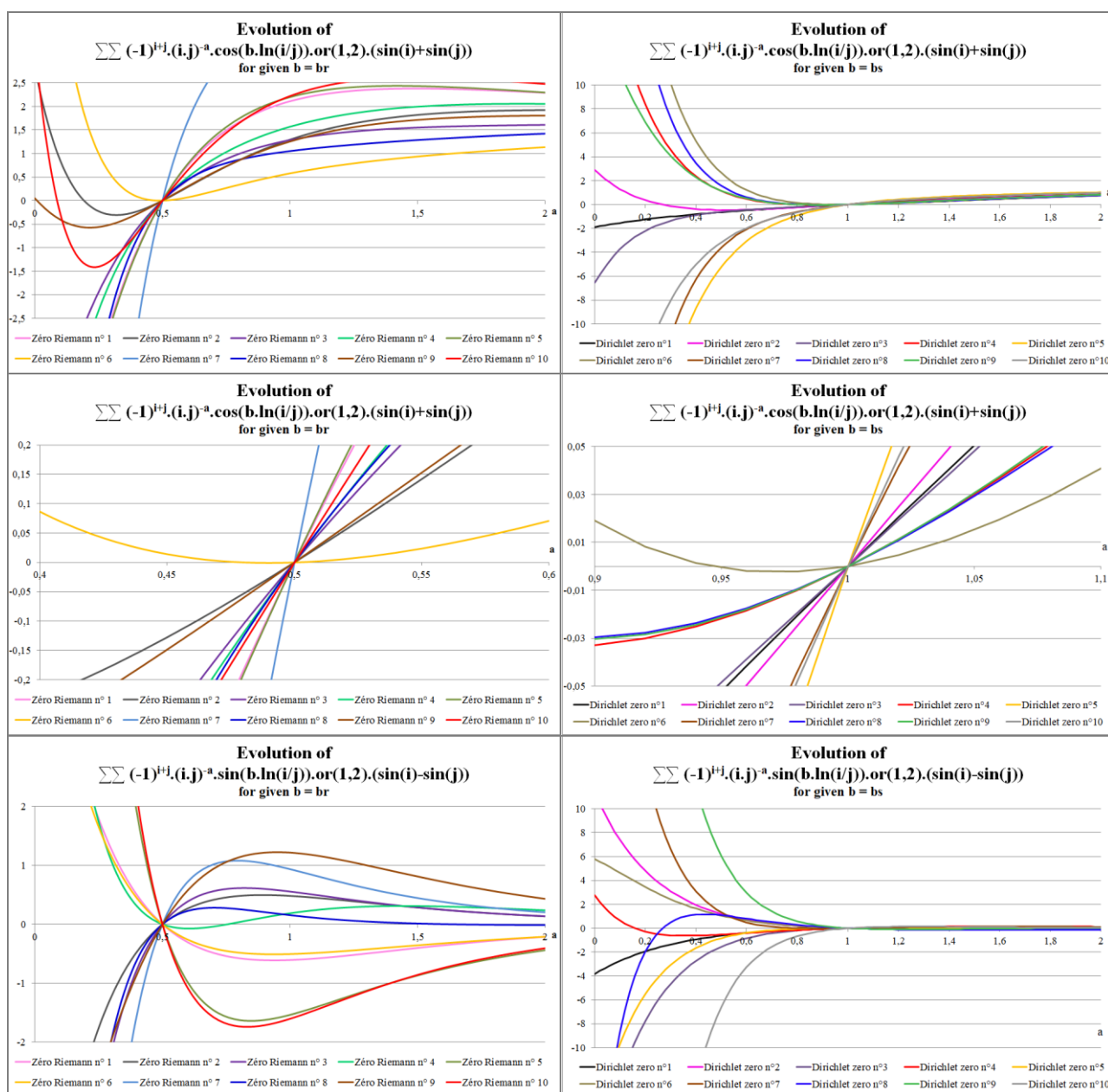
Example 1 : $F(x) = 1/x$

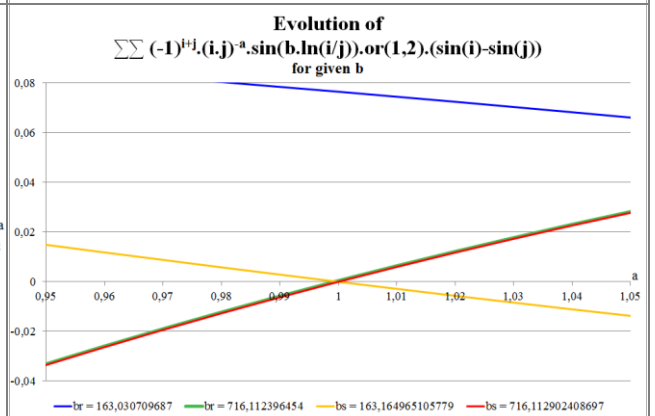
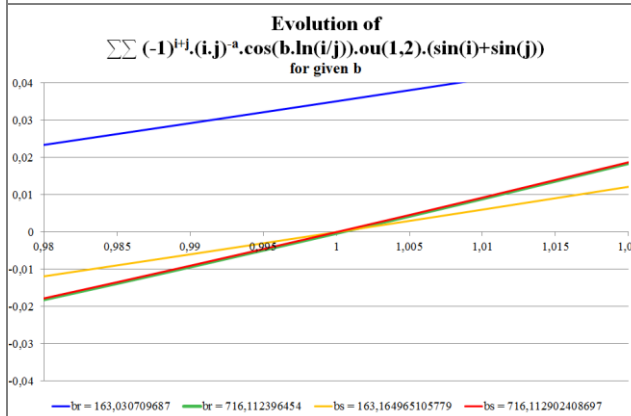
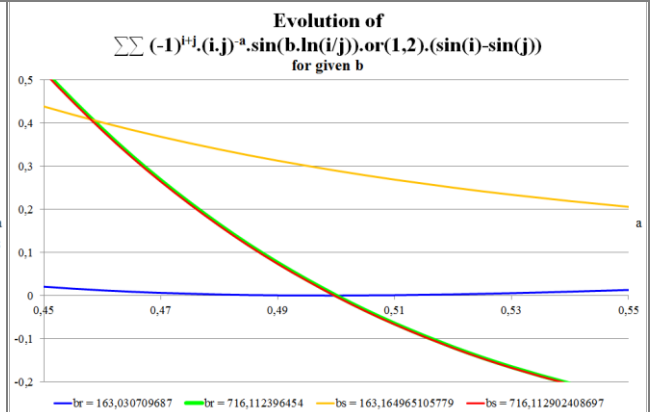
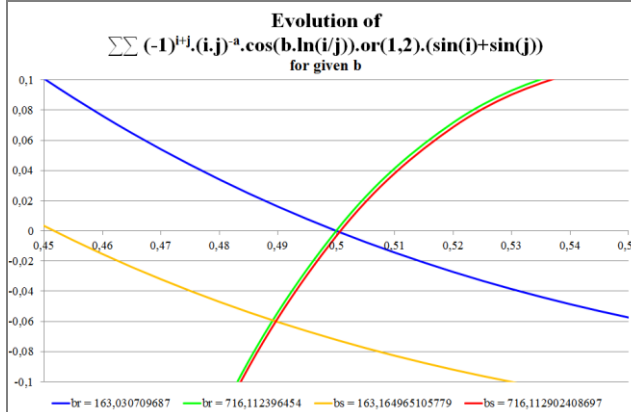
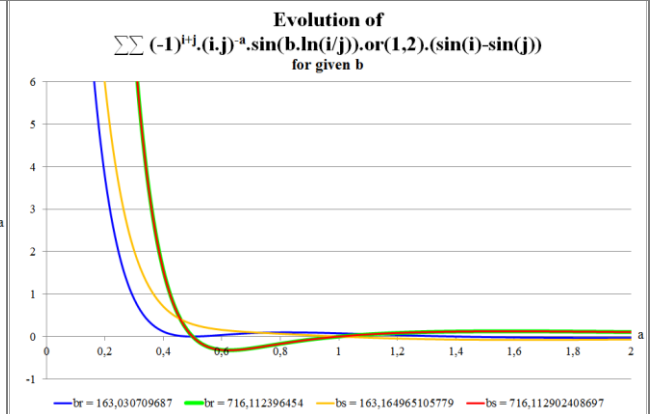
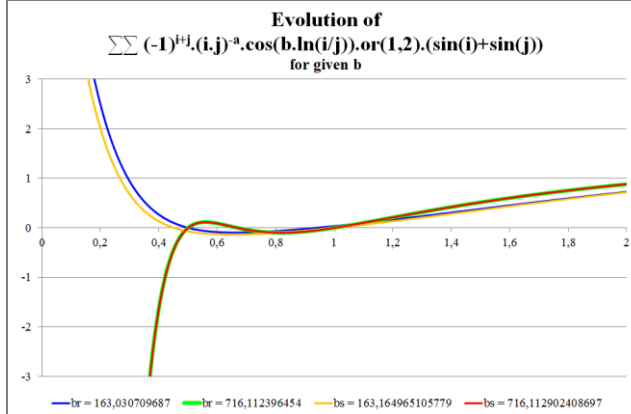
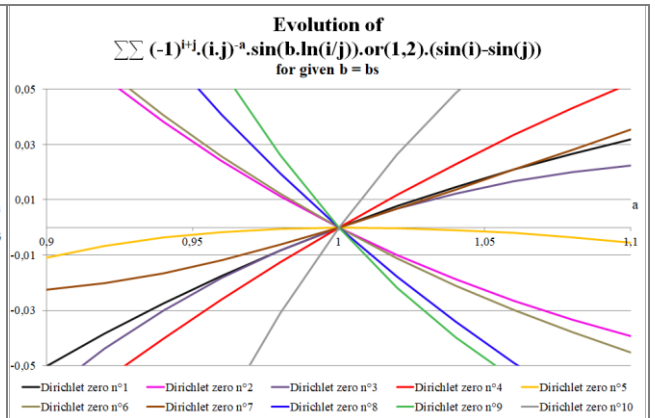
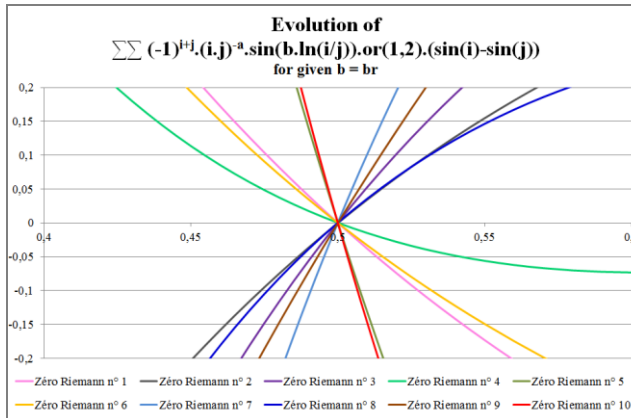




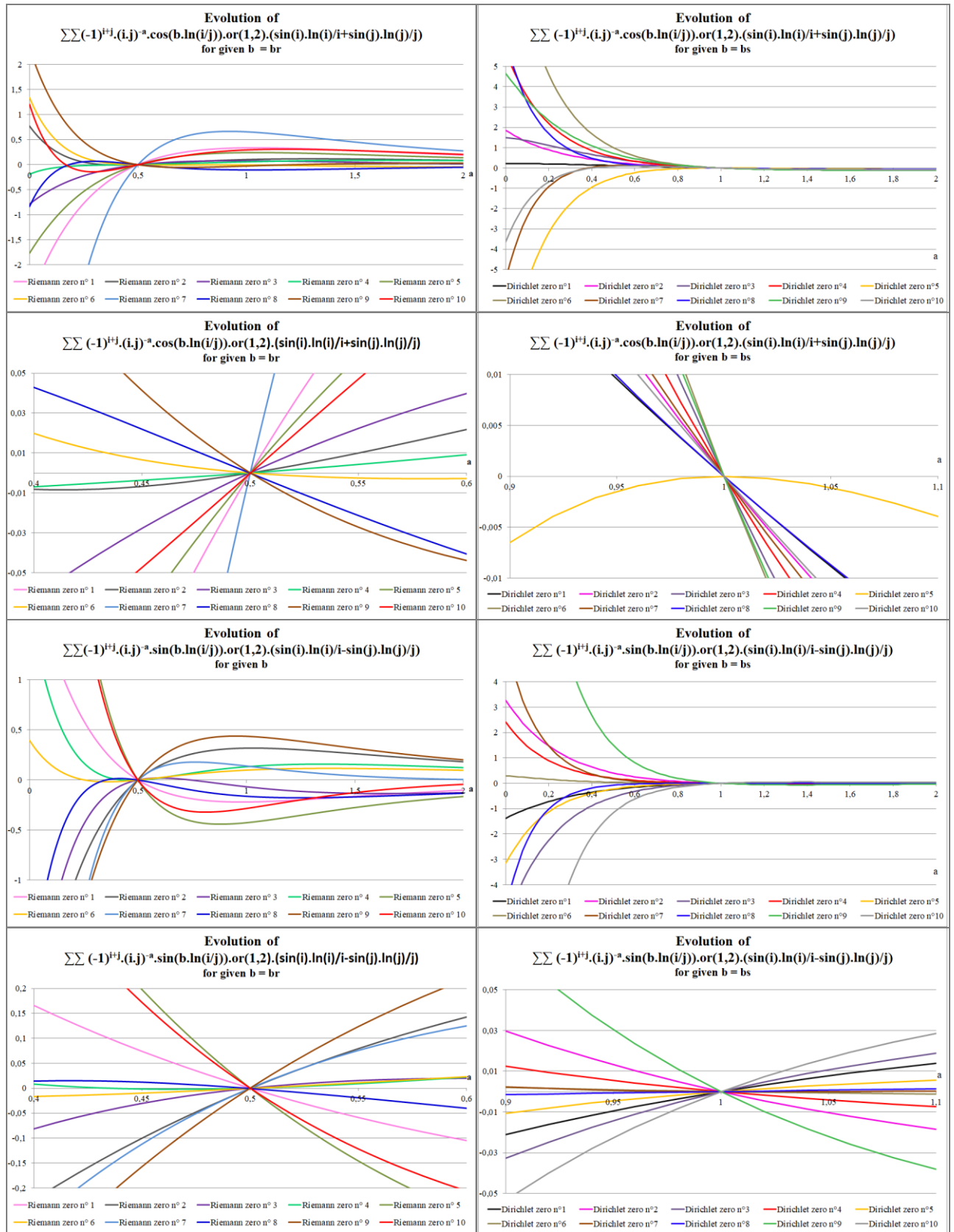


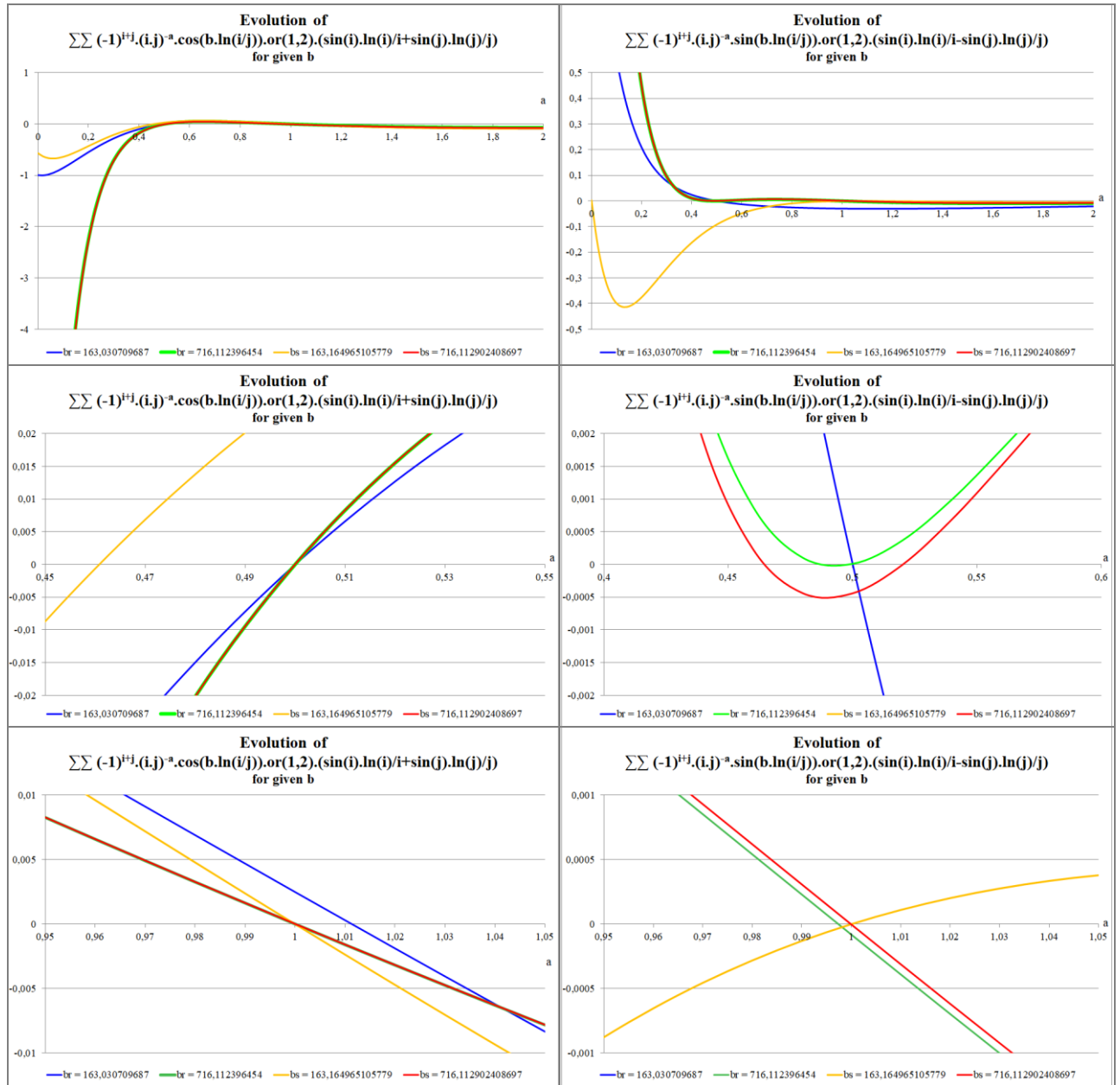
Example 2 : $F(x) = \sin(x)$





Example 3 : $F(x) = \sin(x).Ln(x)/x$





Remarks concerning the positions of the curves at $a = 0.5$ and $a = 1$ are the same as usually.

9.2.2. The second type of general equations.

Formulation

Let us go back to the general function (63) :

$$FG1_{\infty}(s) = \sum_{m=1}^{\infty} \sum_{n=1}^m (m.n)^{-a} \cdot (-1)^{m+n} \cdot \text{or}(1,2) \cdot ((m/n)^{i.b} \cdot F(m) + (m/n)^{-i.b} \cdot F(n)) \quad (68)$$

which write also

$$FG1_{\infty}(s) = \sum_{m=1}^{\infty} \sum_{n=1}^m (-1)^{m+n} \cdot \text{or}(1,2) \cdot ((m^{-a+i.b}/n^{a+i.b}) \cdot F(m) + (n^{-a+i.b}/m^{a+i.b}) \cdot F(n)) \quad (69)$$

Let us choose the particular case of $F(x) = x^{a-i.b}$:

$$FP_{\infty}(s) = \sum_{m=1}^{\infty} \sum_{n=1}^m (-1)^{m+n} \cdot \text{or}(1,2) \cdot ((1/m^{a+i \cdot b}) + (1/n^{a+i \cdot b})) \quad (70)$$

We will prove later on that this function admits effectively the Riemann and Dirichlet zeroes and remarkable proprieties that are easy to verify. We called it the **basic equation** as it is common to the two types of equations that we have identified.

If, instead of the preceding substitutions, we had chosen to do $F(m) \rightarrow m^{a-i \cdot b} \cdot F(n)$ and $F(n) \rightarrow n^{a-i \cdot b} \cdot F(m)$, we would have the general functions of the type :

$$FG2_{\infty}(s) = \sum_{m=1}^{\infty} \sum_{n=1}^m (-1)^{m+n} \cdot \text{or}(1,2) \cdot ((F(n)/m^{a+i \cdot b}) + (F(m)/n^{a+i \cdot b})) \quad (71)$$

It happens that this expression suits to our objectives as we have the theorem.

Theorem 15

Let us have s a Riemann or Dirichlet zero. If $FG2_{\infty}(s)$ converge, then $FG2_{\infty}(s) = 0$.

Proof :

The proof is the same as that used for $FG1_{\infty}(s)$.

The terms collected for $F(r)$, r an integer given in advance, when we develop the expression $FG2_{\infty}(s)$ are :

$$2 \cdot (-1)^r \cdot F(r) \cdot \sum_{n=1}^{\infty} (-1)^n \cdot (1/n)^{a+i \cdot b}$$

Gathering all terms, we get then :

$$FG2_{\infty}(s) = \lim_{r \rightarrow \infty} 2 \sum_{m=1}^r (-1)^m \cdot F(m) \cdot \sum_{n=1}^r (-1)^n \cdot (1/n)^{a+i \cdot b} \quad (72)$$

Again $FG1_{\infty}(s)$ cancels for the Eta function zeros if and only if the first sum does not diverge to fast while the second sum converge. The product $FG2_{\infty}(s)$ either cancels or diverge. Let us consider then the dominant terms of each of the two sums omitting factors with no effect on the module, that is $n^{-i \cdot b}$. This is the same as comparing $\sum (-1)^n \cdot F(n)$ and $\sum (-1)^n \cdot (1/n)^a$. Thus, the divergence of the first sum will be superior to the convergence of the second sum as soon as $F(x) \geq x^a$ asymptotically.

Note : If we have chosen the substitutions $F(m) \rightarrow m^{a-i \cdot b} \cdot F(m)$ and $F(n) \rightarrow n^{a-i \cdot b} \cdot F(n)$, we would not achieve our goal. The formal independence (m and n are obviously linked and it means only here independence in a symbolic way) of variables is essential for these artificial assemblies.

Splitting of real and imaginary parts

Splitting real and imaginary part, we get the deux expressions :

$$FC2_{\infty}(s) = \sum_{m=1}^{\infty} \sum_{n=1}^m (-1)^{m+n} \cdot \text{or}(1,2) \cdot (F(n) \cdot \cos(b \cdot \text{Ln}(m))/m^a + F(m) \cdot \cos(b \cdot \text{Ln}(n))/n^a) \quad (73)$$

et

$$FS2_{\infty}(s) = \sum_{m=1}^{\infty} \sum_{n=1}^m (-1)^{m+n} \cdot \text{or}(1,2) \cdot (F(n) \cdot \sin(b \cdot \text{Ln}(m))/m^a + F(m) \cdot \sin(b \cdot \text{Ln}(n))/n^a) \quad (74)$$

For that last equation, unlike the $FS1_{\infty}(s)$ case, there is no change of sign in front of $F(m)$.

Let us go back to a more precise study.

Basic equations

Let us have

$$FP_{\infty}(s) = \sum_{m=1}^{\infty} \sum_{n=1}^m (-1)^{m+n} \cdot \text{or}(1,2) \cdot (1/m^{a+i.b} + 1/n^{a+i.b}) \quad (75)$$

and the decomposition in real and imaginary parts :

$$FPC_{\infty}(s) = \sum_{m=1}^{\infty} \sum_{n=1}^m (-1)^{m+n} \cdot \text{or}(1,2) \cdot (\cos(b \cdot \text{Ln}(m))/m^a + \cos(b \cdot \text{Ln}(n))/n^a) \quad (76)$$

and

$$FPS_{\infty}(s) = \sum_{m=1}^{\infty} \sum_{n=1}^m (-1)^{m+n} \cdot \text{or}(1,2) \cdot (\sin(b \cdot \text{Ln}(m))/m^a + \sin(b \cdot \text{Ln}(n))/n^a) \quad (77)$$

Then we consider the truncated functions for the first and second of these expressions that we note here $FTPC_m(s)$ and $FTPS_m(s)$.

These two relations cancel regularly in a trivial way for $m = 2k$ and $n = m$, and this even if $a \neq 0,5$ and $\neq 1$ for all b (thus giving no particular information on b).

Proof :

Let us consider the truncations at $m = 2k-1$ and at $m = 2k$, meaning here the internal sums (of the double sum) of $n = 1$ to m for $m = 2k-1$, respectively $m = 2k$ ($k \geq 1$). It suffices to sum up the expressions $\text{coef} = (-1)^{m+n} \cdot \text{or}(1,2) = (-1)^{m+n} \cdot \text{if}(m=n, 1, 2)$ before $\cos(b \cdot \text{Ln}(n))/n^a$, respectively $\sin(b \cdot \text{Ln}(n))/n^a$, for some value n given in advance. Let us thus have such n . One has necessarily $n \leq 2k$. If $1 \leq n \leq 2k-2$, there is a unique increment with value n in the truncation $2k-1$ and a unique one in the truncation $2k$, the values of coef being equal to $2 \cdot (-1)^{2k-1+n}$ and $2 \cdot (-1)^{2k+n}$ will annihilate. If $n = 2k-1$, there are $2k-2$ coef of value $2 \cdot (-1)^{2k-1+r}$ with alternated signs with cancel together (in the truncation $2k-1$), 2 coef of value $(-1)^{2k-1+n}$ (in the truncation $2k-1$) and 1 coef of value $2 \cdot (-1)^{2k+n}$ (in the truncation $2k$) annihilating together. If $n = 2k$, there are $2k-2$ coef of value $2 \cdot (-1)^{2k+r}$ (in the truncation $2k$) of alternated signs annihilating together, 1 coef of value $2 \cdot (-1)^{2k+n-1}$ (in the truncation $2k$) and 2 coef of value $(-1)^{2k+n}$ (in the truncation $2k$) which cancel.

This is illustrated in the table underneath making the sum of the values of « coef » for $m = r$ or $n = r$.

m	1	1	2	1	2	3	1	2	3	4
n	1	2	2	3	3	3	4	4	4	4
coef = $(-1)^{m+n} \cdot (m,n)^{-a} \cdot \text{or}(1,2)$	1	-2	1	2	-2	1	-2	2	-2	1
$\cos(b \cdot \text{Ln}(m))/m^a$	1	1	0,7971	1	0,7971	-0,0095	1	0,7971	-0,0095	0,2708
$\cos(b \cdot \text{Ln}(n))/n^a$	1	0,7971	0,7971	-0,0095	-0,0095	-0,0095	0,2708	0,2708	0,2708	0,2708
$S1 = \text{coef} \cdot \cos(b \cdot \text{Ln}(m))/m^a$	1	-2	0,4740	2	-0,9479	-0,0041	-2	0,9479	0,0083	0,0957
$S2 = \text{coef} \cdot \cos(b \cdot \text{Ln}(n))/n^a$	1	-0,9479	0,4740	-0,0083	0,0083	-0,0041	-0,1915	0,1915	-0,1915	0,0957
$S1+S2$	2	-0,9479	0	1,9917	1,0521	1,0438	-1,1477	-0,0083	-0,1915	0

This table is done for $a = 0.75$ and $b = 10$ without any link to the value of a (Riemann or Dirichlet) zero.

The sum $S1+S2$ returns regularly to 0. However, this does not mean that $S1+S2$ converge.

Let us then consider the truncations, m being fixed :

$$FTPC_{\infty}(s,m) = \sum_{n=1}^m (-1)^{m+n} \cdot \text{or}(1,2) \cdot (\cos(b \cdot \text{Ln}(m))/m^a + \cos(b \cdot \text{Ln}(n))/n^a) \quad (78)$$

and

$$FTPS_{\infty}(s,m) = \sum_{n=1}^m (-1)^{m+n} \cdot \text{or}(1,2) \cdot (\sin(b \cdot \text{Ln}(m))/m^a + \sin(b \cdot \text{Ln}(n))/n^a) \quad (79)$$

The coefficients before $\cos(b \cdot \text{Ln}(m))/m^a$ and $\sin(b \cdot \text{Ln}(m))/m^a$ are equal to ± 2 with alternating signs. The contributions will thus cancel (as m is constant). The terms thus tend, when m tends towards infinity, towards :

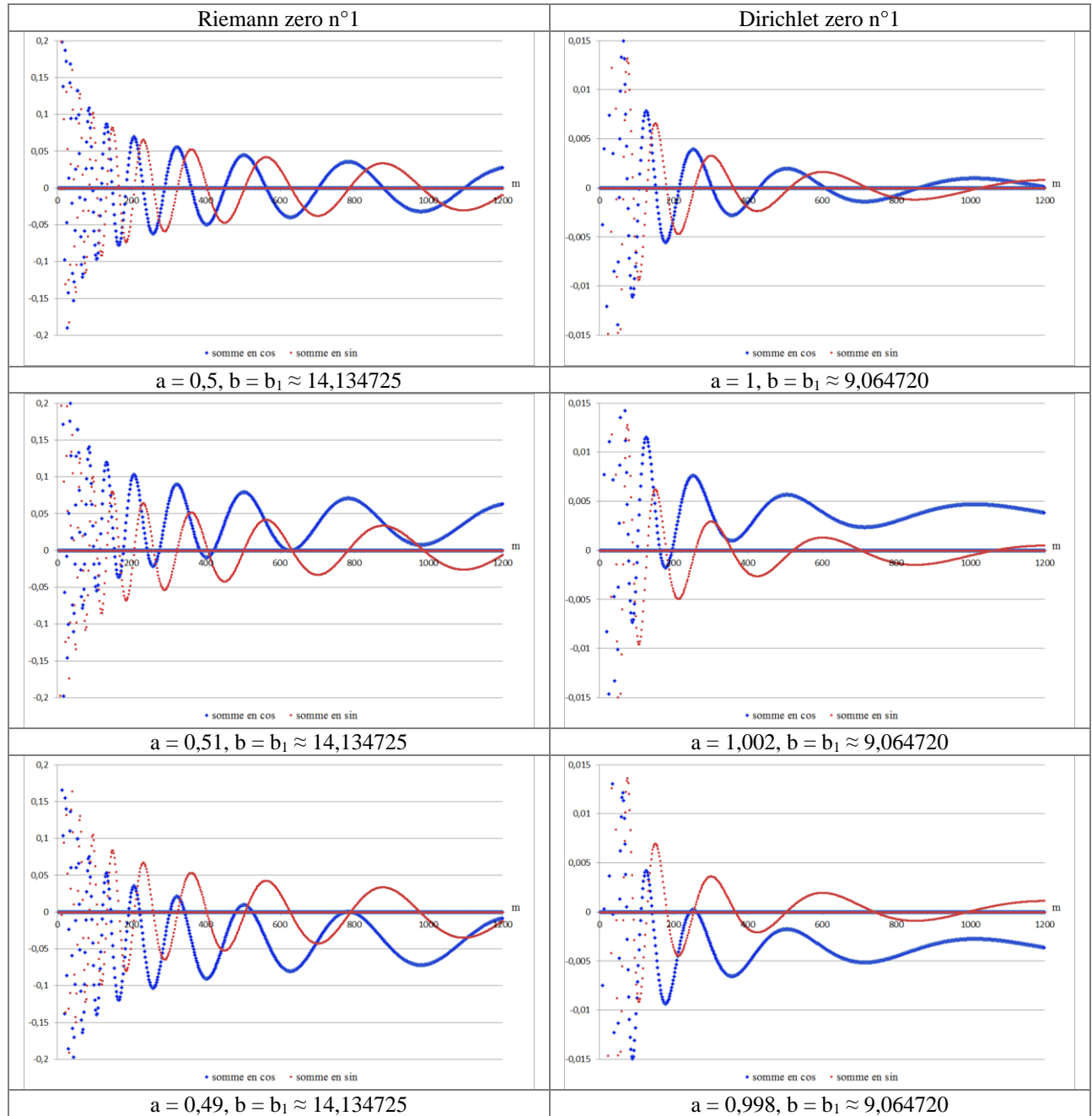
$$\text{FTPC}_{\infty}(s, m \rightarrow +\infty) \rightarrow 2 \cdot (-1)^{\text{or}(0,1)} \sum_{n=1}^{\infty} (-1)^n \cdot \cos(b \cdot \text{Ln}(n)) / n^a \quad (80)$$

et

$$\text{FTPS}_{\infty}(s, m \rightarrow +\infty) \rightarrow 2 \cdot (-1)^{\text{or}(0,1)} \sum_{n=1}^{\infty} (-1)^n \cdot \sin(b \cdot \text{Ln}(m)) / m^a \quad (81)$$

We came back essentially to the initial equations of $\eta(s)$ which cancel exclusively at the Riemann (for $a = 0,5$ a priori) or Dirichlet (for $a = 1$) zeros.

We give the examples underneath of the evolution of the truncated sums $\text{FPC}_m(s)$ and $\text{FPS}_m(s)$ as functions of m for and at the vicinity of a Riemann or Dirichlet zero.



The evolutions are similar but the sensibility to the variation of a is very different from one zero to another. The curves for which the « general move » is large are those associated with the cosines.

Monomial equations

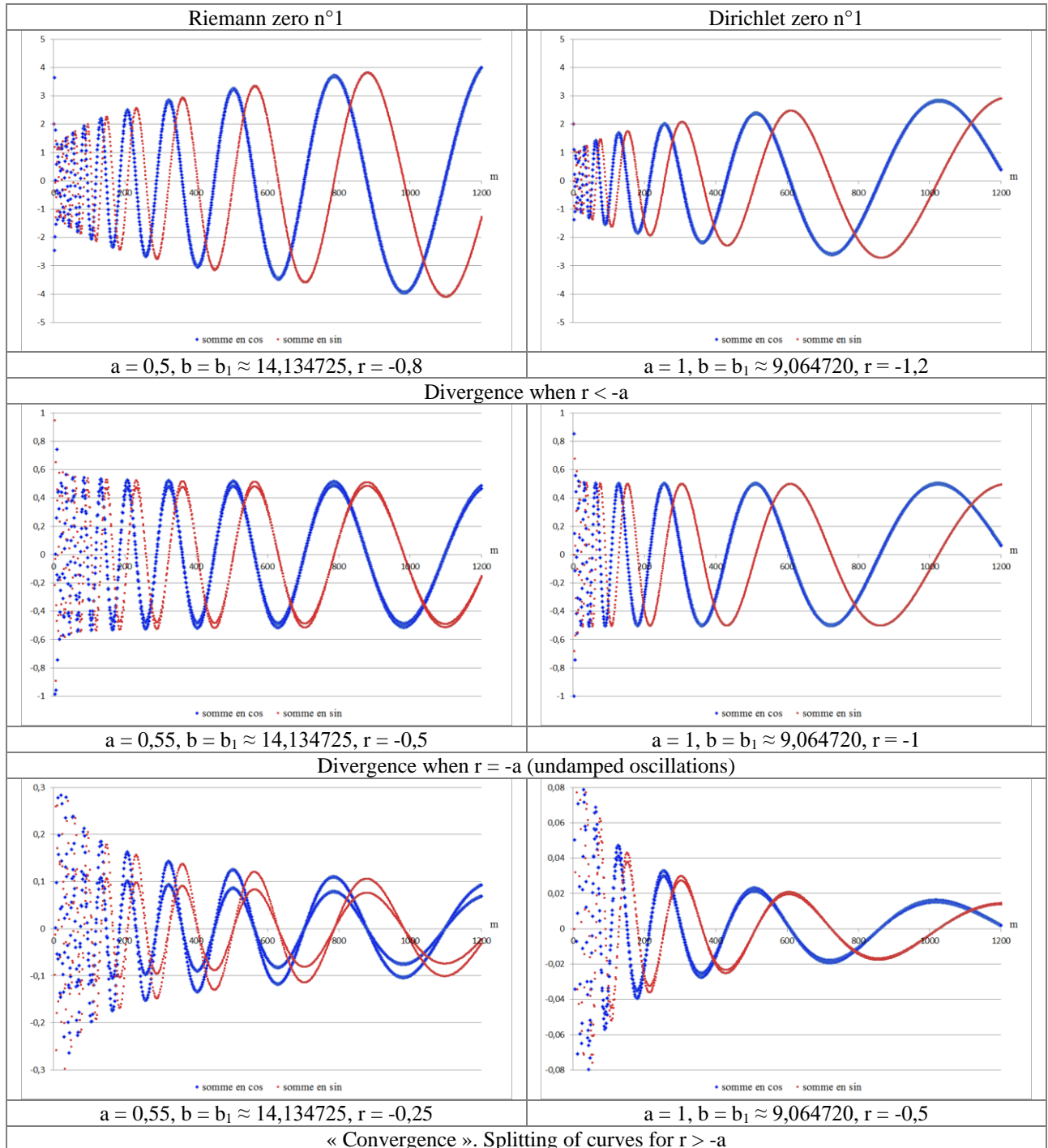
Let us study the case $F(x) = x^r$ for the truncated functions :

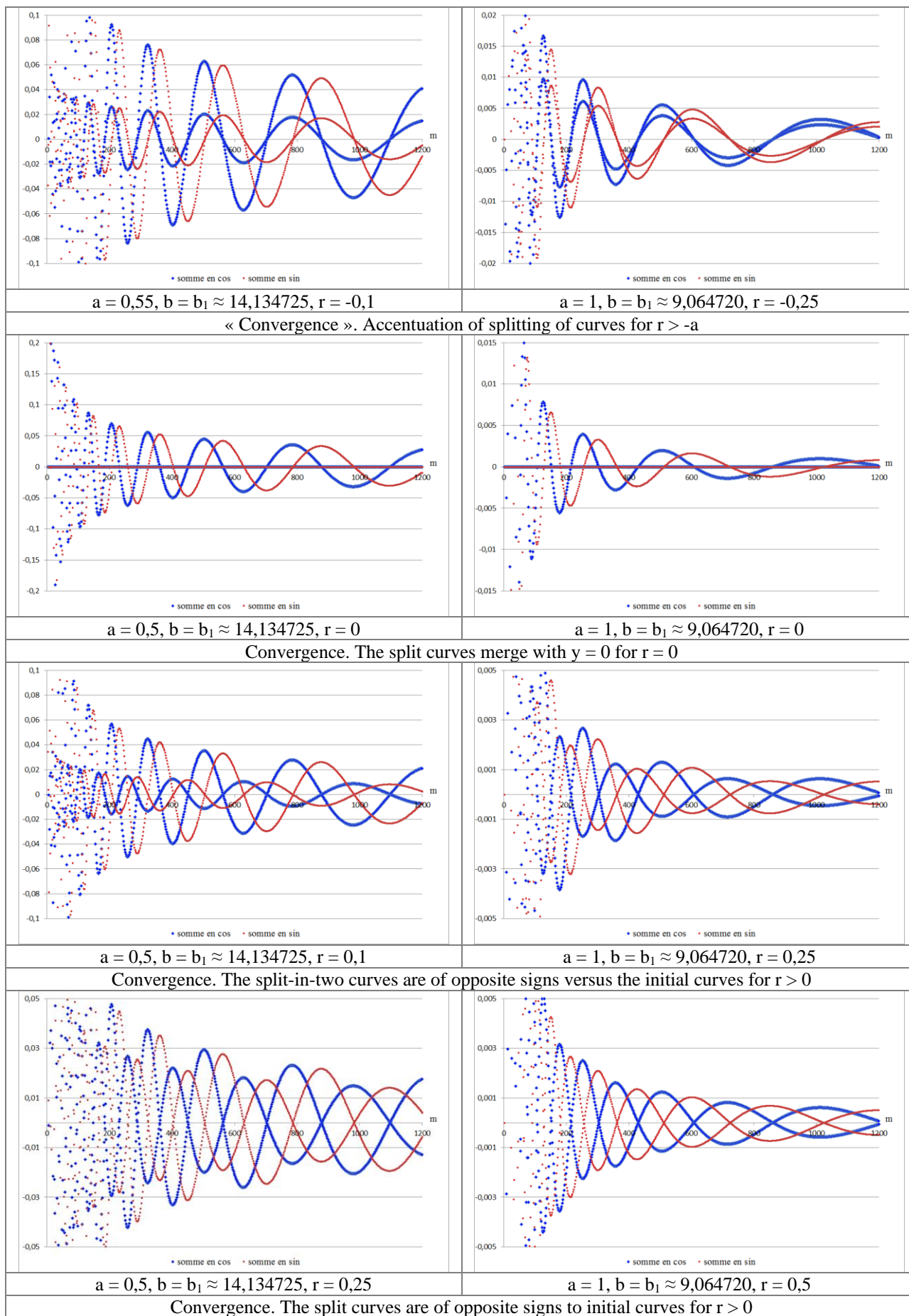
$$FC2_{m_{\max}}(s) = \sum_{m=1}^{m_{\max}} \sum_{n=1}^m (-1)^{m+n} \cdot \text{or}(1,2) \cdot (\cos(b \cdot \text{Ln}(m)) / (m^a \cdot n^r) + \cos(b \cdot \text{Ln}(n)) / (n^a \cdot m^r)) \quad (82)$$

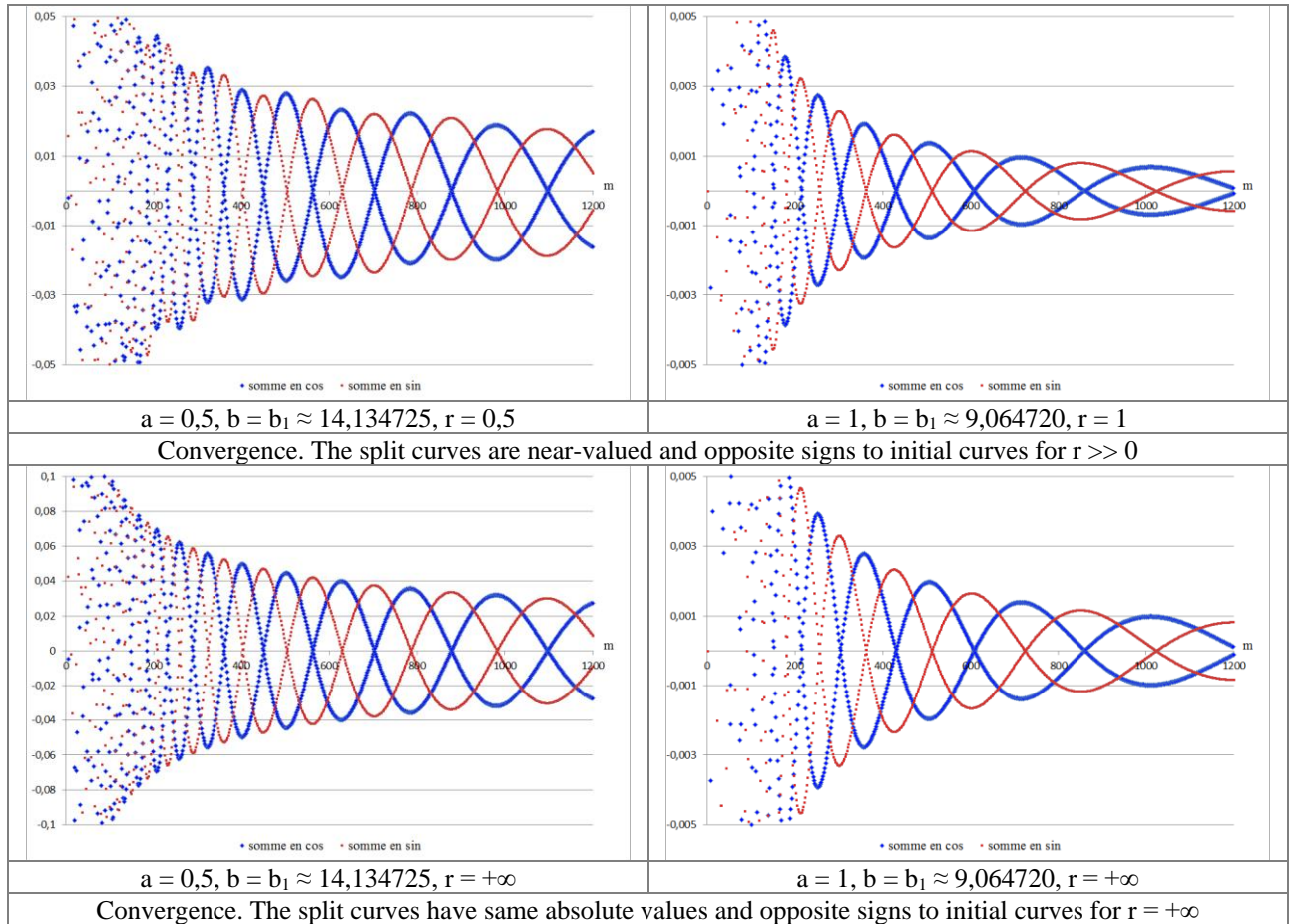
et

$$FS2_{m_{\max}}(s) = \sum_{m=1}^{m_{\max}} \sum_{n=1}^m (-1)^{m+n} \cdot \text{or}(1,2) \cdot (\sin(b \cdot \text{Ln}(m)) / (m^a \cdot n^r) + \sin(b \cdot \text{Ln}(n)) / (n^a \cdot m^r)) \quad (83)$$

For that, we chose again the same zeros (the first of each type) changing only the values of r .
The curves as functions of m are then :







The above represented points correspond to integer truncation, that is a full calculation of a sum inside the double sums $FTPC_m(s)$ et $FTPS_m(s)$. The graphics show what we named initial and double-up curves. The first title (initial curves) corresponds to odd m and the second title to even m . During the entire process of evolution, initial and double-up curves intersect at $y = 0$. We remain however very reserved as for the convergence of double sums for $-a < r < 0$. Indeed, there is well a reduction in the amplitude of the oscillations when calculations are made on whole truncations (comparing the same parity $m = 0 \bmod 2$ or $m = 1 \bmod 2$), but it does not decrease considering the set of intermediate values (the maximum values are higher and higher).

Thus, it is useful and simpler to say: If the function converges, then it converges to 0.

10. The two keys to the Riemann hypothesis.

10.1. First key : The Riemann functional equation.

The first key for resolution of the Riemann hypothesis is the Riemann functional equation mentioned earlier.

Theorem 16

For all $s \neq 0$ and $s \neq 1$, and in particular within the critical strip, we have

$$\zeta(s) = 2^\pi \cdot \pi^{s-1} \cdot \sin(\pi \cdot s/2) \cdot \Gamma(1-s) \cdot \zeta(1-s)$$

This equation has also a more symmetrical form $\Phi(s) = \Phi(1-s)$ with $\Phi(s) = \pi^{-s/2} \cdot \Gamma(s/2) \cdot \zeta(s)$ as specified in [2]. This means, for what interests us here, that to a possible zero s for $(a, b) = (0, 5-\epsilon, b)$, $0 < \epsilon < 1/2$, corresponds another zero s' for $(a', b') = (0, 5+\epsilon, -b)$.

Indeed, we have then

$$0 = \zeta(s) = \zeta(a, b) = \zeta(0, 5-\epsilon, b) = 2^\pi \cdot \pi^{s-1} \cdot \sin(\pi \cdot s/2) \cdot \Gamma(1-s) \cdot \zeta(1-(0, 5-\epsilon, -b)) = 2^\pi \cdot \pi^{s-1} \cdot \sin(\pi \cdot s/2) \cdot \Gamma(1-s) \cdot \zeta(0, 5+\epsilon, -b)$$

As $\sin(\pi \cdot s/2)$ is different from 0 at $(0, 5-\epsilon, b)$ and that $\Gamma(1-s)$ does not vanish, necessarily $\zeta(0, 5+\epsilon, -b) = 0$.

Furthermore

$$\zeta(0, 5+\epsilon, b) = \zeta(0, 5+\epsilon, -b)$$

when

$$\zeta(s=a+ib) = \sum m^{-a} \cdot \cos(b \cdot \ln(m)) + i \cdot \sum m^{-a} \cdot \sin(b \cdot \ln(m)) = 0$$

since this is equivalent to

$$0 = \sum m^{-a} \cdot \cos(b \cdot \ln(m)) - i \cdot \sum m^{-a} \cdot \sin(b \cdot \ln(m)) = \sum m^{-a} \cdot \cos(-b \cdot \ln(m)) + i \cdot \sum m^{-a} \cdot \sin(-b \cdot \ln(m)) = \zeta(a-ib)$$

We can therefore change the sign of ε and b at will.

As $\eta(s) = (1-2^{1-s}) \cdot \zeta(s)$, the previous relationship also applies to $\eta(s)$, as well as the equivalent relation below.

$$\sum_{i=1}^{\infty} \sum_{j=1}^i (i \cdot j)^{-a-\varepsilon} \cdot (-1)^{i+j} \cdot \cos(b \cdot \ln(i/j)) \cdot \text{or}(1,2) = 0$$

Hence what follows :

Theorem 17

Let us have $0 < \varepsilon < 1/2$.

If

$$\sum_{i=1}^{\infty} \sum_{j=1}^i (i \cdot j)^{-a-\varepsilon} \cdot (-1)^{i+j} \cdot \cos(b \cdot \ln(i/j)) \cdot \text{or}(1,2) = 0 \quad (84)$$

then

$$\sum_{i=1}^{\infty} \sum_{j=1}^i (i \cdot j)^{-a+\varepsilon} \cdot (-1)^{i+j} \cdot \cos(-b \cdot \ln(i/j)) \cdot \text{or}(1,2) = 0 \quad (85)$$

that is if $\varepsilon \neq 0$, there are thus two solutions (instead of one if $\varepsilon = 0$).

10.1. Second key : Unicity of the zero

For this end of the article, we are so far obliged to use only the terms of propositions and arguments (instead of theorems and proofs).

Proposition 1

There are no accidental annulations of the function $FG1_{\infty}(s, F)$ for a Riemann or Dirichlet zero.

We mean here that it is impossible that the expression vanishes for an F without peculiar propriety (that is some independent in m and n construction).

Argument

Let us suppose that $FG1_{\infty}(s, F)$ vanishes for $F(n, m)$ some function at a Riemann zero. The accidental annulation induces that if one chooses then $F(n, m) + \varepsilon$, where ε is some constant, the position of the zero will move and is no more a zero for the previous equation. However, as ε is a constant, one can write $FF(m) = \varepsilon/2$ and $FF(n) = \varepsilon/2$ and $FG1_{\infty}(s, F+FF)$ vanishes then still at the said Riemann zero, which is a contradiction. The reasoning hold in the same way for a Dirichlet zero.

Proposition 2

Riemann and Dirichlet zeroes cancel exclusively for functions like $FG1_{\infty}(s, F)$ and $FG2_{\infty}(s, F)$ with F appearing in a sum $F(n) + F(m)$.

Partial argument

The functions are necessarily product of the zeta function by another function to coincide for all of its solutions.

Let us then go back to

$$FC1_{\infty}(s) = \sum_{m=1}^{\infty} \sum_{n=1}^m (m \cdot n)^{-a} \cdot (-1)^{m+n} \cdot \text{ou}(1,2) \cdot \cos(b \cdot \ln(m/n)) \cdot (F(m) + F(n)) \quad (86)$$

$$FS1_{\infty}(s) = \sum_{m=1}^{\infty} \sum_{n=1}^m (m.n)^{-a} \cdot (-1)^{m+n} \cdot \text{ou}(1,2) \cdot \sin(b \cdot \text{Ln}(m/n)) \cdot (F(m) - F(n)) \quad (87)$$

We say that there is no b such that the two expressions are null for any F for two distinct values of a (that is for the $a = 1/2$ et $a = 1$ pair or for any other pair).

Argument

Otherwise, we would have two a_1 and a_2 values (and some b) such that :

$$FC1_{\infty}(s = a_1 + i.b) = 0 \quad (88)$$

and

$$FC1_{\infty}(s = a_2 + i.b) = 0 \quad (89)$$

and the same for $FS1_{\infty}(s)$.

By subtracting, we have then also (trivially) :

$$\Delta FC1_{\infty}(s) = FC1_{\infty}(s_1 = a_1 + i.b) - FC1_{\infty}(s_2 = a_2 + i.b) = 0 \quad (90)$$

that is :

$$\sum_{m=1}^{\infty} \sum_{n=1}^m (-1)^{m+n} \cdot \text{ou}(1,2) \cdot \cos(b \cdot \text{Ln}(m/n)) \cdot \left(\frac{(F_2(m) + F_2(n))}{(m.n)^{a_2}} - \frac{(F_1(m) + F_1(n))}{(m.n)^{a_1}} \right) = 0 \quad (91)$$

and again, the resulting, none-identically null, function $\Delta FC_{\infty}(s)$ encounters the cosine and sine's filter. Consequently, there must be a function $F(x)$ such as for any integers m and n :

$$\frac{F(m) + F(n)}{(m.n)^a} = \frac{(F_2(m) + F_2(n))}{(m.n)^{a_2}} - \frac{(F_1(m) + F_1(n))}{(m.n)^{a_1}}$$

We have of course a choice among an infinite kind of forms F_1 and F_2 to hope to find a function F which meets conditions. However, a is not equal to a_1 , nor to a_2 , otherwise we would be back to trivial identities. We can then rewrite the preceding expression as :

$$F(m) + F(n) = (m.n)^{a-a_2} \cdot \left((F_2(m) + F_2(n)) - \frac{(F_1(m) + F_1(n))}{(m.n)^{a_1-a_2}} \right) \quad (92)$$

However, this equality bears its proper contradiction. The $(m.n)^{a-a_2}$ factor is not trivial as a is different from a_2 . We cannot thus find any function F such that the right member of the equation be independently the sum of a function of m and a function of n and thus especially if the said functions are the same F . Hence the proposition :

Proposition 3

At constant b , $FG1_{\infty}(s = a + i.b)$ vanishes for a single value a at most.

10.2. The Riemann hypothesis.

Proposition 4

The non-trivial zeros of the Riemann function have real value $1/2$.

Argument

There is contradiction, if $a \neq 1/2$, between the theorem 17 meaning two solutions and the argument of the unicity of the solution.

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- [3] http://www.dtc.umn.edu/~odlyzko/zeta_tables/index.html
- [4] Marc Hindry. La preuve par André Weil de l'hypothèse pour une courbe sur un corps fini.
- [5] http://fr.wikipedia.org/wiki/Hypothèse_de_Riemann http://fr.wikipedia.org/wiki/Fonction_zêta_de_Riemann
- [6] http://fr.wikipedia.org/wiki/Fonction_analytique#Les_principaux_théorèmes
- [7] https://fr.wikipedia.org/wiki/Fonction_zêta_de_Riemann#La_bande_critique_et_l'hypothèse_de_Riemann
- [8] Database of L-functions, modular forms, and related objects.
<https://www.lmfdb.org/zeros/zeta/>

Appendix 1

List of the Riemann zeros for imaginary values less than 100.

n	Real values zeros	Imaginary values of zeros	n	Real values zeros	Imaginary values of zeros
1	0,5	14,1347251	16	0,5	67,0798105
2	0,5	21,0220396	17	0,5	69,5464017
3	0,5	25,0108576	18	0,5	72,0671577
4	0,5	30,4248761	19	0,5	75,7046907
5	0,5	32,9350616	20	0,5	77,1448401
6	0,5	37,5861782	21	0,5	79,337375
7	0,5	40,918719	22	0,5	82,9103809
8	0,5	43,3270733	23	0,5	84,735493
9	0,5	48,0051509	24	0,5	87,4252746
10	0,5	49,7738325	25	0,5	88,8091112
11	0,5	52,9703215	26	0,5	92,4918993
12	0,5	56,4462477	27	0,5	94,651344
13	0,5	59,347044	28	0,5	95,8706342
14	0,5	60,8317785	29	0,5	98,8311942
15	0,5	65,112544			

Appendix 2

Values of $C_{\infty}(2,0,s)$ at Riemann zeros (first 500th)

1	7,1305	51	16,5237	101	45,8122	151	78,1467	201	42,3610	251	57,1435	301	29,7604	351	106,995	401	64,3868	451	34,8301
2	10,8741	52	51,4075	102	93,3804	152	43,3887	202	88,3864	252	97,2044	302	71,0911	352	32,3326	402	51,3827	452	59,6033
3	10,7082	53	29,8179	103	91,5001	153	59,2390	203	81,1045	253	130,627	303	64,9111	353	28,8540	403	78,5750	453	13,6989
4	16,2057	54	23,3984	104	42,7579	154	73,8396	204	53,9744	254	74,7229	304	88,4861	354	118,938	404	82,4683	454	9,9060
5	18,7284	55	33,1986	105	16,8979	155	57,6978	205	47,8466	255	40,6040	305	181,835	355	71,2005	405	184,116	455	37,8005
6	9,6855	56	48,0686	106	50,4787	156	40,2400	206	79,8938	256	50,1461	306	73,8915	356	85,7789	406	71,1741	456	179,563
7	25,9004	57	31,2301	107	34,3134	157	46,2825	207	17,6712	257	57,2388	307	31,3936	357	48,4639	407	16,4037	457	120,474
8	16,6645	58	52,9047	108	61,8276	158	151,299	208	35,8997	258	37,8504	308	71,8181	358	63,5827	408	101,530	458	87,9665
9	18,7522	59	62,8352	109	96,3061	159	17,9924	209	114,683	259	75,0688	309	117,636	359	127,217	409	93,6107	459	59,9685
10	23,4928	60	6,2609	110	32,4313	160	11,5061	210	116,471	260	104,819	310	29,1905	360	24,9220	410	65,2995	460	64,7419
11	16,8833	61	73,9558	111	34,0713	161	88,9035	211	110,080	261	109,332	311	39,9033	361	62,2452	411	99,1086	461	131,546
12	29,1320	62	68,2293	112	64,4693	162	103,296	212	12,6380	262	52,5487	312	93,3175	362	154,830	412	58,4737	462	49,1747
13	22,1559	63	21,7610	113	81,9159	163	63,2552	213	11,8205	263	11,5110	313	92,5843	363	16,3791	413	70,8919	463	25,1940
14	20,2364	64	24,3270	114	26,8843	164	39,4108	214	98,2928	264	173,320	314	34,2259	364	13,0487	414	85,4746	464	57,4246
15	19,3622	65	26,0197	115	37,0966	165	72,4477	215	36,4990	265	45,3884	315	11,0417	365	90,7691	415	56,5006	465	203,105
16	33,6980	66	38,8993	116	36,4565	166	15,3596	216	37,8411	266	32,4842	316	14,2761	366	72,7009	416	51,1966	466	107,369
17	41,4214	67	81,4690	117	48,6326	167	37,1614	217	130,436	267	69,5988	317	149,568	367	111,808	417	118,786	467	97,2490
18	5,4539	68	67,3915	118	38,1144	168	62,5777	218	118,104	268	104,993	318	71,9517	368	26,1026	418	164,875	468	45,3518
19	29,7180	69	33,6393	119	91,3718	169	100,356	219	47,7058	269	20,5833	319	57,5844	369	17,5220	419	63,1923	469	49,6426
20	24,7746	70	36,1415	120	83,6489	170	64,9096	220	34,8137	270	73,0014	320	53,9766	370	90,6440	420	41,1678	470	182,966
21	41,2171	71	23,4114	121	12,7365	171	58,6547	221	76,1704	271	34,5420	321	102,081	371	196,776	421	74,3566	471	67,7089
22	17,8033	72	17,6587	122	84,8677	172	14,1269	222	81,6646	272	57,1745	322	103,081	372	85,5619	422	135,387	472	47,9118
23	43,3210	73	73,9196	123	52,0091	173	80,7263	223	27,0686	273	90,8418	323	42,0248	373	18,6085	423	70,1176	473	85,1527
24	30,7658	74	38,4815	124	41,9777	174	25,5207	224	44,2347	274	35,3702	324	39,7802	374	106,345	424	29,0503	474	217,776
25	20,6240	75	28,5775	125	45,8569	175	39,2143	225	101,233	275	41,6614	325	169,443	375	167,571	425	55,0114	475	29,5183
26	39,5112	76	44,3308	126	49,0203	176	137,976	226	59,8400	276	101,332	326	51,1597	376	107,075	426	191,904	476	44,4006
27	24,7283	77	93,1937	127	20,5892	177	74,4099	227	55,2986	277	112,084	327	9,2832	377	78,8779	427	21,3561	477	34,4987
28	31,6490	78	33,7777	128	22,8019	178	27,3820	228	61,7972	278	65,5587	328	119,711	378	22,6482	428	65,9088	478	38,7058
29	16,9129	79	28,2907	129	112,695	179	80,7818	229	142,049	279	52,7478	329	141,059	379	28,8815	429	78,9385	479	247,726
30	34,8771	80	49,8816	130	88,9765	180	145,636	230	107,470	280	102,959	330	137,028	380	24,2094	430	127,484	480	52,0053
31	53,0995	81	44,8745	131	56,0832	181	59,4301	231	38,6919	281	125,015	331	47,6427	381	28,4669	431	133,832	481	30,3354
32	34,4547	82	49,9920	132	18,5247	182	31,1329	232	14,9656	282	14,6513	332	42,9610	382	114,226	432	100,875	482	63,2037
33	31,2399	83	71,1394	133	107,300	183	55,8197	233	41,7031	283	27,8072	333	86,8975	383	135,753	433	32,0846	483	73,7004
34	15,0969	84	16,3288	134	75,2092	184	35,4189	234	34,2649	284	189,955	334	74,1376	384	161,279	434	18,0929	484	69,1358
35	16,8592	85	31,8471	135	17,8558	185	41,0816	235	66,6544	285	79,3687	335	36,1274	385	87,4175	435	172,413	485	28,9377
36	66,9912	86	103,256	136	25,3368	186	23,4146	236	57,3740	286	69,0690	336	55,5672	386	39,4146	436	13,6013	486	31,3119
37	34,4068	87	50,0724	137	71,0826	187	26,4317	237	119,209	287	64,2821	337	194,429	387	49,4528	437	14,5766	487	173,557
38	23,0640	88	42,8989	138	25,5522	188	127,055	238	23,1272	288	35,2889	338	53,9954	388	156,717	438	118,767	488	162,906
39	54,5297	89	53,6766	139	59,1308	189	28,7520	239	39,4303	289	26,3859	339	44,3265	389	74,2108	439	123,715	489	30,2231
40	27,3191	90	54,2810	140	84,5741	190	37,1634	240	123,398	290	20,1689	340	13,4896	390	31,7479	440	148,479	490	40,0572
41	40,1608	91	20,2898	141	50,5164	191	150,662	241	166,579	291	72,3713	341	75,0883	391	49,3037	441	22,3599	491	125,448
42	13,0890	92	29,0996	142	31,4008	192	51,5976	242	27,8349	292	128,476	342	124,028	392	167,237	442	54,2227	492	131,949
43	41,6286	93	51,4867	143	41,1631	193	43,0340	243	26,1697	293	110,146	343	51,9595	393	36,8065	443	185,494	493	36,3023
44	61,8407	94	30,0801	144	59,2287	194	90,6304	244	22,8785	294	44,6175	344	48,9816	394	30,1476	444	194,951	494	50,7273
45	35,7384	95	31,7608	145	39,5274	195	67,0741	245	140,038	295	47,6022	345	98,1925	395	31,1054	445	41,5322	495	114,477
46	17,7363	96	107,064	146	68,4599	196	31,9929	246	51,2638	296	125,710	346	104,172	396	197,434	446	34,2259	496	76,6292
47	55,1902	97	25,5172	147	105,293	197	26,4735	247	27,3317	297	174,198	347	44,4569	397	126,699	447	106,113	497	91,3705
48	35,3289	98	23,1312	148	95,4352	198	105,614	248	68,8090	298	15,3290	348	64,2499	398	23,0290	448	60,3328	498	48,3223
49	45,4383	99	91,8313	149	11,5406	199	64,0531	249	116,958	299	12,1512	349	91,8474	399	28,7367	449	107,927	499	59,4395
50	65,7140	100	20,4283	150	54,7274	200	40,4558	250	25,3253	300	102,384	350	131,329	400	32,3913	450	78,3804	500	169,539

Values of $C_{\infty}(2,0,s)$ at Dirichlet zeroes (first 500th)

1	1,7540	51	2,5012	101	4,6862	151	2,3958	201	1,7899	251	3,6214	301	13,1771	351	3,6405	401	4,3517	451	7,7870
2	3,2443	52	2,8251	102	2,0610	152	8,7376	202	9,9357	252	2,0101	302	3,2952	352	6,6289	402	2,4824	452	1,4905
3	3,6990	53	11,5180	103	5,5234	153	1,9236	203	2,4590	253	6,2274	303	1,8857	353	2,8223	403	2,4809	453	11,2509
4	2,7383	54	1,1717	104	5,5135	154	5,1513	204	2,4849	254	4,3156	304	4,1755	354	13,0210	404	3,3743	454	1,7496
5	5,2534	55	2,1612	105	3,2382	155	4,2520	205	9,4321	255	7,6543	305	3,0469	355	2,6213	405	6,6594	455	3,2001
6	3,1847	56	6,6840	106	8,8636	156	3,2179	206	1,8816	256	1,6723	306	7,5051	356	1,2448	406	1,2413	456	4,3655
7	5,6815	57	3,9079	107	1,3374	157	5,5111	207	8,2704	257	1,7953	307	5,2170	357	10,3177	407	12,0384	457	6,9045
8	1,3412	58	6,0259	108	3,3078	158	1,6892	208	3,2571	258	9,9500	308	3,9153	358	2,2248	408	1,4502	458	5,6817
9	4,5432	59	3,3715	109	7,6160	159	8,5467	209	2,3557	259	1,7049	309	1,8065	359	8,6597	409	2,6090	459	2,0627
10	6,3339	60	3,9779	110	1,9062	160	1,8467	210	3,3590	260	8,3507	310	3,5158	360	2,2609	410	9,9024	460	9,7728
11	2,2782	61	3,0401	111	6,6770	161	9,8207	211	3,1153	261	5,3117	311	15,5711	361	4,2137	411	3,6250	461	2,1700
12	7,5943	62	4,2676	112	2,2890	162	5,1767	212	6,1160	262	2,2092	312	1,2162	362	1,5560	412	4,7938	462	3,8068
13	2,8754	63	7,5320	113	7,9283	163	1,3266	213	2,6014	263	5,4846	313	4,9642	363	5,3356	413	2,1905	463	6,5228
14	2,7086	64	1,0430	114	3,4863	164	5,8251	214	8,0681	264	4,2133	314	2,3851	364	12,6006	414	5,0258	464	2,0292
15	5,5255	65	9,7768	115	6,1381	165	1,9739	215	2,4021	265	8,9319	315	2,0939	365	1,0788	415	3,9666	465	3,7413
16	3,6943	66	1,6378	116	2,7014	166	5,6806	216	1,6093	266	1,5657	316	5,6238	366	7,8366	416	5,4098	466	3,1303
17	5,7588	67	5,3605	117	1,7488	167	2,4986	217	12,2496	267	6,6234	317	4,6214	367	4,3173	417	4,8496	467	5,4146
18	1,0723	68	6,8042	118	11,1889	168	4,3022	218	4,9126	268	2,4221	318	6,3062	368	4,1530	418	1,2905	468	5,1329
19	7,0418	69	2,7570	119	1,5620	169	2,7098	219	3,6414	269	1,4151	319	1,6336	369	4,0753	419	5,5111	469	2,4724
20	3,2332	70	6,0202	120	5,6145	170	3,3539	220	4,0463	270	15,9805	320	7,0695	370	2,9265	420	5,9272	470	5,8045
21	4,1762	71	1,9290	121	4,2895	171	13,9877	221	2,8779	271	1,6832	321	3,0939	371	3,7879	421	2,4894	471	1,5170
22	8,1422	72	9,2282	122	1,6234	172	2,7197	222	3,2738	272	4,0146	322	2,9283	372	1,8406	422	8,2219	472	13,9671
23	1,5261	73	2,6782	123	3,1766	173	3,7164	223	2,1930	273	3,0753	323	12,6332	373	6,5037	423	1,6174	473	2,4579
24	7,6455	74	3,4421	124	5,1850	174	7,0518	224	14,3727	274	5,6692	324	1,8692	374	4,9184	424	7,8411	474	2,0937
25	3,4985	75	2,9969	125	6,8591	175	1,1266	225	1,0977	275	5,8731	325	2,7610	375	2,6069	425	2,5431	475	4,7038
26	4,5921	76	1,5814	126	3,1079	176	9,3695	226	2,9592	276	2,6281	326	4,0486	376	9,3525	426	5,1616	476	1,6532
27	3,8453	77	10,3093	127	2,8838	177	4,6940	227	7,0666	277	9,8082	327	4,3144	377	1,4357	427	3,2241	477	8,7045
28	3,1681	78	4,7428	128	7,1096	178	4,7847	228	1,8383	278	1,0531	328	4,3514	378	6,9442	428	1,9084	478	4,6452
29	4,8968	79	1,9982	129	1,7012	179	1,4136	229	9,8972	279	4,2183	329	2,1388	379	6,4001	429	7,4283	479	5,2371
30	1,7679	80	4,4408	130	13,2951	180	3,8253	230	3,2863	280	5,5435	330	8,3860	380	3,0449	430	1,3269	480	2,5607
31	9,5595	81	3,1027	131	2,0552	181	9,0155	231	4,1661	281	2,4533	331	2,9270	381	3,2727	431	10,8173	481	3,8724
32	2,6567	82	5,6505	132	1,8848	182	2,0869	232	1,5822	282	6,0455	332	6,6036	382	1,7659	432	3,1402	482	7,8688
33	2,6599	83	2,7568	133	5,3656	183	13,5894	233	7,0384	283	3,5127	333	7,0212	383	7,3560	433	1,9065	483	2,2289
34	7,3816	84	9,7867	134	3,9207	184	1,9298	234	5,6829	284	3,1007	334	1,5413	384	1,9096	434	5,4863	484	6,8545
35	1,9755	85	2,3838	135	7,9803	185	3,1096	235	1,7003	285	4,0694	335	8,2415	385	4,9316	435	4,8570	485	3,4252
36	7,2994	86	1,7387	136	2,0091	186	4,9005	236	4,6858	286	5,8391	336	1,8833	386	3,1493	436	9,1899	486	0,8711
37	5,2636	87	11,1221	137	9,5411	187	4,4789	237	2,6874	287	7,0615	337	3,9666	387	2,0284	437	2,9814	487	8,8232
38	2,0817	88	1,9416	138	1,4993	188	3,4273	238	3,6257	288	1,3882	338	4,2619	388	13,5317	438	5,0279	488	3,6549
39	1,7413	89	8,2255	139	2,7236	189	1,8044	239	9,9747	289	9,1017	339	2,7170	389	2,7887	439	2,0823	489	5,2517
40	3,1965	90	4,2148	140	12,4036	190	6,0111	240	3,0522	290	3,1950	340	3,2066	390	6,0456	440	1,9215	490	1,3151
41	10,4870	91	2,6515	141	1,9180	191	2,4799	241	5,4478	291	3,7152	341	1,5262	391	4,2148	441	14,8357	491	5,6238
42	2,0578	92	4,0828	142	3,8699	192	2,9923	242	2,9439	292	4,4727	342	14,4349	392	2,9843	442	2,0177	492	3,4903
43	6,9502	93	5,1155	143	3,0066	193	11,0822	243	6,3468	293	0,9342	343	2,1943	393	3,1421	443	2,8473	493	3,2523
44	4,4554	94	10,7099	144	2,2278	194	1,8131	244	3,7020	294	8,1844	344	5,8317	394	4,5114	444	5,9527	494	14,0574
45	1,4620	95	0,9064	145	3,7425	195	6,4205	245	2,3003	295	2,5392	345	4,0654	395	11,1316	445	2,6078	495	1,8480
46	10,0235	96	6,1811	146	4,9547	196	4,2646	246	8,1041	296	8,2187	346	1,8144	396	1,4651	446	4,4530	496	2,9428
47	3,9105	97	2,2154	147	6,8413	197	4,1749	247	1,0752	297	2,1074	347	9,6442	397	2,9585	447	4,1442	497	4,8940
48	6,6083	98	1,7379	148	1,7586	198	6,5164	248	9,9705	298	4,3798	348	5,1685	398	4,3517	448	9,2997	498	2,5304
49	1,4710	99	9,0818	149	7,1455	199	1,8043	249	2,7632	299	3,4576	349	3,0295	399	1,5658	449	1,1184	499	5,5318
50	5,0024	100	3,1869	150	5,2193	200	5,4704	250	3,5870	300	1,9603	350	2,5080	400	13,6871	450	3,4222	500	3,1275